Hausdorff measures

Def: (i) Let $A \subset \mathbb{R}^n$ $\quad 0 \le s < \infty \quad 0 < \delta \le \infty$

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_j \alpha(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s : A \subset \bigcup_{j=1}^{\infty} C_j ; \operatorname{diam} C_j < \delta \right\}$$

where $\alpha(s) = \dfrac{\pi^{s/2}}{\Gamma(1+s/2)}$ $\qquad \Gamma(s) = \displaystyle\int_0^\infty e^{-x} x^{s-1}\, dx$

(ii) $$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)$$

$\mathcal{H}^s$: s-dimensional Hausdorff measure

Remark: ① $\mathcal{L}^n(B(x,r)) = \alpha(n) r^n \qquad B(x,r) \subset \mathbb{R}^n$

② if $s \in \mathbb{N}$ then $\mathcal{H}^s$ agrees with "s-dimensional surface area" on nice sets

Theorem: For all $0 \le s < \infty$ $\mathcal{H}^s$ is a Borel regular measure in $\mathbb{R}^n$

Warning: $\mathcal{H}^s$ is not Radon for $0 \le s < n$ (in fact $\mathbb{R}^n$ is not even σ-finite w.r.t $\mathcal{H}^s$)

$0 \le s < n$ cover B_1 by ball $\underbrace{B_{r_i}(x_i)}_{B_i}$ $r_i \le \delta$

$$\mathcal{L}^n \left(B_1 \setminus \bigcup_{i=1}^{\infty} B_i \right) = 0 \quad \text{(cor. of Besicovitch)}$$

$$\sum_{i=1}^{\infty} \alpha(n) r_i^n = \sum_{i=1}^{\infty} \mathcal{L}^n(B_i) = \alpha(n) = \mathcal{L}^n(B_1)$$

$$1 = \sum_{i=1}^{\infty} r_i^n = \sum_{i=1}^{\infty} r_i^s r_i^{n-s} \le \delta^{n-s} \sum_{i=1}^{\infty} r_i^s \Rightarrow \sum_{i=1}^{\infty} r_i^s \ge \frac{1}{\delta^{n-s}}$$

$$\Downarrow$$

$$\mathcal{H}^s(B_1) = +\infty$$

Pf: Claim 1: $\mathcal{H}^s_\delta$ is a measure

Claim 2: $\mathcal{H}^s$ is a measure

Claim 3: $\mathcal{H}^s$ is a Borel measure (Caratheodory)

Claim 4: $\mathcal{H}^s$ is a Borel regular measure

Claim 1 : $\mathcal{H}^s_\delta$ is a measure

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_j \alpha(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s : A \subset \bigcup_{j=1}^\infty C_j \, ; \, \operatorname{diam} C_j < \delta \right\}$$

1) $\mathcal{H}^s_\delta(\emptyset) = 0$

2) $\mathcal{H}^s_\delta$ subadditive $\{A_k\} \subset \mathbb{R}^n$ $\mathcal{H}^s_\delta(\bigcup A_k) \leq \sum_k \mathcal{H}^s_\delta(A_k)$

let $\delta > 0$ $\operatorname{diam} C_j^k \leq \delta$ $A_k \subset \bigcup_{j=1}^\infty C_j^k$

$\{C_j^k\}_{j,k}$ cover for $\bigcup_{k=1}^\infty A_k$

$$\mathcal{H}^s_\delta(\bigcup A_k) \leq \sum_{j,k} \alpha(s) \left(\frac{\operatorname{diam} C_j^k}{2} \right)^s = \sum_{k=1}^\infty \underbrace{\sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j^k}{2} \right)^s}$$

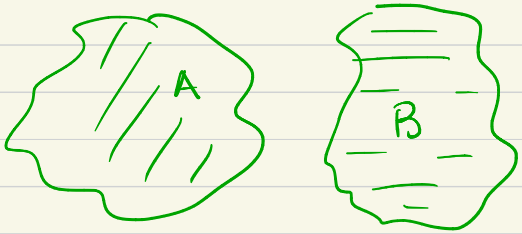
inf of $\{C_j^k\}_j$ covers for A_k

$$\mathcal{H}^s_\delta(\bigcup A_k) \leq \sum_{k=1}^\infty \mathcal{H}^s_\delta(A_k) \quad (*)$$

Claim 2 : letting $\delta \to 0$ in $(*)$ $\mathcal{H}^s(\bigcup A_k) \leq \sum_{k=1}^\infty \mathcal{H}^s(A_k)$

Claim 3: $\mathcal{H}^s$ is a Borel measure (Caratheodory)

show that if $d(A,B) > 0 \Longrightarrow \mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B)$



Choose δ s.t $\frac{1}{4} d(A,B) > \delta > 0$

Cover $A \cup B \subset \bigcup_{k=1}^{\infty} C_k$ diam $C_k \leq \delta$

$\mathcal{A} = \{C_k : C_k \cap A \neq \emptyset\}$ $\qquad$ $\mathcal{B} = \{C_k : C_k \cap B \neq \emptyset\}$

$A \subset \bigcup_{C_k \in \mathcal{A}} C_k$ $\qquad$ $B \subset \bigcup_{C_k \in \mathcal{B}} C_k$

$\mathcal{A} \cap \mathcal{B} = \emptyset$

$$\underbrace{\sum_{k=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_k}{2} \right)^s}_{\text{take inf over all covers of } A \cup B \,\leq \delta} \geq \sum_{C_k \in \mathcal{A}} + \sum_{C_k \in \mathcal{B}}$$

$$\geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$$

$$\Longrightarrow \mathcal{H}^s_\delta(A \cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$$

$\delta \to 0$ ✓

<u>Claim</u> 4 : $\mathcal{H}^s$ is Borel regular for every A $\exists B$ Borel

$A \subset B$ $\quad \mathcal{H}^s(A) = \mathcal{H}^s(B)$ $\quad$ since $\operatorname{diam} C_j = \operatorname{diam} \overline{C_j}$

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum \alpha(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s : A \subset \bigcup C_j \quad \operatorname{diam} C_j \le \delta \right\}$$

C_j closed.

for each k $\quad \{C_j^k\}$ $\quad \operatorname{diam} C_j^k \le 1/k$

$$\underbrace{A \subset \bigcup_{j=1}^\infty C_j^k}_{A \subseteq A_k} \qquad \sum_j \alpha(s) \left(\frac{\operatorname{diam} C_j^k}{2} \right)^s \le \mathcal{H}^s_{1/k}(A) + \frac{1}{k}$$

closed

$$A \subset B = \bigcap_{k=1}^\infty A_k \leftarrow \text{Borel} \qquad \mathcal{H}^s_{1/k}(B) \le \sum_j \alpha(s) \left(\frac{\operatorname{diam} C_j^k}{2} \right)^s$$

$$B \subset \bigcup_{j=1}^\infty C_j^k \qquad \le \mathcal{H}^s_{1/k}(A) + \frac{1}{k}$$

$k \to \infty$ ✓

Theorem (Properties of Hausdorff measure)

(i) $\mathcal{H}^0$ = counting measure

(ii) $\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}$

(iii) $\mathcal{H}^s \equiv 0$ on $\mathbb{R}^n$ for $s > n$

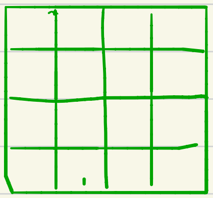
(iv) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A) \quad \forall \lambda > 0$

(v) $\mathcal{H}^s(LA) = \mathcal{H}^s(A)$ for each affine isometry $L : \mathbb{R}^n \circlearrowleft$

Pf : (iii) $\quad m \geq 1 \quad Q \subseteq \mathbb{R}^n \quad Q$ unit cube

decompose Q into 2^{mn} cubes of side length $\frac{1}{2^m}$

and diameter $\frac{\sqrt{n}}{2^m}$



$$\mathcal{H}^s_{\sqrt{n}/2^m}(Q) \leq \sum \alpha(s) \left(\frac{\sqrt{n}}{2^{m+1}} \right)^s = \alpha(s)\, 2^{mn} \frac{(\sqrt{n})^s}{2^{ms}\, 2^s}$$

$$\leq c(n,s)\, 2^{m(n-s)} \xrightarrow[m \to 0]{} 0$$

$$\mathcal{H}^s(Q) = 0$$

Lemma: Suppose $A \subset \mathbb{R}^n$ & $\mathcal{H}^s_\delta(A) = 0$ for some $\delta > 0$

then $\mathcal{H}^s(A) = 0$

Pf $s = 0$ ✓ $s > 0$ $\mathcal{H}^s_\delta(A) = 0$ $\varepsilon > 0$ $\exists \{C_j\}$

$A \subset \bigcup C_j$ diam $C_j \le \delta$

for each j

$$\sum_j \alpha(s) \left(\frac{\text{diam } C_j}{2}\right)^s \le \varepsilon \implies \alpha(s)\left(\frac{\text{diam } C_j}{2}\right)^s \le \varepsilon$$

$$\text{diam } C_j \le 2\left(\frac{\varepsilon}{\alpha(s)}\right)^{1/s} = \delta(\varepsilon) \xrightarrow[\varepsilon \downarrow 0]{} 0$$

$$\implies \mathcal{H}^s_{\delta(\varepsilon)}(A) \le \varepsilon \qquad \varepsilon \to 0$$

$$\downarrow$$

$$\mathcal{H}^s(A) = 0$$

Lemma: Let $A \subset \mathbb{R}^n$ and $0 \leq s < t < \infty$

(i) if $\mathcal{H}^s(A) < \infty \Rightarrow \mathcal{H}^t(A) = 0$

(ii) if $\mathcal{H}^t(A) > 0 \Rightarrow \mathcal{H}^s(A) = +\infty$

Pf $\underbrace{\mathcal{H}^s(A) < \infty}_{\lim_{\delta \to 0} \mathcal{H}^s_\delta(A)}$ $\delta > 0$ $\exists \{C_j\}_{j=1}^\infty$ $\operatorname{diam} C_j \leq \delta$

$$\sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \leq \mathcal{H}^s_\delta(A) + 1 \leq \mathcal{H}^s(A) + 1 < \infty$$

$t > s$

$$\mathcal{H}^t_\delta(A) \leq \sum \alpha(t) \left(\frac{\operatorname{diam} C_j}{2}\right)^t \leq \frac{\alpha(t)}{\alpha(s)} \sum \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \left(\frac{\delta}{2}\right)^{t-s}$$

$$\leq \frac{\alpha(t)}{\alpha(s)} \left(\frac{\delta}{2}\right)^{t-s} (\mathcal{H}^s(A) + 1)$$

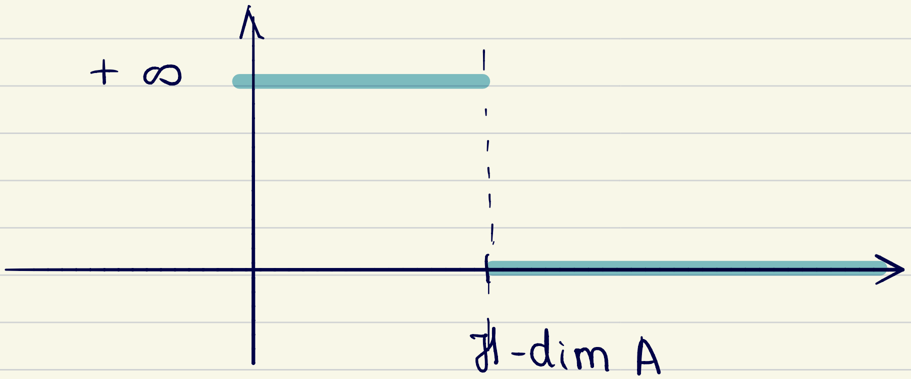
$t - s > 0$

$$\Big\downarrow \delta \to 0$$

$$0$$

Definition: The Hausdorff dimension of $A \subset \mathbb{R}^n$ is

$$\mathcal{H}\text{-}\dim A = \inf\{0 \le s < \infty \quad \mathcal{H}^s(A) = 0\}$$



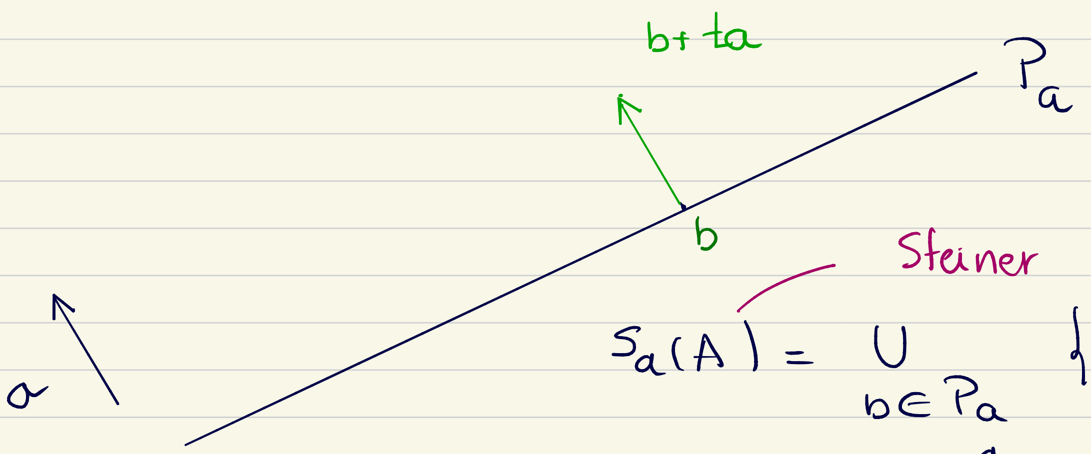
Remarks: (i) $A \subset \mathbb{R}^n$, $\dim A = \mathcal{H}\text{-}\dim A \le n$

(ii) $s = \mathcal{H}\text{-}\dim A \Rightarrow \begin{cases} \mathcal{H}^t(A) = 0 & t > s \\ \mathcal{H}^t(A) = +\infty & t < s \end{cases}$

& $\mathcal{H}^s(A) \in [0, \infty]$

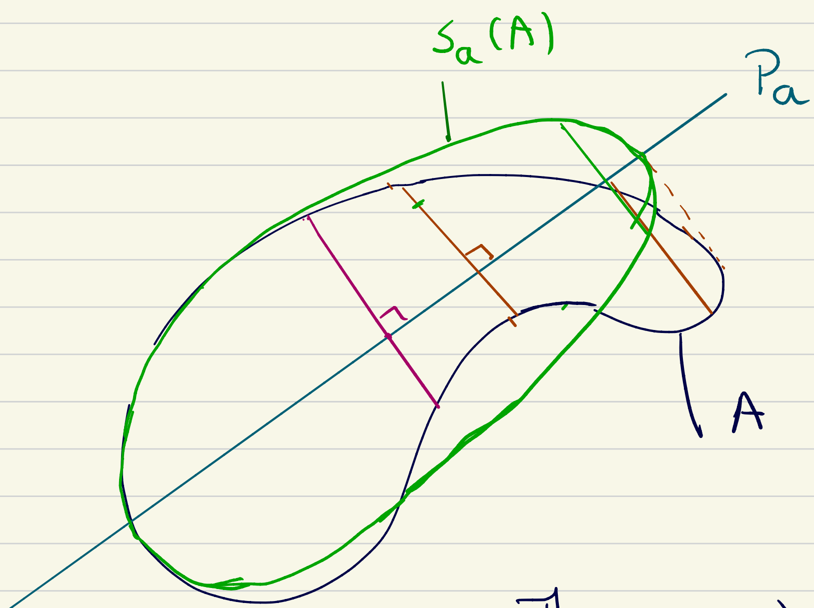
Fix $a, b \in \mathbb{R}^n$ $\quad |a| = 1$

$$P_a = \{x \in \mathbb{R}^n : \langle x, a \rangle = 0\} \qquad L_a^b = \{b + ta : t \in \mathbb{R}^n\}$$



Steiner Symmetrization

$$S_a(A) = \bigcup_{\substack{b \in P_a \\ A \cap L_b^a \neq \emptyset}} \left\{ b + ta : |t| \le \frac{1}{2} \mathcal{H}^1(A \cap L_b^a) \right\}$$

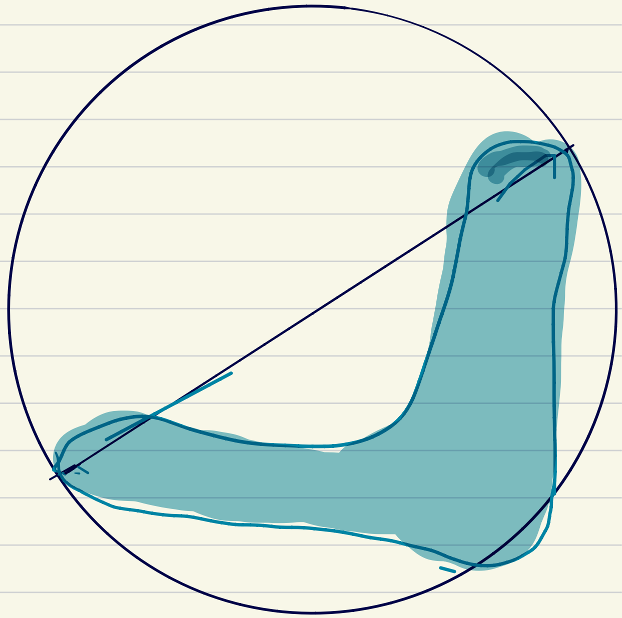


Theorem 1) diam $S_A(A) \le$ diam A

2) A $\mathcal{L}^n$ measurable, $S_a(A)$ also $\quad \mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$

Theorem : (Isodiametric inequality)

For all sets $A \subset \mathbb{R}^n$ $\quad \mathcal{L}^n(A) \leq \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n$



A is not necessarily included in a ball of diam A

Pf : $\operatorname{diam} A = +\infty$ ✓

1) $\operatorname{diam} A < \infty$

$\{e_1 \dots e_n\}$ standard bases for $\mathbb{R}^n$

$A_1 = S_{e_1}(A) \quad A_2 = S_{e_2}(A_1) \quad \dots \quad A^* = A_n = S_{e_n}(A_{n-1})$

Claim 1 : A^* is symmetric w.r.t origin

if $x \in A^* \Rightarrow -x \in A^*$

$2|x| \leq \operatorname{diam} A^*$

$A^* \subset B(0, \operatorname{diam} A^*)$

$\operatorname{diam} A^* \leq \operatorname{diam} A$

Claim 2 $\quad \mathcal{L}^n(A^*) \leq \alpha(n)\left(\frac{\operatorname{diam} A^*}{2}\right)^n$

Claim 3 : $\mathcal{L}^n(A) = \mathcal{L}^n(A^*) \leq$

$\leq \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n$

Theorem : $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbb{R}^n$

Pf Claim 1 : $\mathcal{L}^n(A) \le \mathcal{H}^n(A) \quad \forall A \subset \mathbb{R}^n$

Claim 2 : $\mathcal{H}^n \ll \mathcal{L}^n \quad (\mathcal{H}^n(A) \le c(n)\, \mathcal{L}^n(A))$

claim 3 : $\mathcal{H}^n(A) = \mathcal{L}^n(A)$.

Pf : claim 1 Fix $\delta > 0$ $\{C_j\}$ $A \subset \bigcup C_j$ $\operatorname{diam} C_j \le \delta$

$$\mathcal{L}^n(A) \le \sum_j \mathcal{L}^n(C_j) \underset{\uparrow \text{isodiametric}}{\le} \sum \alpha(n) \left(\frac{\operatorname{diam} C_j}{2}\right)^n$$

$\le$ inf on the covers

$$\mathcal{L}^n(A) \le \mathcal{H}^n_\delta(A) \qquad \delta \to 0 \quad ✓$$

Claim 2 $\mathcal{L}^n(A) = \inf \left\{ \sum_i \mathcal{L}^n(Q_i) : A \subset \bigcup Q_i \right\}$ cubes // coord

diam $Q_i \le \delta$

Q cube

$$\alpha(n) \left(\frac{\operatorname{diam} Q}{2}\right)^n = \underbrace{\alpha(n) \frac{1}{2^n} (\sqrt{n})^n}_{c(n)} l(Q)^n \quad \mathcal{L}^n(Q)$$

$$\mathcal{H}^n_\delta(A) \le \sum \alpha(n) \left(\frac{\operatorname{diam} Q_i}{2}\right)^n$$

$$\le c_n \sum \mathcal{L}^n(Q_i) = c_n \mathcal{L}^n(A)$$

$\mathcal{H}^n(A) \le c_n \mathcal{L}^n(A) \rightsquigarrow \mathcal{H}^n(A) \le \mathcal{L}^n(A)$

$\mathcal{L}^n(A) \le \mathcal{H}^n(A)$

<u>Claim 3</u>: fix $\delta > 0$ $\varepsilon > 0$ $\{Q_i\}$ s.t. $A \subset \bigcup Q_i$ (cubes) $\operatorname{diam} Q_i < \delta$

$$\sum \mathcal{L}^n(Q_i) \le \mathcal{L}^n(A) + \varepsilon$$

by Vitali for each i "cover" Q_i $\{B_i^k\}$ disjoint balls $\operatorname{diam} B_i^k \le \delta$

$$\mathcal{L}^n\Big(Q_i \setminus \bigcup_k B_i^k\Big) = 0 \overset{C2}{\Longrightarrow} \mathcal{H}^n\Big(Q_i \setminus \bigcup_k B_i^k\Big) = 0$$

$$\mathcal{H}^n_\delta(A) \le \sum_i \mathcal{H}^n_\delta(Q_i) \le \sum_{i=} \sum_k \underbrace{\mathcal{H}^n_\delta(B_i^k)}_{\le \alpha(n)\left(\frac{\operatorname{diam} B_i^k}{2}\right)^n}$$

$\rightsquigarrow$ def

$$\mathcal{H}^n_\delta(A) \le \sum_i \sum_k \mathcal{L}^n(B_i^k) = \sum_i \mathcal{L}^n\Big(\bigcup B_i^k\Big)$$

$$\le \sum_i \mathcal{L}^n(Q_i) \le \mathcal{L}^n(A) + \varepsilon \; ; \quad \delta \to 0 \text{ then } \varepsilon \to 0.$$