Riesz Representation Theorem

① m = Lebesgue measure in $\mathbb{R}^n$

$$(L^2(m))^* = L^2(m)$$

$\Uparrow$

$\forall\ \psi \in (L^2(m))^*$ $\quad \psi : L^2(m) \longrightarrow \mathbb{R}$ bdd linear functional

$\exists\ f_0 \in L^2(m)$

$$\psi(g) = \langle g, f_0 \rangle = \int f_0 g \, dm$$

② $1 \leq p < \infty$

$$(L^p(m))^* = L^q(m) \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1$$

Riesz Representation Theorem

Let $L : C_c(\mathbb{R}^n, \mathbb{R}^m) \longrightarrow \mathbb{R}$ be a linear functional satisfying

$$\sup \{ L(f) : f \in C_c(\mathbb{R}^n, \mathbb{R}^m) : |f| \leq 1 \quad \operatorname{spt} f \subseteq K \} < \infty$$

for each compact set $K \subset \mathbb{R}^n$.

Then there exist a Radon measure μ on $\mathbb{R}^n$ and a μ-measurable function $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t

(i) $|\sigma(x)| = 1 \quad \mu\text{-a.e } x \in \mathbb{R}^n$

(ii) $L(f) = \int f \cdot \sigma \, d\mu \qquad \forall f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$

Divergence theorem $\quad F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad F \subset C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$

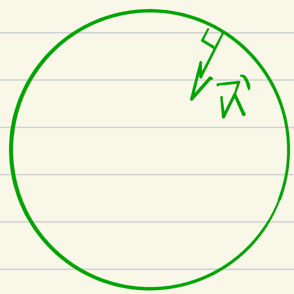
$F = (F_1, F_2, F_3) \qquad L(F) = \int_B \operatorname{div} F \, dx$

$$\int_B \operatorname{div} F \, dx = \int \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) dx = \int_{\partial B} F \cdot \vec{n} \, d\sigma$$

unit normal

$$LF = \int_B \operatorname{div} F \, dx = \int_{\partial B} F \cdot \vec{n} \, d\sigma \le |\partial B|$$

$|F| \le 1$



$|\vec{n}| \le 1$

Does this say anything about $(L^\infty)^*$? $\mathbb{R}^n$

$L \in (L^\infty(m))^*$ $\quad f \in C_c(\mathbb{R}^n, \mathbb{R}) \subset L^\infty(m)$

$|f| \le 1$

$|L(f)| \le \|L\|$ $\quad \hat{L} = L|_{C_c(\mathbb{R}^n, \mathbb{R})}$

$\exists \mu$ measure $\quad \sigma : \mathbb{R}^n \to \mathbb{R} \quad |\sigma(x)| = 1$

$$\hat{L}(f) = \int f \sigma \, d\mu$$

Riesz Representation Theorem

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Then there exist a Radon measure μ on $\mathbb{R}^n$ and a μ-measurable function $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t

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Definition: We call μ the variation measure associated with L. It is defined as follows: for each open set $V \subset \mathbb{R}^n$

(*) $$\mu(V) := \sup \{ L(f) : f \in C_c(\mathbb{R}^n, \mathbb{R}^m) : |f| \leq 1, \ spt\, f \subset V \}$$

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Pf let $A \subset \mathbb{R}^n$

$$\mu A = \inf \{ \mu U : A \subset U \quad U \text{ open} \}$$

Claim 1 : μ is a measure

Claim 2 : μ is a Radon measure

Claim 3 : there is an associated non-negative linear functional λ.

Claim 4 : λ can be represented as an integral w.r.t μ

Claim 5 : $\exists\ \sigma : \mathbb{R}^n \to \mathbb{R}^m$ μ-measurable satisfying

$$Lf = \int f \cdot \sigma \, d\mu \qquad \forall f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$$

Claim 6 : $|\sigma| = 1$ μ a.e x.

Claim 1 : μ is a measure : i) $\mu(\phi) = 0$ $(L0 = 0)$

ii) subadditivity : a) for open sets

b) for general sets

a) $\{V_i\}_{i=1}^{\infty}$ open sets $V \subset \bigcup_{i=1}^{\infty} V_i$

open

$$\mu(V) := \sup \{ L(f) : f \in C_c(\mathbb{R}^n, \mathbb{R}^m) : |f| \le 1, \; \text{spt } f \subset V \}$$

choose $g \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ $|g| \le 1$ $\text{spt } g \subset V \subset \bigcup_{i=1}^{\infty} V_i$

$\{\xi_i\}_{i=1}^{\infty}$ partition of unity subordinate to $\{V_i\}$

$\text{spt } \xi_j \subset V_j$ $0 \le \xi_j \le 1$ $\sum_{j=1}^{M} \xi_j = 1$ on $\text{spt } g$

compact set

since $\text{spt } g$ compact

$\text{spt } g \subset \bigcup_{i=1}^{M} V_i$

$$g = g \sum_{i=1}^{\cancel{\infty}\, M} \xi_i \qquad \cancel{L(g) = \sum_{i=1}^{\infty} L(g\xi_i)}$$

infinite sum
an issue

$$g = \sum_{i=1}^{M} g\xi_i \qquad \operatorname{spt}(g\xi_i) \subset V_i \qquad |g\xi_i| \le 1$$

$$Lg = \sum_{i=1}^{M} L(g\xi_i) \le \sum_{i=1}^{M} \mu V_i \le \sum_{i=1}^{\infty} \mu V_i$$

$$\forall\, g \in C_c(\mathbb{R}^n, \mathbb{R}^m) \quad |g| \le 1 \qquad \text{sup over } g$$

$$\mu V \le \sum_{i=1}^{\infty} \mu V_i$$

let $\{A_i\}_{i=1}^{\infty}$ $\quad A \subset \bigcup_{j=1}^{\infty} A_i$ $\quad$ sets. $\quad V_i \supset A_i$ open

$$\mu V_i \ge \mu A_i \ge \mu V_i - \varepsilon/2^i$$

$$\mu A \le \mu\left(\bigcup_{i=1}^{\infty} V_i\right) \le \sum_{i=1}^{\infty} \mu V_i \le \sum_{i=1}^{\infty} \mu A_i + \varepsilon \qquad \forall \varepsilon > 0$$

$\varepsilon \to 0$

Claim 2 : μ is Radon : Borel regular & finite on compact sets.

① Caratheodory critiria : $\mu(A_1 \cup A_2) = \mu A_1 + \mu A_2$ if $d(A_1, A_2) > 0$

② measures of all Balls finite $\Rightarrow$ Borel measure

enough to show $\mu(\underbrace{B(0,R)}_{\text{open}})$ is finite

$$\mu(B(0,R)) = \sup \{ Lg : \operatorname{spt} g \subset B(0,R) : |g| \le 1 \} < \infty$$

Regular $\quad A \quad$ find Borel set $B \quad A \subset B$

$$\mu A = \mu B$$

$$\mu A = \inf \{ \mu U : U \text{ open } U \supset A \} \qquad \overset{\text{open}}{V_k} \supset A$$

$$\mu V_k - \frac{1}{2^k} \le \mu A \le \mu V_k \qquad V = \bigcap_{k=1}^{k} V_k \supset A$$

check $\qquad \mu A = \mu V$

$C_c^+(\mathbb{R}^n) = \{ f \in C_c(\mathbb{R}^n) : f \geq 0 \}$ for $f \in C_c^+(\mathbb{R}^n)$

$$\lambda(f) = \sup \{ |Lg| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m) : |g| \leq f \}$$

Properties : 1) $f_1, f_2 \in C_c^+(\mathbb{R}^n)$ $\quad f_1 \leq f_2$

$$\lambda(f_1) \leq \lambda(f_2)$$

2) $c \geq 0$ $\quad \lambda(cf) = c\lambda(f)$

Claim 3 : for $f_1, f_2 \in C_c^+(\mathbb{R}^n)$ $\quad \lambda(f_1 + f_2) = \lambda f_1 + \lambda f_2$

Pf : $g_1, g_2 \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ $\quad |g_i| \leq f_i$

$|g_1 + g_2| \leq f_1 + f_2$ $\quad Lg_i \geq 0$

$$|Lg_1| + |Lg_2| = Lg_1 + Lg_2 = L(g_1 + g_2) \leq |L(g_1 + g_2)| \leq$$

sup over g_1 & g_2 / $\quad \lambda(f_1) + \lambda(f_2) \leq \lambda(f_1 + f_2)$

$g \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ $\quad |g| \leq f_1 + f_2$

$$g_i = \begin{cases} f_i g / f_1 + f_2 & \text{if } f_1 + f_2 > 0 \\ 0 & \text{if } f_1 + f_2 = 0 \end{cases} \qquad i = 1, 2$$

$g_1, g_2 \in C_c(\mathbb{R}^n, \mathbb{R}^m)$

$g = g_1 + g_2$ (because $|g| \leq f_1 + f_2$)

$$|g_i| \leq \frac{f_i |g|}{f_1 + f_2} \leq f_i$$

$|Lg| = |Lg_1 + Lg_2| \leq Lg_1 + Lg_2 \leq \lambda(f_1) + \lambda(f_2)$

take sup on g

$$\lambda(f_1 + f_2) \leq \lambda(f_1) + \lambda(f_2)$$

Problem 2

$u \in L^p(B)$ $\qquad$ $\left(\fint_{B_r(x)} |u - u_{x,r}|^p \, dy \right)^{1/p} \le K r^\gamma$

Claim 1 $\quad \{ u_{x, r2^{-j}} \}_{j=1}^{\infty}$ $\qquad x \in B$

$\bar{u}(x) = \lim_{j \to \infty} u_{x, r2^{-j}}$ exists

Claim 2 $\quad \{ \underbrace{u_{x, r2^{-j}}}_{r_j} \}_{j=1}^{\infty}$ Cauchy with bounds

$|u_{x, r_{j+l}} - u_{x, r_j}| \le C(n,p) K r^\gamma 2^{-j\gamma}$

$l \to \infty$ $\quad |\bar{u}(x) - u_{x,r_j}| \le$

hint $\quad |x - y| \sim r 2^{-j}$ $\qquad |\bar{u}(x) - \bar{u}(y)|$ using

$|\bar{u}_{x,r_j} - u_{y,r_j}|$