Math 525

Homework due 01/21/2015

Reading from Folland: Chapter 4: sections 1, 2, 4 and 5.

Exercises from Folland: Chapter 3: 3 and 9

Problem 1 (Prelim 2000): Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n\geq 1}$ be a sequence of non-negative measurable functions such that

$$\int_X f_n \, d\mu \leq 1.$$

Prove that

$$\limsup_{n\to\infty} (f_n(x))^{1/n} \leq 1 \quad \mu \text{ a.e. } x \in X.$$

Problem 2: Let $f : \mathbb{R}^d \to (0, \infty]$ be an $L^1(\mathbb{R}^d)$ function. Prove that for any given $E \in \mathcal{B}_{\mathbb{R}^d}$ with $m(E) > 0$ and any $\alpha \in (0, 1)$ there exists a closed cube Q such that

$$\int_{E\cap Q} f \, dm \geq \alpha \int_Q f \, dm.$$

Here m denotes the Lesbegue measure in $\mathbb{R}^d$.

Hint: Show that for $g \in L^1_{loc}(\mathbb{R}^d)$

$$\lim_{x\in Q,\ m(Q)\to 0} \frac{1}{m(Q)} \int_Q g(y) \, dy,$$

where the Q's belong to the collection of dyadic cubes containing x.

Problem 3: For $f \in L^1_{loc}(\mathbb{R}^n)$, and $0 < s < n$ define

$$\Lambda_s = \{x \in \mathbb{R}^n : \liminf_{r\to 0} \frac{1}{r^s} \int_{B(x,r)} |f|(y) \, dy > 0.\}$$

Prove that $m(\Lambda_s) = 0$, for all $s \in (0, n)$.