Math 525

Homework due 01/14/2015

Reading from Folland: Chapter 3: sections 1 and 2.

Exercises from Folland: Chapter 3: 40, 42

Problem 1: Let f be a continuous real-valued function on [0, 1]. Let k be a fixed positive integer. Suppose that for each $y \in \mathbb{R}$ the set $\{x \in [0,1] : f(x) = y\}$ contains at most k points. Prove that $f \in BV[0,1]$.

Problem 2: Show that if F is of bounded variation in $[a,b]$, then:

$$\int_a^b |F'(x)|\, dx \leq T_a^b(F)$$

and

$$\int_a^b |F'(x)|\, dx = T_a^b(F)$$

if and only if F is absolutely continuous in $[a,b]$.

Problem 3: A bounded function $F : \mathbb{R} \to \mathbb{R}$ is said to be of bounded variation on $\mathbb{R}$ if F is of bounded variation on any finite interval $[a,b]$ and $\sup_{[a,b]} T_a^b(F) < \infty$. Prove that F enjoys the following properties:

1. $\int_{\mathbb{R}} |F(x+h) - F(x)|\, dx \leq A|h|$, for some constant A and all $h \in \mathbb{R}$.
2. $|\int_{\mathbb{R}} F(x)\varphi'(x)|\, dx| \leq A$, where φ ranges over all C^1 functions with compact support satisfying $\sup_{x\in\mathbb{R}} |\varphi(x)| \leq 1$.