Densities

Recall that for $E \subset \mathbb{R}^n$ $\mathcal{L}^n$ measurable

$$\lim_{r \to 0} \frac{\mathcal{L}^n(E \cap B(x,r))}{\alpha(n) r^n} = \begin{cases} 1 & \mathcal{L}^n \text{ a.e } x \in E \\ 0 & \mathcal{L}^n \text{ a.e } x \notin E \end{cases}$$

Theorem: Assume $E \subset \mathbb{R}^n$, E $\mathcal{H}^s$ measurable &

$\mathcal{H}^s(E) < \infty$ then

$$\lim_{r \to 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{\alpha(s) r^s} = 0 \qquad \mathcal{H}^s \text{ a.e } x \in \mathbb{R}^n \setminus E$$

we want

$\mathcal{H}^s(A_t) = 0$

$\Uparrow$

Pf: fix $t > 0$

$$A_t = \left\{ x \in \mathbb{R}^n \setminus E : \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} > t \right\}$$

Since $\mathcal{H}^s \llcorner E$ is Radon, given $\varepsilon > 0$ $\exists K \subset E$ s.t

$\mathcal{H}^s(E \setminus K) < \varepsilon$, $U = \mathbb{R}^n \setminus K$ open

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$A_t \subset U$ fix $\delta > 0$ consider

$$\mathcal{F}_\delta = \{ B(x,r) : x \in A_t, \ 0 < r < \delta : \mathcal{H}^s(B(x,r) \cap E) > \alpha(s) t r^s, \ B(x,r) \subset U \}$$

$B(x,r)$ ↑ closed

$\mathcal{F}$ cover A_t, $\operatorname{diam} B \le 2\delta$; $\exists \{B_i\}_{i \in \mathbb{N}} \subset \mathcal{F}$ Vitali

$$A_t \subset \bigcup_{i=1}^\infty \hat{B}_i \qquad \hat{B}_i = 5B_i$$

$$\mathcal{H}^s_{10\delta}(A_t) \le \sum_{i=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} \hat{B}_i}{2} \right)^s = \sum_{i=1}^\infty \alpha(s)(5 r_i)^s$$

$$\le 5^s \sum_{i=1}^\infty \alpha(s) r_i^s \le \frac{5^s}{t} \sum_{i=1}^\infty \mathcal{H}^s(B_i \cap E)$$

$$\mathcal{H}^s_{10\delta}(A_t) \leq \frac{5^s}{t} \sum_{i=1}^{\infty} \mathcal{H}^s(B_i \cap E) \leq \frac{5^s}{t} \mathcal{H}^s\left(\bigcup_i B_i \cap E\right) = \frac{5^s}{t} \mathcal{H}^s(U \cap E)$$

$$U = \mathbb{R}^n \setminus K \Rightarrow \mathcal{H}^s_{10\delta}(A_t) \leq \frac{5^s}{t} \mathcal{H}^s(E \setminus K) < \frac{5^s}{t} \varepsilon$$

$$\delta \to 0 \qquad \mathcal{H}^s(A_t) \leq 5^s/t \; \varepsilon$$

$$\varepsilon \to 0 \qquad \mathcal{H}^s(A_t) = 0 \qquad \forall t > 0$$

<u>Theorem</u>: Assume $E \subset \mathbb{R}^n$, $\mathcal{H}^s$ measurable +

$0 < \mathcal{H}^s(E) < \infty$ then

(*) $$\frac{1}{2^s} \leq \limsup_{r \to 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{\alpha(s) r^s} \leq 1$$

for $\mathcal{H}^s$ a.e. $x \in E$

<u>Remarks</u>: ① 2^{-s} is known to be best possible for $s \leq 1$.
For $s > 1$, Besicovitch conjecture is that the best lower bound is $1/2$.

② It is possible to have E $\mathcal{H}^s$-measurable s.t.

$$\limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} < 1$$

and for $\mathcal{H}^s$ a.e.

$$\liminf_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} = 0 \qquad x \in E.$$

Pf ① $t > 1$ $\quad B_t = \{ x \in E : \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} > t \}$

Goal : show $\mathcal{H}^s(B_t) = 0$

② $B_0 = \{ x \in E : \limsup_{r \to 0} \frac{\mathcal{H}^s_\infty(B(x,r) \cap E)}{\alpha(s) r^s} < 2^{-s} \}$

content

$\mathcal{H}^s(\cdot) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(\cdot) = \sup_{\delta > 0} \mathcal{H}^s_\delta(\cdot) \geq \mathcal{H}^s_\infty(\cdot)$

Goal : show that $\mathcal{H}^s(B_0) = 0$ $\qquad t < 1$

$A_t = \{ x \in B_0 : \limsup_{r \to 0} \frac{\mathcal{H}^s_\infty(B(x,r) \cap E)}{\alpha(s) r^s} < t 2^{-s} \}$

$x \in A_t \quad \exists \delta_x > 0 \quad \forall r \in (0, \delta_x)$

$\frac{\mathcal{H}^s_\infty(B(x,r) \cap E)}{\alpha(s) r^s} < t 2^{-s} \Rightarrow \mathcal{H}^s_\infty(B(x,r) \cap E) \leq t \alpha(s) \left(\frac{r}{2}\right)^s$

$$E(\delta, \tau) = \left\{ x \in E : \mathcal{H}^s_\delta (E \cap C) \leq \tau \alpha(s) \left(\frac{\operatorname{diam} C}{2} \right)^s : x \in C \right.$$

$$\left. \operatorname{diam} C \leq \delta \right\}$$

$\tau < 1$

$$\mathcal{H}^s (E(\delta, \tau)) = 0$$

Goal : show if $x \in B_0$ $\exists$ δ, τ s.t $x \in E(\delta, \tau)$

① $t > 1$ $\quad B_t = \{x \in E : \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} > t\}$

Goal : show $\mathcal{H}^s(B_t) = 0$ $\qquad \delta > 0$

$\mathcal{F} = \{ B(x,r) : x \in B_t : \mathcal{H}^s(B(x,r) \cap E) > t\alpha(s) r^s , 0 < r < \delta \}$

$B(x,r) \subset U$

where U open s.t $B_t \subset U$ $\quad \mathcal{H}^s(E \cap U) \le \mathcal{H}^s(B_t) + \varepsilon$

Corollary to Vitali $\{B_i\}$ disjoint in $\mathcal{F}$

$$B_t \setminus \bigcup_{i=1}^{m} B_i \subset \bigcup_{i=m+1}^{\infty} \hat{B}_i \qquad \left(B_t \subset \bigcup_{i=1}^{m} B_i \cup \bigcup_{i=m+1}^{\infty} \hat{B}_i \right)$$

$$\mathcal{H}^s_{10\delta}(B_t) \le \sum_{i=1}^{m} \alpha(s) r_i^s + 5^s \sum_{i=m+1}^{\infty} \alpha(s) r_i^s$$

$$\le \sum_{i=1}^{m} \frac{1}{t} \mathcal{H}^s(B_i \cap E) + \frac{5^s}{t} \sum_{i=m+1}^{\infty} \mathcal{H}^s(B_i \cap E)$$

$$\le \underbrace{\frac{1}{t} \mathcal{H}^s\left(\bigcup_{i}^{m} B_i \cap E\right)}_{\text{find ind of the balls}} + \underbrace{\frac{5^s}{t} \sum_{i=m+1}^{\infty} \mathcal{H}^s(B_i \cap E)}_{\text{want this small}}$$

$$\sum_{i=1}^{\infty} \mathcal{H}^s(B_i \cap E) \le \mathcal{H}^s(E) < \infty$$

$$\sum_{i=m+1}^{\infty} \mathcal{H}^s(B_i \cap E) \xrightarrow[m \to \infty]{} 0$$

$$\sum_{i=1}^{m} \mathcal{H}^s(B_i \cap E) = \mathcal{H}^s\left(\bigcup_{i=1}^{m} B_i \cap E\right) \le \mathcal{H}^s(U \cap E)$$

$$\mathcal{H}^s_{10\delta}(B_t) \le \frac{1}{t}\mathcal{H}^s(U \cap E) + \frac{5^s}{t}\sum_{i=m+1}^{\infty} \mathcal{H}^s(B_i \cap E)$$

$m \to \infty$ $\qquad \mathcal{H}^s_{10\delta}(B_t) \le \frac{1}{t}\mathcal{H}^s(E \cap U) \le \frac{1}{t}\left(\mathcal{H}^s(B_t) + \varepsilon\right)$

$\delta \to 0$ $\qquad \mathcal{H}^s(B_t) \le \frac{1}{t}\left(\mathcal{H}^s(B_t) + \varepsilon\right)$ $\qquad \mathcal{H}^s(B_t) = 0$

$\varepsilon \to 0$ $\qquad \mathcal{H}^s(B_t) \le \frac{1}{t}\mathcal{H}^s(B_t)$ $\qquad t > 1$