Weak Convergence of Measure

Theorem: Let μ, μ_k $k \in \mathbb{N}$ be Radon measures on $\mathbb{R}^n$. T.f.a.e.

(i) $\displaystyle\lim_{k \to \infty} \int f \, d\mu_k = \int f \, d\mu \qquad \forall f \in C_c(\mathbb{R}^n)$

(ii) $\displaystyle\limsup_{k \to \infty} \mu_k(K) \leq \mu(K) \qquad \forall K \subset\subset \mathbb{R}^n$ compact

$\displaystyle\mu U \leq \liminf_{k \to \infty} \mu_k(U) \qquad \forall U \subset \mathbb{R}^n$ open

(iii) $\displaystyle\lim_{k \to \infty} \mu_k(B) = \mu(B) \qquad \forall B \subset \mathbb{R}^n$ bounded Borel set B s.t. $\mu(\partial B) = 0$

Definition: If (i) - (iii) hold we say that the measures $\{\mu_k\}_{k=1}^{\infty}$ converge weakly to the measure μ.

$$\mu_k \longrightarrow \mu$$

i) $\Rightarrow$ ii) $\int f\,d\mu_k \longrightarrow \int f\,d\mu \quad \forall f \in C_c(\mathbb{R}^n)$

$U \supset K$ choose $f \in C_c(\mathbb{R}^n)$

$\uparrow$ open

$\operatorname{spt} f \subset U \quad 0 \le f \le 1 \quad f = 1$ on K

$$\mu K \le \int f\,d\mu = \lim_{k\to\infty} \int_K f\,d\mu_k \le \liminf_{k\to\infty} \mu_k(U)$$

$$\mu K \le \liminf_{k\to\infty} \mu_k(U) \quad \forall K \subset U \text{ compact}$$

$$\mu U = \sup\{\mu K : K \text{ compact} : K \subset U\}$$

ii) $\Rightarrow$ (iii) B bounded Borel $\mu(\partial B) = 0$

$$\mu B = \mu(\operatorname{int} B) \le \liminf_{k\to\infty} \mu_k(\operatorname{int} B) \le \liminf_{k\to\infty} \mu_k(B)$$

$$\le \limsup_{k\to\infty} \mu_k(B) \le \limsup_{k\to\infty} \mu_k(\bar{B}) \le \mu(\bar{B}) = \mu(B)$$

iii) $\Rightarrow$ i) $\quad f \in C_c^+(\mathbb{R}^n) \quad R>0 \quad spt\, f \subset B(0,R) = B$

$\mu(\partial B(0,R)) = 0 \quad$ choose $\varepsilon > 0$

$0 = t_0 < t_1 < \dots < t_N \quad t_N = 2\|f\|_\infty$

$0 < t_i - t_{i-1} < \varepsilon \quad \mu(f^{-1}\{t_i\}) = 0 \quad i=1,\dots,N$

$B_i = f^{-1}(t_{i-1}, t_i) \quad \mu(\partial B_i) = 0 \quad i = 2, \dots N$

$$\underbrace{\sum_{i=2}^N t_{i-1}\mu_k(B_i)} = \sum_{i=1}^N t_{i-1}\mu_k(B_i) \le \int f\,d\mu_k \le \sum_{i=1}^N t_i\,\mu_k(B_i)$$

$$\sum_{i=2}^N t_i\,\mu_k(B_i) + t_1\mu_k(B)$$

$$\sum_{i=1}^N t_{i-1}\mu(B_i) \le \int f\,d\mu \le \underbrace{\sum_{i=2}^N t_i\,\mu(B_i)} + \underbrace{t_1\mu(B)}$$

$$\left|\int f\,d\mu - \int f_k\,d\mu\right| \le \sum_{i=2}^N |t_i\mu(B_i) - t_{i-1}\mu_k(B_i)| + t_1\mu_k(B)$$

$$+ \sum_{i=2}^N |t_i\mu_k(B_i) - t_{i-1}\mu(B_i)| + t_1\mu(B)$$

$$\left| \int f\,d\mu - \int f_k\,d\mu \right| \leq \sum_{i=2}^{N} \left| t_i \mu(B_i) - t_{i-1} \mu_k(B_i) \right| + t_1 \mu_k(B)$$

$$+ \sum_{i=2}^{N} \left| t_i \mu_k(B_i) - t_{i-1} \mu(B_i) \right| + t_1 \mu(B)$$

$k \to \infty$ $\qquad \mu_k(B_i) \to \mu B_i \qquad \mu(\partial B_i) = 0$

$\qquad \mu_k(B) \to \mu B \qquad \mu(\partial B) = 0$

$$\limsup_{k \to \infty} \left| \int f\,d\mu - \int f_k\,d\mu \right| \leq 2 \sum_{i=2}^{N} (t_i - t_{i-1})\, \mu(B_i)$$

$$+ 2 t_1 \mu(B)$$

$$< 2\varepsilon \underbrace{\sum_{i=2}^{N} \mu(B_i)}_{\leq \mu(B)} + 2\varepsilon \mu(B) \qquad B_i \cap B_j = \emptyset$$

$$\leq 4\varepsilon \mu(B) \qquad \varepsilon \to 0.$$

Theorem: (Weak compactness for measures)

Let $\{\mu_k\}_k$ be a sequence of Radon measures on $\mathbb{R}^n$ satisfying

$$\sup_k \mu_k(K) < \infty \quad \text{for each compact set } K \subset\subset \mathbb{R}^n$$

Then there exists a subsequence $\{\mu_{k_j}\}_j$ and a Radon measure μ such that

$$\mu_{k_j} \longrightarrow \mu$$

① $M = \sup_k \mu_k(\mathbb{R}^n) < \infty$, $C_c(\mathbb{R}^n)$ is separable

$\{f_k\}$ countable dense set of $C_c(\mathbb{R}^n)$

$$\left| \int f_i \, d\mu_k \right| \leq \int |f_i| \, d\mu_k \leq \|f_i\|_\infty \mu_k(\mathbb{R}^n)$$

$$\leq M \|f_i\|_\infty < \infty$$

$i = 1$ $\quad \exists \{\mu_k^1\} \subset \{\mu_k\} \quad \rightsquigarrow \quad \{\mu_k^2\} \subset \{\mu_k^1\} \subset \{\mu_k\}$

$$\int f_1 \, d\mu_k^1 \xrightarrow[k \to \infty]{} a_1 \qquad \int f_2 \, d\mu_k^2 \to a_2$$

$\{\mu_j^k\}_{j=1}^{\infty} \subset \{\mu_j^{k-1}\}$ $\quad a_k \in \mathbb{R}$

$$\int f_k \, d\mu_j^k \to a_k \qquad \int f_\ell \, d\mu_j^k \longrightarrow a_\ell \qquad \ell \leq k$$

$$\mu_j^j = \nu_j$$

Claim 1: $\displaystyle \int f_k \, d\nu_j \xrightarrow[j\to\infty]{} a_k \quad \forall k$

$$L(f_k) = a_k \qquad L(af_k + bf_j) = aLf_k + bLf_j$$

$$|Lf_k| \leq \|f_k\|_\infty M$$

Claim 2 $\{f_k\}$ dense L can be extended uniquely to a linear functional $\bar{L}$

$\left(\text{ie} \quad f_k \to f \text{ in } C_c(\mathbb{R}^n) \qquad \bar{L}f = \lim_{k\to\infty} Lf_k \right)$

$|\bar{L}f| \leq \|f\|_\infty M \qquad \bar{L}$ bounded linear functional

RRT $\exists \mu$ s.t

$$\bar{L} f = \int f \, d\mu$$

$$\bar{L} f_k = L f_k = \lim_{j \to \infty} \int f_k \, d\nu_j$$

use $\{f_k\}$ dense in $C_c(\mathbb{R}^n)$ $\forall f \in C_c(\mathbb{R}^n)$

$$\int f \, d\nu_j \longrightarrow \int f \, d\mu.$$

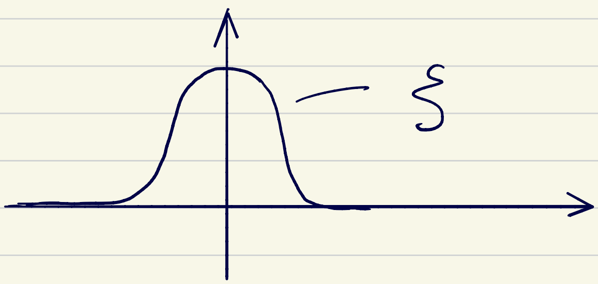
Regularization of Radon measures

A *regularizing kernel* is a function $\xi \in C_c^\infty(B_1)$ $\quad 0 \le \xi \le 1$

$$\int \xi \, dx = 1 \qquad \xi(-x) = \xi(x) \qquad \text{for } \varepsilon > 0$$

$$\xi_\varepsilon(x) = \varepsilon^{-n}\, \xi\left(\frac{x}{\varepsilon}\right) \qquad \text{for } x \in \mathbb{R}^n$$

then $\xi_\varepsilon \in C_c^\infty(B_\varepsilon)$ & $\int \xi_\varepsilon(x)\, dx = 1$



The *ε-regularization* μ_ε of μ is a Radon measure on $\mathbb{R}^n$

$$\langle \mu_\varepsilon, \varphi \rangle = \int \varphi(x)\, \mu * \xi_\varepsilon(x)\, dx \qquad \forall \varphi \in C_c(\mathbb{R}^n)$$

where

$$\mu * \xi_\varepsilon(x) = \int \xi_\varepsilon(x-y)\, d\mu(y) \; ; \qquad \mu * \xi_\varepsilon \in C^\infty(\mathbb{R}^n)$$

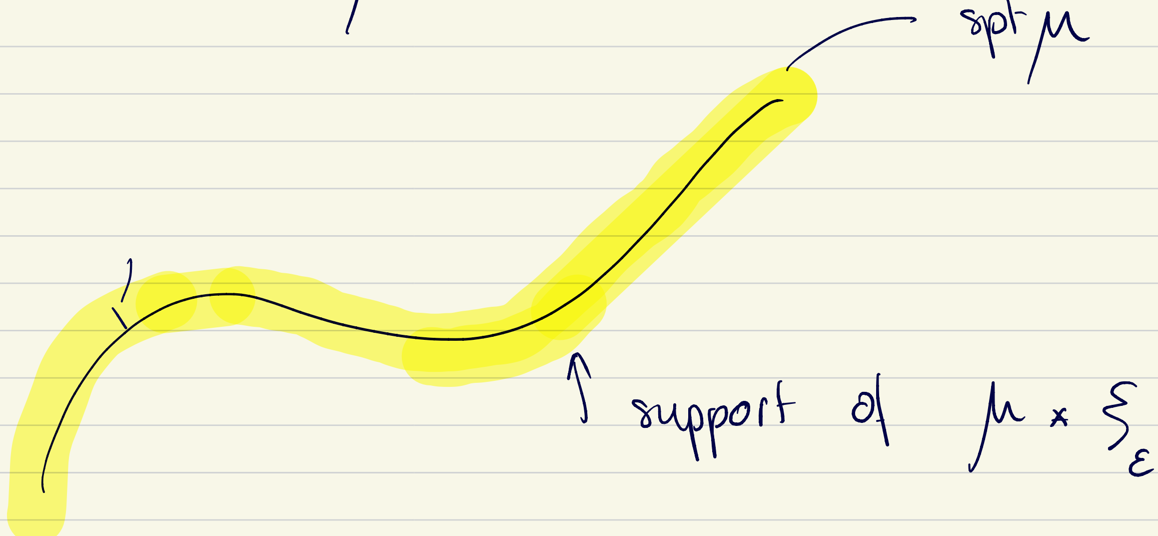
Theorem: If μ is a Radon measure on $\mathbb{R}^n$

$$\mu_\varepsilon \to \mu \quad \text{as } \varepsilon \downarrow 0$$

Moreover for every Borel set $E \subset \mathbb{R}^n$ if

$$N_\varepsilon(\bar{E}) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \bar{E}) < \varepsilon\}$$

then
$$\mu_\varepsilon(\bar{E}) \leq \mu(N_\varepsilon(E))$$



Pf $\quad \varphi \in C_c(\mathbb{R}^n)$

$$\int \varphi \, d\mu_\varepsilon = \int \varphi \, (\mu * \xi_\varepsilon) \, dx = \int \varphi(x) \int \xi_\varepsilon(x-y) \, d\mu(y) \, dx$$

Fubini

$$= \int \underbrace{\left(\int \varphi(x) \underbrace{\xi_\varepsilon(x-y)}_{\xi_\varepsilon(y-x)} dx \right)}_{\varphi_\varepsilon(y)} d\mu(y)$$

$$\varphi_\varepsilon(y) = \underbrace{\int \varphi(x) \xi_\varepsilon(y-x) \, dx}_{\substack{\varepsilon \to 0 \\ \downarrow \\ \varphi(y)}}$$

nice conv. $\rightsquigarrow$

$$= \int \varphi_\varepsilon(y) \, d\mu(y)$$

$$\int \varphi \, d\mu_\varepsilon \xrightarrow[\varepsilon \to 0]{} \int \varphi \, d\mu$$

$$\mu_\varepsilon(E) = \int_E dx \left(\int_{B(x,\varepsilon)} \xi_\varepsilon(x-y) \, d\mu(y) \right) \le \mu(N_\varepsilon(E))$$

Weak convergence of functions

Let $U \subset \mathbb{R}^n$ open & $1 \leq p < \infty$

Definition: A sequence $\{f_k\} \subset L^p(U)$ converges weakly to f in $L^p(U)$

$$f_k \rightharpoonup f \quad \text{in } L^p(U)$$

provided $\int f_k g \, dx \to \int f g \, dx \quad \forall g \in L^q(U)$ with

$1 < q \leq \infty$ $\qquad \frac{1}{p} + \frac{1}{q} = 1$ $\qquad$ (why does $f \in L^p(U)$?)

Theorem: (Weak compactness in L^p): Suppose $\boxed{1 < p < \infty}$ and let $\{f_k\} \subset L^p(U)$ s.t.

$$\sup_k \|f_k\|_{L^p(U)} < \infty$$

then there exists a subsequence $\{f_{k_j}\}$ and a function $f \in L^p(U)$ s.t.

$$f_k \rightharpoonup f \quad \text{in } L^p(U)$$

Pf $\quad U = \mathbb{R}^n \quad f_k \geq 0 \quad M_K = \sup \|f_k\|_{L^p(K)} < \infty$

$$\mu_k(E) = \int_E f_k \, dx \qquad K \text{ compact}$$

$$\mu_k(K) = \int_K f_k \, dx \leq \left(\int_K f_k^p \, dx\right)^{1/p} \mathcal{L}^n(K)^{1-1/p} \quad (*)$$

$$\sup_k \mu_k(K) \leq M_K \, \mathcal{L}^n(K)^{1-1/p} < \infty$$

$$\exists \mu \quad \mu_{k_j} \to \mu \quad \left(\forall \varphi \in C_c(\mathbb{R}^n) \quad \int \varphi f_k \, dx \longrightarrow \int \varphi \, d\mu \right)$$

Claim 1: $\mu \ll \mathcal{L}^n$

$\varepsilon > 0$

$\mathcal{L}^n(A) = 0 \quad \exists V$ open $\quad A \subset V \quad \mathcal{L}^n(V) < \varepsilon$

$$\mu V \leq \liminf_{k_j \to \infty} \mu_{k_j}(V) \leq \underbrace{\liminf_{k_j \to \infty} \|f_{k_j}\|_{L^p(V)}}_{} \underbrace{\mathcal{L}^n(V)^{1-1/p}}_{C\,\varepsilon^{1-1/p}}$$

$\mathcal{L}^n(A) = 0 \qquad A \subset V \qquad (V \text{ bounded})$

$$\mu V \leq C_V \, \varepsilon^{1-1/p} \qquad \Rightarrow \mu A = 0$$

$$\forall \varepsilon > 0$$

LRN $\quad \exists f \in L^1_{loc}(\mathcal{L}^n) \quad$ s.t $\quad d\mu = f \, dx$

<u>Claim 2</u>: $\quad f \in L^p(\mathbb{R}^n) \qquad \varphi \in C_c(\mathbb{R}^n)$

$$\int f \varphi \, dx = \lim_{k_j \to \infty} \int \varphi f_{k_j} \, dx \leq \sup \| f_{k_j} \|_{L^p} \, \| \varphi \|_{q\,(\mathrm{spt}\,\varphi)}$$

<u>duality</u> $\Rightarrow \quad f \in L^p$

$$\| f \|_{L^p} \leq \liminf_{k_j \to \infty} \| f_{k_j} \|_{L^p}$$

Theorem (Uniform integrability & weak convergence in L^1)

assume U bounded open set & $\{f_k\}_k \subset L^1(U)$ s.t

① $\sup_k \|f_k\|_1 < \infty$

② $\lim_{l \to \infty} \sup \int_{|f_k| \geq l} |f_k| \, dx = 0$ (uniform integrability)

$\exists \{f_{k_j}\} \subset \{f_k\}$ & $f \in L^1(U)$ s.t $f_{k_j} \rightharpoonup f$ in $L^1(U)$