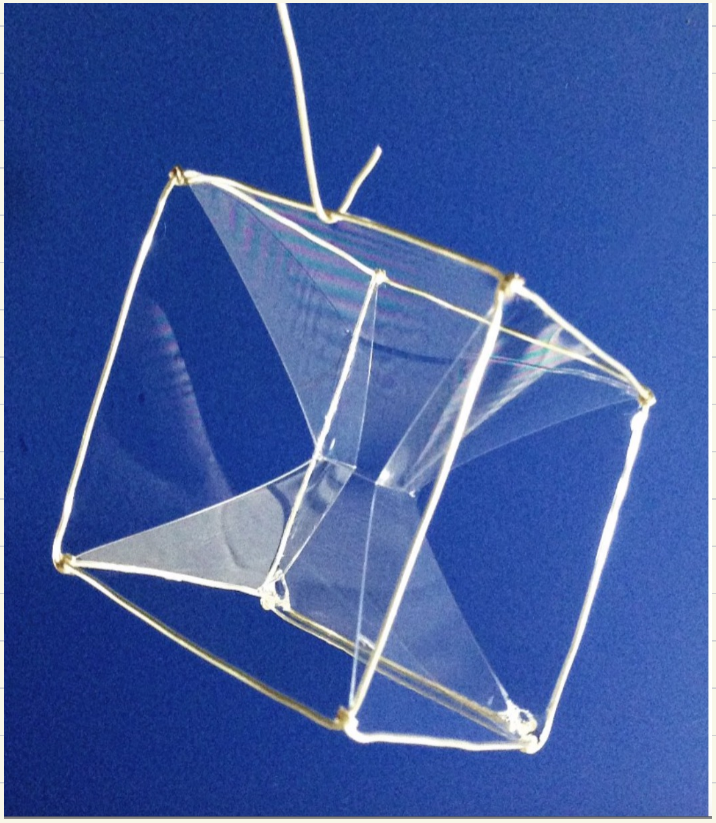
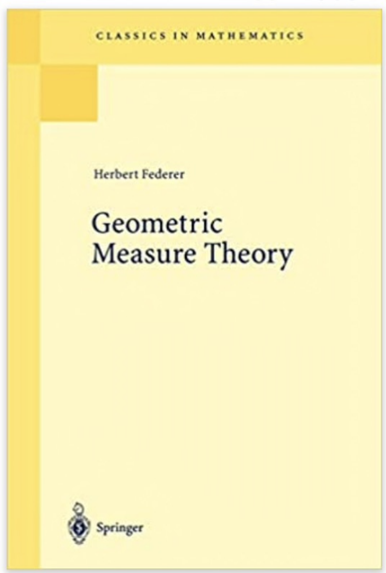
Introduction to Geometric Measure Theory





Definition 1: Let $X \neq \emptyset$, a mapping $\mu : 2^X \longrightarrow [0,\infty]$ is a measure on X provided

i) $\mu \emptyset = 0$

ii) (subadditivity) if $A \subset \bigcup_{k=1}^{\infty} A_k$ then $\mu A \leq \sum_{i=1}^{\infty} \mu A_i$

Definition 2: A set $A \subset X$ is μ-measurable if for each $B \subset X$

$$\mu B = \mu(B \cap A) + \mu(B \setminus A)$$

Definition 3: $\mathcal{U} \subset 2^X$ is a σ-algebra if

i) $\emptyset, X \in \mathcal{U}$

ii) $A \in \mathcal{U} \implies A^c \in \mathcal{U}$

iii) if $A_k \in \mathcal{U}$ then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{U}$

Definition 4: The Borel σ-algebra of a topological (metric) space X is the σ-algebra generated by the open sets of X

Definitions : A measure μ in $\mathbb{R}^n$ is

i) a Borel measure if all Borel sets are measurable

ii) Borel regular if it Borel and $A \subset \mathbb{R}^n$ $\exists B$ Borel set s.t
$A \subset B$ & $\mu A = \mu B$.

iii) a Radon measure is Borel regular and $\mu K < \infty$ for every
$K \subset \mathbb{R}^n$ compact.

Theorem : Let μ be a Borel regular measure in $\mathbb{R}^n$. If
$A_1 \subset A_2 \subset \dots A_k \subset A_{k+1} \subset \dots$ then

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

What is the point of this theorem? The A_k's do not need to be measurable!

<u>Theorem</u> : Let μ be a Borel regular measure on $\mathbb{R}^n$. Suppose $A \subset \mathbb{R}^n$ is μ-measurable and $\mu A < \infty$, then $\mu \llcorner A$ is Radon.

<u>Lemma</u> : Let μ be a Borel measure on $\mathbb{R}^n$ & let B be a Borel set

i) if $\mu B < \infty$ $\forall \varepsilon > 0$ $\exists C$ closed set such that $C \subset B$ & $\mu(B \setminus C) < \varepsilon$

ii) if μ is a Radon measure $\forall \varepsilon > 0$ $\exists U$ open set such that $B \subset U$ & $\mu(U \setminus B) < \varepsilon$

Compare with Lebesgue measure!

$\tilde{i})$ if $\mu = m$ i) does <u>not</u> need $mB < \infty$

if $\mu = m$ in i) $mB < \infty$ then C can be taken compact.

if μ is Radon $\tilde{i})$ holds

Theorem : (approximation by open and compact sets)

Let μ be a Radon measure in $\mathbb{R}^n$, then

i) for $A \subset \mathbb{R}^n$

$$\mu A = \inf \{ \mu U : A \subset U \text{ open} \}$$

A does not need to be measurable

ii) for each μ - measurable set $A \subset \mathbb{R}^n$

$$\mu A = \sup \{ \mu K : K \subset A : K \text{ compact} \}$$

" compact nicer than closed".

What is the point of this theorem ?

Theorem: (Caratheodory's criterion)

Let μ be a measure on $\mathbb{R}^n$. If for all sets $A, B \subset \mathbb{R}^n$ we have

$$\mu(A \cup B) = \mu A + \mu B \quad \text{whenever } d(A,B) > 0$$

then μ is a Borel measure.

$$d(A,B) = \inf\{|a-b| : a \in A, b \in B\}$$

Definition: Let μ be a Borel regular measure in $\mathbb{R}^n$

$$\underbrace{\operatorname{spt}}_{\text{support of } \mu} \mu = \{x \in \mathbb{R}^n : \mu(B(x,r)) > 0 \quad \forall r > 0\}$$

Claim: $\operatorname{spt} \mu$ is a closed set.

Covering theorems

A. Vitali's covering lemma

$B = B(x, r)$ = closed ball in $\mathbb{R}^n$ $\qquad \hat{B} = 5B = B(x, 5r)$

Definitions: i) A collection $\mathcal{F}$ of closed balls in $\mathbb{R}^n$ is a cover of a set $A \subset \mathbb{R}^n$ if

$$A \subset \bigcup_{B \in \mathcal{F}} B$$

ii) $\mathcal{F}$ is a fine cover of A if it is a cover and
$\inf \{ \operatorname{diam} B : x \in B : B \in \mathcal{F} \} = 0$ for each $x \in A$.

Theorem : (Vitali's covering theorem) Let $\mathcal{F}$ be a collection of non-degenerate closed balls in $\mathbb{R}^n$ with

$$D = \sup \{ \operatorname{diam} B : B \in \mathcal{F} \} < \infty$$

Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} \hat{B}$$

Moreover for each $B \in \mathcal{F}$ $\exists B' \in \mathcal{G}$ s.t $B \cap B' \neq \emptyset$ and $B \subset \hat{B'}$

Why is the diameter hypothesis necessary? $\mathcal{F} = \{B(0,m)\}_{m \in \mathbb{N}}$ does not satisfy theorem

How do you prove this?

How did we use this theorem?

Corollary: Assume that $\mathcal{F}$ is a fine cover of A by closed balls and that

$$\sup\{\operatorname{diam} B : B \in \mathcal{F}\} < \infty$$

Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that for each finite subset $\{B_1, \dots B_m\} \subseteq \mathcal{F}$

$$A \setminus \bigcup_{k=1}^{m} B_k \subset \bigcup_{B \in \mathcal{G} \setminus \{B_1, \dots B_m\}} \hat{B}$$

How do you prove this?

<u>Theorem</u> (Filling open sets with balls)

Let $U \subset \mathbb{R}^n$ be an open set, let $\delta > 0$. There exists a countable collection $\mathcal{G}$ of <u>disjoint</u> closed balls in U s.t. $\operatorname{diam} B < \delta$ $\forall B \in \mathcal{G}$ and $\mathcal{L}^n(U \setminus \bigcup_{k=1}^{\infty} B_k) = 0$ ($\mathcal{L}^n$ Lebesgue measure)

<u>Pf</u>: Fix θ : $1 - \frac{1}{5^n} < \theta < 1$ assume $\mathcal{L}^n(U) < \infty$

<u>Claim</u> There exists a finite collection $\{B_i\}_{i=1}^{M_1}$ of disjoint closed balls $\operatorname{diam} B_i \not\leq \delta$ $\forall i = 1 \dots M_1$

$$\mathcal{L}^n(U \setminus \bigcup_{i=1}^{M_1} B_i) \leq \theta \mathcal{L}^n(U)$$

<u>Pf of claim</u>: $\mathcal{F}_1 = \{ \underset{\text{closed}}{B \subset U} : \operatorname{diam} B < \delta \} \neq \emptyset$

by Vitali $\exists\, \mathcal{G}_1$ countable disjoint subfamily of $\mathcal{F}_1$

$$U \subset \bigcup_{B \in \mathcal{G}_1} \hat{B} \qquad L^n(U) \leq \sum_{B \in \mathcal{G}_1} L^n(\hat{B}) = \underbrace{5^n \sum_{B \in \mathcal{G}_1} L^n(B)}_{5^n L^n(U) < \infty} \leq$$

$$5^n \sum_{B \in \mathcal{G}_1} L^n(B) = 5^n L^n\Big(\bigcup_{B \in \mathcal{G}_1} B\Big) \geq L^n(U)$$

$$L^n\Big(U \setminus \bigcup_{B \in \mathcal{G}_1} B\Big) = L^n(U) - L^n\Big(\bigcup_{B \in \mathcal{G}_1} B\Big) \leq \left(1 - \frac{1}{5^n}\right) L^n(U)$$

$$L^n\Big(U \setminus \bigcup_{B \in \mathcal{G}_1} B\Big) \leq \left(1 - \frac{1}{5^n}\right) L^n(U)$$

There are finitely many balls in $\mathcal{G}_1$ s.t $\left(1 - \frac{1}{5^n} < \theta\right)$

$$L^n\Big(U \setminus \bigcup_{i=1}^{M_1} B_i\Big) \leq \theta L^n(U)$$

$U = U_1 \qquad U_2 = U \setminus \bigcup_{i=1}^{M_1} B_i$ open & $L^n(U_2) < \infty$

$\mathcal{F}_2 = \{ B \subset U_2 \text{ closed} \quad \operatorname{diam} B < \delta \}$ use same argument

$\{B_i\}_{i=M_1+1}^{M_2} \subset \mathcal{F}_2$ disjoint

$$\mathcal{L}^n \Big(U_2 \setminus \bigcup_{i=M_1+1}^{M_2} B_i \Big) \le \theta \, \mathcal{L}^n (U_2)$$

$$\mathcal{L}^n \Big(U \setminus \bigcup_{i=1}^{M_2} B_i \Big) \le \theta \, \mathcal{L}^n \Big(U \setminus \bigcup_{i=1}^{M_1} B_i \Big) \le \theta^2 \mathcal{L}^n (U)$$

by induction

$$\mathcal{L}^n \Big(U \setminus \bigcup_{i=1}^{M_k} B_i \Big) \le \theta^k \underbrace{\mathcal{L}^n (U)}_{< \infty} \xrightarrow[k \to \infty]{} 0$$

If $\mathcal{L}^n(U) = \infty$ apply construction to

$U_m = \{ x \in U : \quad m < |x| < m+1 \}$ or

$\hat{U}_j = U \cap Q_j$. $\quad \{Q_j\}$ open cubes of sidelength 1

$\bigcup \overline{Q}_j = \mathbb{R}^n$

$\mathcal{L}^n (\partial (B_{m+1} \setminus B_m)) = 0 \qquad \mathcal{L}^n (\partial Q_j) = 0$