Definition: A measure μ in $\mathbb{R}^n$ is doubling if $\exists C > 1$ s.t $\forall x \in \operatorname{spt} \mu$ $\forall r \in (0, \operatorname{diam} \operatorname{spt} \mu)$

$$\mu(B(x,2r)) \leq C \mu(B(x,r))$$

Problem 1: Let μ be a doubling measure on $\mathbb{R}^n$, assume $x_0 \in \operatorname{spt} \mu$ and

$$\mu(B(x_0,1)) < \infty$$

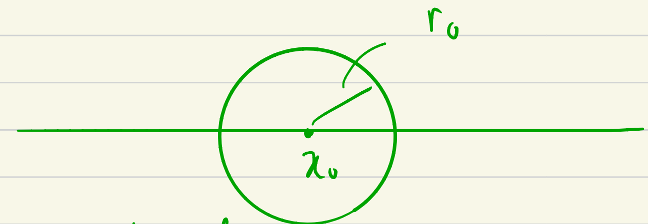
Show that for every $x \in \operatorname{spt} \mu$ there exists $E_x \subset (0, \operatorname{diam} \operatorname{spt} \mu)$

$m((0, \operatorname{diam} \operatorname{spt} \mu) \setminus E_x) = 0$ $\quad \forall r \in E_x$

$$\mu(\partial B(x,r)) = 0$$

Given an example of a doubling measure in $\mathbb{R}^2$, $x_0 \in \operatorname{spt} \mu$ and $r_0 > 0$ such that

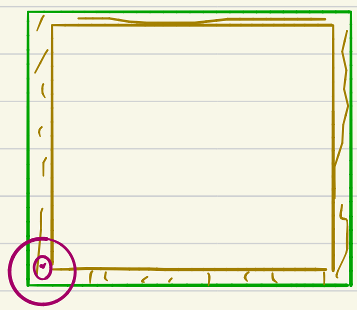
$$\mu(\partial B(x_0, r_0)) \neq 0$$



$\mu =$ arc length

Problem 2 : Let μ be a Borel measure in $\mathbb{R}^n$, such that $\operatorname{spt}\mu = \mathbb{R}^n$, assume μ is doubling and $\mu(B(0,1)) < \infty$ show that for any dyadic cube Q in $\mathbb{R}^n$

$$\mu(\partial Q) = 0.$$



unit cube

Besicovitch covering theorem

Theorem (Besicovitch covering theorem)

There exists a constant N_n depending only on the dimension n, with the following property: If $\mathcal{F}$ is a collection of non-degenerate closed balls in $\mathbb{R}^n$ with

$$\sup\{\operatorname{diam} B : B \in \mathcal{F}\} < \infty$$

and if A is the set of centers of the balls in $\mathcal{F}$ then there exist N_n countable collections $\mathcal{G}_1, \dots, \mathcal{G}_{N_n}$ of disjoint balls in $\mathcal{F}$, such that

$$A \subset \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}_i} B$$

① Read proof in Evans & Gariepy

② Does Besicovitch hold on metric spaces ?

Corollary: If μ is a measure in $\mathbb{R}^n$ and $\mathcal{F}$ & A are as in Besicovitch covering theorem, then there exists a countable disjoint subfamily of $\mathcal{F}$, $\mathcal{F}'$, such that

$$\mu(A) \le c(n) \sum_{B \in \mathcal{F}'} \mu(A \cap B)$$

Warning: $\mathcal{F}'$ does not necessarily cover A. !

Pf

$$A \subset \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}_i} B \cap A$$

$$\mu(A) \le \sum_{i=1}^{N_n} \underbrace{\sum_{B \in \mathcal{G}_i} \mu(A \cap B)}_{\text{choose } \mathcal{G}_i \text{ s.t}} \le N_n \sum_{B \in \mathcal{F}'} \mu(A \cap B)$$

$$\sum_{B \in \mathcal{G}_i} \mu(A \cap B) \text{ is the maximum} \qquad \mathcal{G}_i = \mathcal{F}'$$

Theorem : (More on filling open sets with balls)

Let μ be a Borel measure on $\mathbb{R}^n$, $\mathcal{F}$ a non-degenerate collection of closed balls. Let A denote the set of centers of balls in $\mathcal{F}$.

Assume : i) $\mu(A) < \infty$

ii) $\inf\{r : B(a,r) \in \mathcal{F}\} = 0 \quad \forall a \in A$

Then for each open set $U \subset \mathbb{R}^n$ there exists a countable disjoint subcollection $\mathcal{G} \subset \mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{G}} B \subset U \quad \&$$

$$\mu\Big(A \cap U \setminus \bigcup_{B \in \mathcal{G}} B\Big) = 0$$

Remark : A does not need to be μ-measurable.

Proof : $1 - \frac{1}{N_n} < \theta < 1$

Claim : $\exists \{B_1, \dots B_{M_1}\}$ disjoint closed balls in $\mathcal{F}$; $B_i \subset U$

$$\mu\left(A \cap U \setminus \bigcup_{i=1}^{M_1} B_i\right) \leq \theta \mu(A \cap U)$$

Pf : Wr $\mathcal{F}_1 = \{B \in \mathcal{F} : \operatorname{diam} B \leq 1 ; B \subset U\}$

$\mathcal{F}_1$ covers A by Besicovitch $\exists \mathcal{G}_1, \dots \mathcal{G}_{N_n}$ disjoint subfamilies

$$A \cap U \subset \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}_i} B$$

$\exists \mathcal{G}_i$

$$\mu(A \cap U) \leq N_n \sum_{B \in \mathcal{G}_i} \mu(A \cap U \cap B)$$

$$\frac{1}{N_n} \mu(A \cap U) \leq \mu\left(A \cap U \cap \bigcup_{B \in \mathcal{G}_i} B\right)$$

Claim : $B_1 \dots B_{M_1}$ $\quad (1-\theta)\mu(A \cap U) \leq \mu\left(A \cap U \cap \bigcup_{i=1}^{M_1} B_i\right)$

$$\mu(A \cap U) = \mu\left(A \cap U \cap \bigcup_{i=1}^{M_1} B_i\right) + \mu\left(A \cap U \setminus \bigcup_{i=1}^{M_1} B_i\right)$$

$$\geq (1-\theta)\,\mu(A \cap U) + \mu\left(A \cap U \setminus \bigcup_{i=1}^{M_1} B_i\right)$$

(2) $U_2 = U \setminus \bigcup_{i=1}^{M_1} B_i$ repeat.

$$\mu\left(A \cap U \setminus \bigcup_{i=1}^{M_k} B_i\right) \leq \theta^k \mu(A \cap U) \leq \theta^k \underbrace{\mu(A)}_{<\infty}$$

$k \to \infty$ ✓

Differentiation of Radon measures

Definition: Let μ & ν be Radon measures. For $x \in \mathbb{R}^n$ define

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup\limits_{r \to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } x \in \operatorname{spt}\mu \\ +\infty & \text{otherwise} \end{cases}$$

upper density of ν w.r.t μ

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf\limits_{r \to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } x \in \operatorname{spt}\mu \\ +\infty & \text{otherwise} \end{cases}$$

lower density of ν w.r.t μ

Definition: If $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) < \infty$ we say that ν is differentiable w.r.t μ a x and write

$$\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) = D_\mu \nu(x)$$

derivative or density of ν w.r.t μ at x

Relationship with Radon-Nikodym ?