Math 525

Homework due 02/04/2015

Reading from Folland: Chapter 4: sections 6 and 7.

Problems from Folland: Chapter 4: 63 and 64.

Problem 1: Show that a Hausdorff space is normal if and only if it satisfies the conclusion of Urysohn's lemma.

Problem 2: Consider $\mathbb{R}^d$ with the topology given by the Euclidean metric.

1. Let $K \subset U \subset \mathbb{R}^d$, where K is compact and U is open. Prove that there exists a precompact open set V such that $K \subset V \subset \overline{V} \subset U$.
2. Let $K \subset U \subset \mathbb{R}^d$, where K is compact and U is open. Prove that there exists $f \in C(\mathbb{R}^d, [0,1])$ such that $f = 1$ on K and $f = 0$ outside a compact subset of U.
3. Let K be a compact subset of $\mathbb{R}^d$, and let $\{U_j\}_{j=1}^n$ be an open cover of K. Prove that there exists a family of compactly supported continuous functions $\{h_j\}_{j=1}^n \subset C(\mathbb{R}^d, [0,1])$ such that the support of h_j is contained in U_j and $\sum_{j=1}^n h_j(x) = 1$ for all $x \in K$. (The family $\{h_j\}_{j=1}^n$ is a partition of unity on K subordinated to the open cover $\{U_j\}_{j=1}^n$.)

Problem 3: Let X be a topological space, (Y, ρ) a complete metric space, and $\{f_n\}_n$ a sequence of functions from X to Y such that

$$\sup_{x \in X} \rho(f_n(x), f_m(x)) \to 0 \text{ as } m,\ n \to \infty.$$

There is a unique function $f : X \to Y$ such that

$$\sup_{x \in X} \rho(f_n(x), f(x)) \to 0 \text{ as } n \to \infty.$$

If each f_n is continuous, so is f.