Math 525

Homework due 03/11/2015

Reading from Folland, 2nd edition: Chapter 6: section 2 and 3.

Problems from Folland: Chapter 6: problems 20 and 22.

Problem 1 (20 points): Let $(X, \mathcal{M}, \mu)$ be a complete σ-finite measure space.

1. Show that for $p \geq 2$ and $x \in [0, 1]$
$$\left(\frac{1+x}{2}\right)^p + \left(\frac{1-x}{2}\right)^p \leq \frac{1}{2}(1 + x^p)$$

2. Prove that for $2 \leq p < \infty$, if $f,\ g \in L^p(\mu)$ then
$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \leq \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right)$$
This is Clarkson inequality.

3. Show that for $p \geq 2$, L^p is **uniformly convex**. Recall this means that given $\epsilon > 0$ there is $\delta = \delta(\epsilon) \in (0, 1)$ with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ s so that whenever
$$\|f\|_p = \|g\|_p = 1 \ \text{ then } \ \|f - g\|_p \geq \epsilon \text{ implies that } \left\|\frac{f+g}{2}\right\|_p \leq 1 - \delta.$$

4. Using the result above prove that for $2 \leq p < \infty$ the following statement holds. If $\{f_n\}_n \in L^p$ converges weakly to f and $\|f_n\|_p \to \|f\|_p$ then $\{f_n\}_n$ converges strongly to f, that is $\|f_n - f\|_p \to 0$ as $n \to \infty$.

Problem 2 (Reals 2011): Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ a one-to-one bounded linear map whose range $T(\mathcal{T})$ is closed in $\mathcal{Y}$. Show that for each bounded linear functional ϕ on $\mathcal{X}$ there is a bounded linear functional ψ on $\mathcal{Y}$ such that $\phi = \psi \circ T$, and there is a constant C (independent of ϕ) such that ψ can be chosen to satisfy $\|\psi\| \leq C\|\phi\|$.