Math 524

Homework due 12/08/2010

Reading from Stein & Shakarchi: Chapter 6: §3. Chapter 3: §1

Exercises from Stein & Shakarchi: Chapter 2: 19, 21.

Problem 1: Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, μ=Lebesgue measure and ν=counting measure. If $D = \{(x,x) : \ x \in [0,1]\}$ is the diagonal in $X \times Y$, show that

$$\iint \chi_D \, d\mu d\nu, \quad \iint \chi_D \, d\nu d\mu, \quad \iint \chi_D \, d(\mu \times \nu)$$

are all unequal. To compute $\iint \chi_D \, d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.

Problem 2: Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in \mathcal{L}^+$. Let

$$G_f = \{(x,y) \in X \times [0,\infty] : \ y \leq f(x)\}.$$

1. Prove that G_f is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$ measurable. (Hint: consider the map $(x,y) \to f(x) - y$).
2. Prove that $\mu \times m(G_f) = \int f \, d\mu$.

Problem 3: Let $(X\mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.

1. If $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable, $g : Y \to \mathbb{R}$ is $\mathcal{N}$-measurable, and $h(x, y) = f(x)g(y)$, prove that h is $\mathcal{M} \otimes \mathcal{N}$-measurable.

2. If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, prove that $h \in L^1(\mu \times \nu)$ and

$$\int h \, d(\mu \times \nu) = \left[\int f \, d\mu \right] \left[\int g \, d\nu \right].$$

Problem 4: Let (X, d) be a metric space. Let μ be a doubling measure on $\mathcal{B}_X$, i.e. there exists a constant $C \geq 1$ such that $\forall x \in X$ and $r > 0$

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Assume that $\mu(B(x_0, 1)) < \infty$ for some $x_0 \in X$.

1. Show that if K is compact then $\mu(K) < \infty$.

2. Assume $X = \mathbb{R}^n$ and d is the Euclidean distance. Prove that if $E \in \mathcal{B}_{\mathbb{R}^n}$ with $\mu(E) < \infty$ then given $\epsilon > 0$ there exists a compact set K with $K \subset E$ and $\mu(E \backslash K) < \epsilon$.

   Note: μ is not necessarily Lebesgue measure.