Differentiation of Radon measures

Definition: Let μ & ν be Radon measures. For $x \in \mathbb{R}^n$ define

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup\limits_{r \to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } x \in \operatorname{spt}\mu \\ +\infty & \text{otherwise} \end{cases}$$

upper density of ν w.r.t μ

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf\limits_{r \to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } x \in \operatorname{spt}\mu \\ +\infty & \text{otherwise} \end{cases}$$

lower density of ν w.r.t μ

Definition: If $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) < \infty$ we say that ν is differentiable w.r.t μ a x and write

$$\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) = D_\mu \nu(x)$$

derivative or density of ν w.r.t μ. at x

Lemma: Fix $0 < \alpha < \infty$. Then if

i) $A \subset \{ x \in \mathbb{R}^n : \underline{D}_\mu \nu(x) \le \alpha \} \Rightarrow \nu(A) \le \alpha \mu(A)$ $(*)$

ii) $A \subset \{ x \in \mathbb{R}^n : \overline{D}_\mu \nu(x) \ge \alpha \} \Rightarrow \nu(A) \ge \alpha \mu(A)$

Remark: A does not need to be measurable.

Pf: Assume $\nu(\mathbb{R}^n) < \infty$ & $\mu(\mathbb{R}^n) < \infty$

Fix ε, U open $A \subset U$ A satisfies i)

$$\mathcal{F} = \{ B \underset{\substack{\uparrow \\ \text{closed} \\ \text{balls}}}{=} B(a,r) : a \in A \quad B(a,r) \subset U \quad r \le 1 \quad \frac{\nu(B)}{\mu(B)} \le \alpha + \varepsilon \}$$

$$\Big(\underline{D}_\mu \nu(a) = \liminf_{r \to 0} \frac{\nu(B(a,r))}{\mu(B(a,r))} \le \alpha$$

$\mathcal{F}$ is a fine cover of A, $B \in \mathcal{F}$ diam $B \le 2$

↑

$\inf \{ r : B(a,r) \in \mathcal{F} \} = 0 \quad \forall a \in A$

By cor. of Besicovitch $\exists \mathcal{G} \subset \mathcal{F}$ disjoint s.t

$\nu\big(A \cap U \setminus \bigcup_{B\in\mathcal{G}} B\big) = 0$ $\qquad A \cap U = A \quad (A \subset U)$

$\nu\big(A \setminus \bigcup_{B\in\mathcal{G}} B\big) = 0$

$B \subset U$
$+$ disjoint $\mathcal{G}$

$$\nu(A) \leq \sum_{B\in\mathcal{G}} \nu(B) \leq (\alpha+\varepsilon) \sum_{B\in\mathcal{G}} \mu(B) \leq (\alpha+\varepsilon)\mu(U)$$

$$\nu(A) \leq (\alpha+\varepsilon) \underbrace{\inf\{\mu(U) : U \text{ open } \& \ A \subset U\}}_{\mu A}$$

$\forall \varepsilon > 0 \qquad \varepsilon \to 0$

$\nu(A) \leq \alpha \mu(A).$ $\qquad \blacksquare$

Theorem (Differentiating measures)

Let μ & ν Radon measures in $\mathbb{R}^n$. Then

i) $D_\mu \nu$ exists and is finite μ a.e.

ii) $D_\mu \nu$ is μ-measurable

Pf μ, & ν finite

claim 1: $D_\mu \nu$ exists & is finite μ-a.e

$$I = \{x : \overline{D}_\mu \nu(x) = +\infty\} \subset \{x : \overline{D}_\mu \nu(x) \geq \alpha\} \quad \forall \alpha > 0$$

$$\nu(I) \geq \alpha \mu I \Rightarrow \mu I \leq \frac{1}{\alpha} \nu(I) \qquad \alpha \to \infty$$

$$\Rightarrow \mu I = 0 \Longrightarrow \overline{D}_\mu \nu \text{ is finite } \mu\text{-a.e.}$$

$0 < a < b < \infty$

$$R(a,b) = \{x : \underbrace{\underline{D}_\mu \nu(x) < a}_{\nu(R(a,b)) \leq a \mu(R(a,b))} < \underbrace{b < \overline{D}_\mu \nu(x)}_{\nu(R(a,b)) \geq b \mu(R(a,b))} < \infty\}$$

$\nu(R(a,b)) \le a\,\mu(R(a,b))$ $\qquad$ $\nu(R(a,b)) \ge b\,\mu(R(a,b))$

$$b\,\mu(R(a,b)) \le \nu(R(a,b)) \le a\,\mu(R(a,b))$$

$\Rightarrow$ $\mu(R(a,b)) = 0$ bec $b > a$.

$$\{x : \underline{D}_\mu\nu(x) < \overline{D}_\mu\nu(x)\} = \bigcup_{\substack{a,b\in\mathbb{Q}\\ a<b}} R(a,b)$$

$$\mu\{x : \underline{D}_\mu\nu(x) < \overline{D}_\mu\nu(x)\} = 0$$

$\Rightarrow$ $D_\mu\nu$ exists μ. a.e. (we want $D_\mu\nu$ be a limit of measurable functions)

Claim 2 : for $x \in \mathbb{R}^n$ $\quad r > 0$

$$\limsup_{y\to x} \mu(B(y,r)) \le \mu(B(x,r))$$

(same thing holds for ν) ↗

($\mu(B(x,r))$ is upper semi cont as a function of x)

↑ fixed

Pf of Claim 2 : $y_k \to x$ w.w. $\limsup_{k\to\infty} \mu(B(y_k, r)) \leq \mu(B(x,r))$

$$\mu(B(y_k,r)) = \int f_k \, d\mu \qquad f_k = \chi_{B(y_k, r)}$$

k large enough

$B(y_k, r) \subset B(x, 2r)$

$0 \leq 1 - f_k$ Fatou

$$\int_{B(x,2r)} \liminf_{k\to\infty} (1-f_k) \, d\mu \leq \liminf_{k\to\infty} \int_{B(x,2r)} (1-f_k) \, d\mu.$$

$f_k \to f$ μ a.e (true a.e $r>0$)

$$\int_{B(x,2r)} (1 - \chi_{B(x,r)}) \, d\mu \leq \liminf_{k\to\infty} \mu(B(x,2r)) - \mu(B(y_k, r))$$

$$\underbrace{\mu(B(x,2r)}_{<\infty} - \mu(B(x,r)) \leq \mu B(x,2r)) - \limsup_{k\to\infty} \mu(B(y_k, r))$$

Claim 3: $D_\mu V$ is μ measurable

$x \longmapsto \mu(B(x,r))$, $V(B(x,r))$ μ-measurable functions

$$f_r(x) = \begin{cases} V(B(x,r))/\mu(B(x,r)) & x \in \operatorname{spt}\mu \\ +\infty & \mu(B(x,r)) = 0 \end{cases}$$

$\uparrow$

μ measurable

$r_k \to 0$ $\quad D_\mu V(x) = \lim_{r_k \to 0} f_{r_k}(x)$ is μ-measurable

Definitions : Assume μ & ν Borel measures in $\mathbb{R}^n$

(i) ν is absolutely continuous w.r.t μ : $\nu \ll \mu$ if

$$\mu A = 0 \implies \nu(A) = 0$$

(ii) μ & ν are mutually singular : $\mu \perp \nu$ if $\exists$ B Borel s.t

$$\mu(\mathbb{R}^n \setminus B) = \nu(B) = 0$$

Theorem (Differentiation of Radon measures

Let μ & ν be Radon measures in $\mathbb{R}^n$ with $\nu \ll \mu$ then

$$\nu(A) = \int_A D_\mu \nu(x)\, d\mu(x) \qquad A \ \mu\text{-meas}$$

Pf : If A is μ-measurable then A is ν-measurable

(def of ν measurability)

w log μ & ν finite

$$\left.\begin{array}{l} Z = \{x : D_\mu \nu(x) = 0\} \\ I = \{x : D_\mu \nu(x) = +\infty\} \end{array}\right\} \mu \text{ measurable.}$$

$$\uparrow$$

$$\mu I = 0 \quad \overset{}{\underset{\nu \ll \mu}{\Longrightarrow}} \quad \nu I = 0 \qquad\qquad \nu(I) = \int_I \overset{\overset{0}{\sim}}{D_\mu \nu}\, d\mu = 0$$

$$Z \subset \{x : D_\mu \nu(x) < \alpha\} \qquad \forall\, \alpha > 0$$

$$\Longrightarrow \quad \nu(Z) \leq \alpha\, \mu(Z) \qquad \forall \alpha > 0 \qquad \Longrightarrow \nu(Z) = 0$$

$$f \quad A \cap Z = \emptyset$$

A μ-measurable, fix $1 < t < \infty$ $\quad m \in \mathbb{Z}$

$$A_m = A \cap \{x : t^m \leq D_\mu \nu(x) < t^{m+1}\}$$

$$A \setminus \bigcup_{m \in \mathbb{Z}} A_m \subset \cancel{Z} \cup I \cup \{\overline{D}_\mu \nu(x) \neq \underline{D}_\mu \nu(x)\}$$

$$\mu\Big(A \setminus \bigcup_{m \in \mathbb{Z}} A_m\Big) = 0 \quad \overset{\nu \ll \mu}{\Longrightarrow} \quad \nu\Big(A \setminus \bigcup_{m \in \mathbb{Z}} A_m\Big) = 0$$

$$\nu(A) = \sum_{m\in\mathbb{Z}} \nu(A_m) \le \sum_{m=-\infty}^{\infty} t^{m+1}\mu(A_m) = t\sum_{m=-\infty}^{\infty} t^{m}\mu(A_m)$$

$$\le t \sum_{m=-\infty}^{\infty} \int_{A_m} D_\mu \nu \, d\mu = t\int_A D_\mu \nu \, d\mu$$

$$\nu(A) \le t\int_A D_\mu \nu \, d\mu \qquad \forall\, t>1 \qquad t \to 1$$

$$\nu(A) \le \int_A D_\mu \nu \, d\mu$$

similarly $\nu(A) \ge \int_A D_\mu \nu \, d\mu$

if $A \cap Z = \emptyset$ $\quad \nu(Z) = 0$

for general A $\quad \nu(A) \overset{\downarrow}{=} \nu(A\setminus Z) = \int_{A\setminus Z} D_\mu \nu \, d\mu + \overset{0}{\overset{\parallel}{\int_{Z\cap A} D_\mu \nu \, d\mu}}$

$$= \qquad\qquad = \int_A D_\mu \nu \, d\mu$$

Theorem (Lebesgue Decomposition Theorem)

Let μ & ν be Radon measures in $\mathbb{R}^n$, then

i) $\nu = \underset{\nwarrow}{\nu_{ac}} + \underset{\uparrow}{\nu_s}$; $\nu_{ac} \ll \mu$ & $\nu_s \perp \mu$

Radon measures

ii) Moreover $D_\mu \nu = D_\mu \nu_{ac}$ & $D_\mu \nu_s = 0$ μ-a.e

$$\nu(A) = \int_A D_\mu \nu \, d\mu + \nu_s(A)$$

Pf: $\mu(\mathbb{R}^n) < \infty$ $\nu(\mathbb{R}^n) < \infty$

$\mathcal{E} = \{ A \subset \mathbb{R}^n : \mu(\mathbb{R}^n \setminus A) = 0 \}$ $\exists B \in \mathcal{E}$

$\nearrow \nu(B) = \inf_{A \in \mathcal{E}} \nu(A)$ $\nu_{ac} = \nu \llcorner B$

$\nu_s = \nu \llcorner (\mathbb{R}^n \setminus B)$

Claim: if $A \subset B$ and $\mu A = 0 \Rightarrow \nu(A) = 0 \Rightarrow \nu_{ac} \ll \mu$

$\mu(\mathbb{R}^n \setminus B) = 0$ bec $B \in \xi$ $\quad \nu_s \perp \mu$

$$C = \{ x \in B : D_\mu \nu_s \geq \alpha \}$$

$$\begin{array}{l} = \nu_s(C) \geq \alpha \mu(C) \\ \quad \parallel \\ \nu(C \setminus B) \geq \alpha \mu(C) \Rightarrow \mu(C) = 0 \\ \qquad \parallel \\ \qquad 0 \end{array}$$

$D_\mu \nu_s = 0$ $\quad \mu$. a.e $\quad D_\mu \nu_{ac} = D_\mu \nu$

Theorem (Lebesgue - Besicovitch differentiation theorem)

Let μ be a Radon measure in $\mathbb{R}^n$ & $f \in L^1_{loc}(\mathbb{R}^n, \mu)$ then

$$\lim_{r \to 0} \underbrace{\fint_{B(x,r)} f(y)\, d\mu(y)}_{f_{x,r}} = f(x) \quad \text{for } \mu\text{. a.e } x \in \mathbb{R}^n$$

Corollary : $E \subset \mathbb{R}^n$ $\mathcal{L}^n$ measurable then

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} = 1 \quad \text{for } \mathcal{L}^n \text{ a.e } x \in E$$

measure theoretic interior

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} = 0 \quad \text{for } \mathcal{L}^n \text{ a.e } x \in \mathbb{R}^n \setminus E$$

measure theoretic exterior

Theorem: (Lebesgue points for Radon measures)

Let μ be a Radon measure in $\mathbb{R}^n$, suppose $f \in L^p_{loc}(\mu)$ for some $1 \leq p < \infty$ then

$$\lim_{r \to 0} \fint_{B(x,r)} |f - f_{x,r}|^p \, d\mu = 0 \qquad \mu \text{ a.e } x \in \mathbb{R}^n$$

↑

x is a Lebesgue point of f.