Riesz Representation Theorem

Let $L : C_c(\mathbb{R}^n, \mathbb{R}^m) \longrightarrow \mathbb{R}$ be a linear functional satisfying

$$\sup \{ L(f) : f \in C_c(\mathbb{R}^n, \mathbb{R}^m) : |f| \leq 1 \quad \operatorname{spt} f \subseteq K \} < \infty$$

for each compact set $K \subset \mathbb{R}^n$.

Then there exist a Radon measure μ on $\mathbb{R}^n$ and a μ-measurable function $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t

(i) $|\sigma(x)| = 1 \quad \mu\text{-a.e } x \in \mathbb{R}^n$

(ii) $L(f) = \int f \cdot \sigma \, d\mu \qquad \forall f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$

<u>Definition</u>: We call μ the variation measure associated with L. It is defined as follows: for each open set $V \subset \mathbb{R}^n$

$$(*) \qquad \mu(V) := \sup \{ L(f) : f \in C_c(\mathbb{R}^n, \mathbb{R}^m) : |f| \leq 1, \ \operatorname{spt} f \subset V \}$$

$A \subset \mathbb{R}^n$

$\mu(A) = \inf \{ \mu U : U \text{ open } A \subset U \}$

Claim 1 : μ is a measure

Claim 2 : μ is a Radon measure

Claim 3 : there is an associated non- negative linear functional λ.

$$C_c^+(\mathbb{R}^n) = \{ f \in C_c(\mathbb{R}^n) : f \geq 0 \} \qquad \text{for } f \in C_c^+(\mathbb{R}^n)$$

$$\lambda(f) = \sup \{ |Lg| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m) : |g| \leq f \}$$

Claim 4 : λ can be represented as an integral w.r.t μ

Claim 5 : $\exists\, \sigma : \mathbb{R}^n \to \mathbb{R}^m$ μ- measurable satisfying

$$Lf = \int f \cdot \sigma \, d\mu \qquad \forall f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$$

Claim 6 : $|\sigma| = 1$ μ a. e x.

Claim 4

$$\lambda(f) = \int_{\mathbb{R}^n} f \, d\mu \qquad \forall f \in C_c^+(\mathbb{R}^n)$$

Given $\varepsilon > 0$ $\quad 0 \leq t_0 < t_1 < \dots < t_N$ s.t $t_N = 2\|f\|_\infty$

$$0 < t_i - t_{i-1} < \varepsilon \qquad \mu(f^{-1}(t_i)) = 0 \quad i = 1 \dots N$$

$$\left(\mu(f^{-1}(\mathbb{R})) \right) = \mu(f^{-1}(0, t_N]) = \mu(\underbrace{\operatorname{spt} f}_{\text{compact}}) < \infty$$

$U_j = f^{-1}(t_{j-1}, t_j)$ open, disjoint $\mu(U_j) < \infty$

By approx of Radon measures of sets of finite measure

$\exists K_j$ compact $\quad K_j \subset U_j \qquad \mu(U_j \setminus K_j) < \varepsilon/N \quad \leftarrow$

$\exists g_j \in C_c(\mathbb{R}^n, \mathbb{R}^m) \qquad |g_j| \leq 1 \qquad \operatorname{spt} g_j \subset U_j$

$$|Lg_j| \geq \mu(U_j) - \varepsilon/N \quad (\text{def})$$

$\exists h_j \in C_c^+(\mathbb{R}^n)$ s.t $\operatorname{spt} h_j \subset U_j \qquad 0 \leq h_j \leq 1$

$h_j = 1$ on $K_j \cup \operatorname{spt} g_j$

$|g_j| \leq h_j$

$|g_j| \le h_j \qquad \lambda(h_j) \ge |Lg_j| \ge \mu(U_j) - \frac{\varepsilon}{N}$

$$\lambda(h_j) = \sup\{|Lg| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m) : |g| \le h_j\}$$

$$\le \sup\{|Lg| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m), |g| \le 1 \ \& \ \operatorname{spt} g \subset U_j$$

$$\le \mu U_j$$

$$\boxed{\mu(U_j) - \frac{\varepsilon}{N} \le \lambda(h_j) \le \mu U_j}$$

$$A = \{x : f(x)\Big(1 - \sum_{j=1}^{N} h_j(x)\Big) > 0\} \subset \operatorname{spt} f$$

open

$$\downarrow \ f(x) > 0 \quad \& \quad h_j(x) \ne 1$$

$$\mu A = \mu\Big(\bigcup_{j=1}^{N} (U_j - \{h_j = 1\})\Big) \qquad \text{bec} \quad \mu(\partial U_j) = 0$$

bec disj

$$\le \sum_{j=1}^{N} \mu(U_j \setminus K_j) = \varepsilon$$

$K_j \subset \{h_j = 1\}$

$$f - f\sum_{j=1}^{N} h_j \geq 0$$

$$\lambda\big(\underbrace{f - f\sum_{j=1}^{N} h_j}_{\geq 0}\big) = \sup\Big\{ |Lg| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m),\ |g| \leq f - f\sum_{j=1}^{N} h_j \Big\}$$

$$\leq \sup\{ |Lg| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m) : |g| \leq \|f\|_\infty \chi_A \}$$

A open

$$\leq \|f\|_\infty \mu A \leq \varepsilon \|f\|_\infty$$

$$\lambda(f) = \lambda\Big(f\sum_{j=1}^{N} h_j\Big) + \lambda\Big(f - f\sum_{j=1}^{N} h_j\Big)$$

$$\leq \sum_{j=1}^{N} \lambda(f h_j) + \varepsilon \|f\|_\infty \qquad \text{in spt } h_j \quad t_{j-1} < f < t_j$$

$$\geq \sum_{j=1}^{N} \lambda(f h_j) \geq \sum_{j=1}^{N} t_{j-1} \lambda(h_j) \geq \sum_{j=1}^{N} t_{j-1}\Big(\mu U_j - \frac{\varepsilon}{N}\Big)$$

$$\leq \sum_{j=1}^{N} t_j \lambda(h_j) + \varepsilon \|f\|_\infty$$

$$t_N \varepsilon \geq \sum_{j=1}^{N} t_{j-1} \frac{\varepsilon}{N}$$

$$\boxed{\sum_{j=1}^{N} t_{j-1} \mu(U_j) - 2\varepsilon \|f\|_\infty \leq \lambda(f) \leq \sum_{j=1}^{N} t_j \mu U_j + \varepsilon \|f\|_\infty}$$

$$\underbrace{\sum_{j=1}^{N} t_{j-1}\,\mu U_j}\leq \int_{\mathbb{R}^n} f\,d\mu = \sum_{j=1}^{N}\int_{U_j} f\,d\mu \leq \underbrace{\sum_{j=1}^{N} t_j\,\mu U_j}$$

$$\lambda(f) - \int f\,d\mu \leq \underbrace{\varepsilon\|f\|_\infty + 2\varepsilon\|f\|_\infty} + \underbrace{\sum_{j=1}^{N}\underbrace{(t_j - t_{j-1})}_{\varepsilon}\mu(U_j)}$$

$$\int f\,d\mu - \lambda(f) \leq 3\varepsilon\|f\|_\infty +$$

$$\left|\lambda(f) - \int f\,d\mu\right| \leq 3\varepsilon\|f\|_\infty + \varepsilon\,\mu\underbrace{(\operatorname{spt} f)}_{<\infty} \xrightarrow[\varepsilon\to 0]{} 0$$

Claim 5: $e \in S^{m-1}$ $\quad e\in\mathbb{R}^m$ $\quad |e|=1$ $\quad f\in C_c(\mathbb{R}^n)$

$$\lambda_e(f) = L(fe)$$

λ_e is linear

$$|\lambda_e(f)| \leq |L(fe)| \leq \sup\{|Lg| : g\in C_c(\mathbb{R}^n,\mathbb{R}^m) : |g|\leq|f|\}$$

$$\leq \lambda(|f|) = \int |f|\,d\mu = \|f\|_{L^1(\mu)}$$

$$|\lambda_e(f)| \leq \|f\|_{L^1(\mu)}$$

λ_e bounded linear functional in $L^1(\mu)$, μ Radon $\exists \sigma_e \in L^\infty$

s.t.
$$\lambda_e(f) = \int f \sigma_e \, d\mu \qquad \forall f \in C_c(\mathbb{R}^n)$$

$e_1 \dots e_m$ o.n basis in $\mathbb{R}^m$ $\qquad \sigma = \sum_{i=1}^m \sigma_{e_i} e_i$

$g \in C_c(\mathbb{R}^n, \mathbb{R}^m)$

$$g = \sum_{i=1}^m \langle g, e_i \rangle e_i$$

$$L(f) = \sum_{i=1}^m \underbrace{L\langle g, e_i \rangle e_i}_{\lambda_{e_i}(\langle g, e_i \rangle)} = \sum_{i=1}^m \int \langle g, e_i \rangle \sigma_{e_i} \, d\mu$$

$$= \int \langle g, \sigma \rangle \, d\mu = Lg$$

<u>Claim</u> 6 : $|\sigma| = 1$ μ.a.e

<u>Pf</u> U open $\quad \mu U < \infty$

$$\mu U = \sup \left\{ \int g \cdot \sigma \, d\mu : g \in C_c(\mathbb{R}^n, \mathbb{R}^m), \ \operatorname{spt} g \subset U, \ |g| \le 1 \right\}$$

Take $f_k \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ $\quad |f_k| \le 1$ $\quad \operatorname{spt} f_k \subset U$

$$f_k \cdot \sigma \longrightarrow |\sigma| \quad \mu \text{ a.e in } U \quad (\text{Lusin's theo})$$

$$\left(\text{"} f_k \longrightarrow \frac{\sigma}{|\sigma|} \quad \text{if } \sigma \neq 0 \text{"} \right)$$

$$\int_U |\sigma| \, d\mu \overset{\uparrow}{=} \lim_{k \to \infty} \int_U f_k \cdot \sigma \, d\mu \le \mu U$$

$$|f_k \cdot \sigma| \le \|\sigma\|_\infty < \infty$$

L.D.C.T

$$\int f \cdot \sigma \, d\mu \le \int_U |\sigma| \, d\mu \qquad \text{sup on } f$$

$$\mu U \le \int_U |\sigma| \, d\mu$$

$$\mu U = \int_U |\sigma| \, d\mu$$

$$\Downarrow$$

$$|\sigma| = 1 \quad \mu\text{.a.e}$$

Theorem: (Non-negative linear functionals)

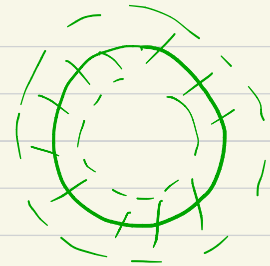
Assume $L : C_c^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is linear & non-negative i.e.

$L(f) \geq 0 \quad \forall f \in C_c^\infty(\mathbb{R}^n) \quad f \geq 0$. Then there exists

a Radon measure μ on $\mathbb{R}^n$ s.t

$$L(f) = \int f \, d\mu \qquad \forall f \in C_c^\infty(\mathbb{R}^n)$$

Example : Let $\Omega \subset \mathbb{R}^n$ $(\Omega = B_1(0))$



Dirichlet problem : given $f \in C(\partial\Omega)$ does there exist $u_f \in C(\bar{\Omega}) \cap C^2(\Omega)$

$$\begin{cases} \Delta u_f = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 & \text{in } \Omega \\ u_f = f & \text{on } \partial\Omega \end{cases} \quad ?$$

Maximum principle

$$\|u_f\|_{L^\infty(\Omega)} \leq \max_{\partial\Omega} |f|$$

$x \in \Omega$ $\qquad |u_f(x)| \leq \|f\|_\infty$

$$u_f(x) + u_g(x) = u_{f+g}(x)$$

$L_x : C(\partial\Omega) \to \mathbb{R}$ $\qquad L_x f = u_f(x)$

bounded linear functional $\qquad$ RRT $\qquad \exists\, \omega^x$ Radon measure

$$u_f(x) = \int_{\partial\Omega} f \, d\omega^x$$

$$|u_f(x)| = \left|\int_{\partial\Omega} f\, d\omega^x\right| \le \|f\|_\infty \implies f = 1$$

$$\omega^x(\partial\Omega) \le 1$$

$$u_1(x) = 1 \quad — \quad \omega^x(\partial\Omega) = 1$$

$\{\omega^x\}$ probability measures $\quad \omega^x$ harmonic measure with pole x.

$\omega^x(E)$ = probability that a Random walk starting at x will first hit the boundary at a pt in $\bar{E}$