Math 524

Homework due 12/01/10

Reading from Stein & Shakarchi: Chapter 2: §3. Chapter 6: §3.

Exercises from Stein & Shakarchi: Chapter 2: 4.

Problem 1: Show that in Egorov's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by $|f_n| \leq g$ for all n, where $g \in L^1(\mu)$.

Problem 2: (A generalized Dominated Convergence Theorem) Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Assume that $f_n,\ g_n,\ f,\ g \in L^1(\mu)$, $f_n \to f$ and $g_n \to g$ μ-a.e., $|f_n| \leq g_n$ and $\int g_n \, d\mu \to \int g \, d\mu$. Prove that $\int f_n \, d\mu \to \int f \, d\mu$ and that $f_n \to f$ in $L^1(\mu)$.

Problem 3: The goal of this problem is to prove the following result.

Theorem: Let f be a bounded real-valued function on $[a, b]$. f is Riemman integrable if and only if $\{x \in [a, b] : f \text{ is discontinuous at } x\}$ has Lebesgue measure 0.

Given a bounded function $f : [a, b] \to \mathbb{R}$, let

$$H(x) = \lim_{\delta \to 0} \sup_{|x-y| \leq \delta} f(y), \qquad h(x) = \lim_{\delta \to 0} \inf_{|x-y| \leq \delta} f(y).$$

Using the notation introduced in class which is the same one as the one used in the proof of Theorem 2.28 a) in Folland Chapter 2 (page 57), establish the following lemmas.

- $H(x) = h(x)$ is f is continuous at x.
- $H = G$ a.e. and $h = g$ a.e. Thus H and h are Lebesgue measurable, and

$$\int_{[a,b]} H \, dm = \overline{I}_a^b(f) \qquad \text{and} \qquad \int_{[a,b]} h \, dm = \underline{I}_a^b(f)$$

($*$) Problem: Let X be a compact metric space, and let μ be a finite positive Borel measure on X. Suppose that $\mu\{x\} = 0$ for every $x \in X$. Prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that, if E is any Borel subset of X having diameter less than δ, then $\mu(E) < \epsilon$.