Theorem: Assume $E \subset \mathbb{R}^n$, $\mathcal{H}^s$ measurable &

$0 < \mathcal{H}^s(E) < \infty$ then

$$(*) \quad \frac{1}{2^s} \le \limsup_{r \to 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{\alpha(s) r^s} \le 1$$

for $\mathcal{H}^s$ a.e. $x \in E$.

Claim 2 $\quad \mathcal{H}^s \ge \mathcal{H}^s_\infty$

$$B_0 = \left\{ x \in E : \limsup_{r \to 0} \frac{\mathcal{H}^s_\infty(B(x,r) \cap E)}{\alpha(s) r^s} < \frac{1}{2^s} \right\}$$

Goal: show $\mathcal{H}^s(B_0) = 0$, for $x \in B_0$ $\exists \delta_x > 0$

s.t $0 < \rho < \delta_x$

$$\mathcal{H}^s_\infty(B(x,\rho) \cap E) < \frac{1-\delta_x}{2^s} \alpha(s) r^s \longleftarrow$$

$$B_0 = \bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} \underbrace{\{ x \in B_0 : \delta_x > 2^{-j} \}}_{A_j}$$

$\delta > 0 \qquad \tau \in (0,1)$

$$E(\delta,\tau) = \left\{ x \in E : \underline{\mathcal{H}^s_\delta(E \cap C) \le \tau\,\alpha(s)} \left(\frac{\operatorname{diam} C}{2}\right)^s \right.$$

$$\left. x \in C \;\; \forall \;\; \operatorname{diam} C \le \delta \right\}$$

let $\{C_i\}$ $\quad \operatorname{diam} C_i \le \delta \qquad E(\delta,\tau) \subset \bigcup_{i=1}^\infty C_i$

$C_i \cap E(\delta,\tau) \neq \emptyset$

$$\mathcal{H}^s_\delta(E(\delta,\tau)) \le \sum_{i=1}^\infty \mathcal{H}^s_\delta(E(\delta,\tau) \cap C_i)$$

$$\le \sum_{i=1}^\infty \mathcal{H}^s_\delta(E \cap C_i) \le \sum_{i=1}^\infty \tau\,\alpha(s)\left(\frac{\operatorname{diam} C_i}{2}\right)^s$$

$$\le \tau \sum_{i=1}^\infty \alpha(s)\left(\frac{\operatorname{diam} C_i}{2}\right)^s$$

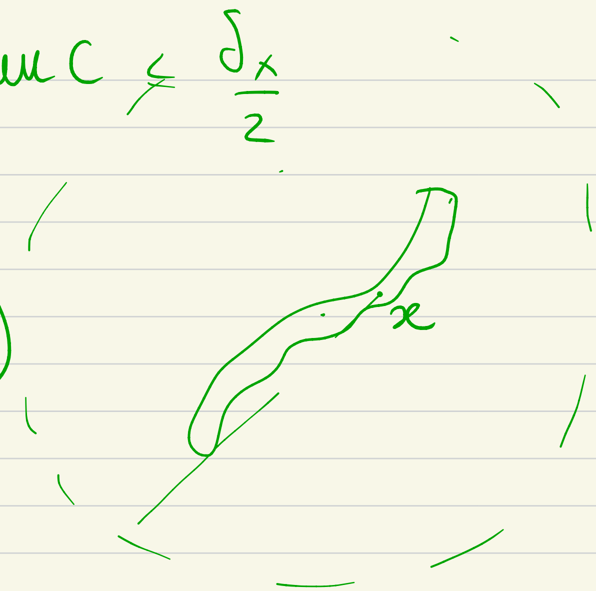
inf over $\{C_i\}$

$$\mathcal{H}^s_\delta(E(\delta,\tau)) \le \tau\,\mathcal{H}^s_\delta(E(\delta,\tau))$$

$\tau < 1$

$$\Rightarrow \underline{\mathcal{H}^s_\delta(E(\delta,\tau)) = 0} \implies \mathcal{H}^s(E(\delta,\tau)) = 0$$

Let $x \in B_0$, recall $\delta_x = \delta$ $\quad x \in C \quad \operatorname{diam} C \le \frac{\delta_x}{2}$



$$\mathcal{H}^s_\infty(E \cap C) \le \mathcal{H}^s_\delta(E \cap C)$$

$$\le \mathcal{H}^s_\delta(E \cap B(x, \operatorname{diam} C))$$

$$\le \frac{1-\delta_x}{2^s}\, \alpha(s)\, (\operatorname{diam} C)^s$$

$\longrightarrow$
$$\le (1-\delta_x)\, \alpha(s) \left(\frac{\operatorname{diam} C}{2}\right)^s$$

$$x \in E\left(\frac{\delta}{2}, 1-\delta\right)$$

$$B_0 \subset \bigcup_{j=1}^{\infty} \underbrace{E\left(\frac{1}{2j}, 1-\frac{1}{j}\right)}_{\mathcal{H}^s \text{ measure } 0} \implies \mathcal{H}^s(B_0) = 0$$

Notation:

$$\Theta^{*,s}(E,x) = \limsup_{r\to 0} \frac{\mathcal{H}^s(E\cap B(x,r))}{\alpha(s)\, r^s}$$ upper s-density of E at x

$$\Theta^s_*(E,x) = \liminf_{r\to 0} \frac{\mathcal{H}^s(E\cap B(x,r))}{\alpha(s)\, r^s}$$ lower s-density of E at x

(*)

Theorem: Let $f \in L^1_{loc}(\mathbb{R}^n)$ $\quad 0 \le s < n$ &

$$\Lambda_s = \left\{ x\in\mathbb{R}^n : \limsup_{r\to 0} \frac{1}{r^s}\int_{B(x,r)} |f| > 0 \right\}$$

then

$$\mathcal{H}^s(\Lambda_s) = 0$$

Pf: HW in 524 $\quad \mathcal{L}^n(\Lambda_s) = 0$

+ covering

Lipschitz functions

Definitions: i) A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz (continuous) if $\exists L > 0$ s.t

$$|f(x) - f(y)| \leq L|x-y| \qquad \forall x, y \in A$$

ii) The smallest such L is the Lipschitz constant of f

$$Lip(f) = \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|} : x, y \in \mathbb{R}^n,\ x \neq y \right\}$$

(iii) A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz (continuous) if for each $K \subset A$ compact $\exists C_K > 0$ s.t

$$|f(x) - f(y)| \leq C_K |x-y| \qquad \forall x, y \in K$$

Examples: ① $f : \bar{B} \to \mathbb{R}$ C^1 (diff + deriv. continuous)

$$|f(x) - f(y)| \leq |f'(\xi)|\,|x-y| \qquad \xi \in \text{seg}[x,y]$$

② $f : \mathbb{R}^n \to \mathbb{R}$ $\quad f(x) = \|x\|$ $\quad f : \mathbb{R} \to \mathbb{R}$

$f(x) = |x|$

Theorem: (i) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz, $A \subset \mathbb{R}^n$ $0 \le s < \infty$ then

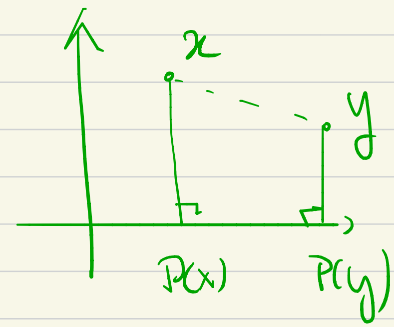
$$\mathcal{H}^s(f(A)) \le (\mathrm{Lip}\, f)^s \, \mathcal{H}^s(A)$$

(ii) Suppose $n > k$ & $P : \mathbb{R}^n \to \mathbb{R}^k$ denotes the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^k$ then for $A \subset \mathbb{R}^n$ $0 \le s < \infty$

$$\mathcal{H}^s(P(A)) \le \mathcal{H}^s(A)$$

Pf: (ii) from (i) $|P(x) - P(y)| \le |x - y|$

$\mathrm{Lip}\, P = 1$ apply (i)



(i) $\mathrm{diam}\, f(C) \le \mathrm{Lip}\, f \, (\mathrm{diam}\, C)$ $*$

$A \subset \bigcup C_i$ $\mathrm{diam}\, C_i \le \delta$ $f(A) \subset \bigcup_{i=1}^{\infty} f(C_i)$

$$\mathcal{H}^s_{(\mathrm{Lip} f)\delta}(f(A)) \le \sum_{i=1}^{\infty} \alpha(s) \left(\frac{\mathrm{diam}\, f(C_i)}{2} \right)^s$$

inf $\{C_i\}$

$$\le (\mathrm{Lip}\, f)^s \sum_{i=1}^{\infty} \alpha(s) \left(\frac{\mathrm{diam}\, C_i}{2} \right)^s$$

<u>Theorem</u> : (Extension of Lipschitz mappings)

Assume $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz. Then there exists a Lipschitz function $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that

(i) $\bar{f} = f$ in A

(ii) $\operatorname{Lip} \bar{f} \leq \sqrt{m} \operatorname{Lip} f$

<u>Theorem</u> : (Kirszbraun's theorem)

Assume $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz. Then there exists a Lipschitz function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ s.t

(i) $\tilde{f} = f$ in A

(ii) $\operatorname{Lip} \hat{f} = \operatorname{Lip} f$

Pf $f: A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, for $x \in \mathbb{R}^n$

$\mathrm{Lip} f = L$

$$\bar{f}(x) = \inf_{a \in A} \{ f(a) + \mathrm{Lip} f \, |x-a| \} \leftarrow$$

$b \in A$ $\quad \bar{f}(b) \leq f(a) + \mathrm{Lip} f \, |b-a| \quad \forall a \in A$

$\bar{f}(b) \leq f(b) \quad a = b$

since f is Lip on A $\quad |f(b) - f(a)| \leq L|b-a|$

$f(b) \leq f(a) + L|b-a| \quad \forall a \in A$ inf

$f(b) = \bar{f}(b)$

$f(b) \leq \bar{f}(b)$

$x, y \in \mathbb{R}^n$

$$\bar{f}(x) = \inf_{a \in A} \{ f(a) + L|x-a| \} \leq \inf_{a \in A} \{ f(a) + L|y-a| + L|x-y| \}$$

$$\leq \underbrace{\inf_{a \in A} \{ f(a) + L|y-a| \}} + L|x-y|$$

$$\leq \bar{f}(y) + L|x-y| \implies \bar{f}(x) - \bar{f}(y) \leq L|x-y| \checkmark$$

Definition: The function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if there exists a linear mapping

$$L: \mathbb{R}^n \to \mathbb{R}^m \quad \text{s.t.}$$

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y-x)|}{|x-y|} = 0$$

or equiv.

$$f(y) = f(x) + L(y-x) + o(|x-y|)$$

Notation: If such linear map L exists it is unique & we write

$$L = Df(x) \quad \text{the derivative of } f \text{ at } x.$$

Rademacher's theorem: Assume that $f: \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz function. Then f is differentiable $\mathcal{L}^n$ a.e.

Assume $m = 1$, f is Lipschitz

For $v \in S^{n-1}$ $|v| = 1$, $v \in \mathbb{R}^n$, $x \in \mathbb{R}^n$

$$D_v f(x) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} \quad \text{provided the limit exist.}$$

<u>Claim</u> 1: $D_v f(x)$ exists $\mathcal{L}^n$ a.e $x \in \mathbb{R}^n$

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n} \right) \qquad \frac{\partial f}{\partial x_k} = D_{e_k} f$$

<u>Claim</u> 2: $D_v f = \langle \operatorname{grad} f ; v \rangle$

<u>Claim</u> 3: f is diff. $\mathcal{L}^n$ a.e. $x \in \mathbb{R}^n$

Claim 1

$$\overline{D}_v f(x) = \limsup_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$ f continuous

Borel measurable

$$\underline{D}_v f(x) = \liminf_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

$\operatorname{Lip} f = L$

$$|\overline{D}_v f|, \ |\underline{D}_v f| \le L$$

$$A_v = \{x \in \mathbb{R}^n : D_v f \text{ does not exist}\} = \{\overline{D}_v f(x) \neq \underline{D}_v f(x)\}$$

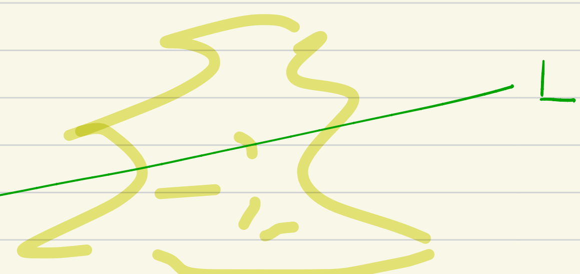
$x \in \mathbb{R}^n \qquad v \in S^{n-1}$

$$\phi : \mathbb{R} \to \mathbb{R} \qquad \phi(t) = f(x+tv)$$ absol. cont. diff a.e

$$|\phi(t) - \phi(s)| = |f(x+tv) - f(x+sv)| \le L|sv - tv| = L|s-t|$$

$L \parallel v \qquad \mathcal{H}^1(A_v \cap L) = 0$

By Fubini $\mathcal{H}^n(A_v) = 0 \Rightarrow Df$ exists a.e



<u>Claim 2</u> $\quad \xi \in C_c^\infty(\mathbb{R}^n) \qquad v \in S^{n-1}$

$$\int_{\mathbb{R}^n} \frac{f(x+tv)-f(x)}{t}\,\xi\,dx = \frac{1}{t}\left(\int f(\overset{y}{\widetilde{x+tv}})\,\xi(\overset{y-tv}{\uparrow x})\,dx - \int f(x)\,\xi(x)\,dx\right)$$

$t = 1/k$

change of variables

$$= -\int_{\mathbb{R}^n} \frac{f(x)\xi(x) - \xi(x-tv)f(x)}{t}$$

$$= -\int_{\mathbb{R}^n} f(x)\,\frac{\xi(x)-\xi(x-tv)}{t}$$

$t = 1/k$

$$\Bigg|\underbrace{\frac{f(x+\frac{1}{k}v)-f(x)}{1/k}}_{g_k}\Bigg| \le L|v| = L$$

$$|g_k(x)\,\xi(x)| \le \underbrace{L|\xi(x)|}_{\in L^1} \qquad g_k(x) \longrightarrow D_v f(x) \quad \text{a.e}$$

$$\int \frac{f(x+\frac{1}{k}v) - f(x)}{1/k}\, \xi(x)\, dx = -\int f(x)\, \frac{\xi(x) - \xi(x - 1/k\, v)}{1/k}\, dx$$

LDCT $\quad k \to \infty \quad\quad\quad \xi \in C_c^\infty(\mathbb{R}^n)$

$$\int D_v f(x)\, \xi(x)\, dx = -\int f(x)\, \underbrace{D_v \xi(x)}\, dx$$

$$= -\int f(x) \sum_{i=1}^{n} v_i \frac{\partial \xi}{\partial x_i}\, dx$$

$$\underbrace{\sum_{i=1}^{n} v_i \int \frac{\partial f}{\partial x_i}}_{D_{e_i} f}\, \xi\, dx = -\sum_{i=1}^{n} v_i \int f(x)\, \underbrace{\frac{\partial \xi}{\partial x_i}}_{D_{e_i}\xi}\, dx$$

$$\int \xi \langle \operatorname{grad} f, v \rangle = \int D_v f(x)\, \xi(x)\, dx$$

$\forall \xi \in C_c^\infty(\mathbb{R}^n)$

$$\Rightarrow \langle \operatorname{grad} f, v \rangle = D_v f(x)$$

Corollary: (i) Let $f: \mathbb{R}^n \to \mathbb{R}^m$ locally Lipschitz &

$Z = \{x \in \mathbb{R}^n : f(x) = 0\}$ then $Df(x) = 0$ a.e $x \in Z$

(ii) $f, g: \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz

$Y = \{x \in \mathbb{R}^n : g(f(x)) = x\}$

then $Dg(f(x))\, Df(x) = Id$ a.e $x \in Y$.

Questions: ① How do you compute the length of a C^1 curve?

② How do you compute the area of a C^1 graph?

③ What does integration using spherical coordinates tell us?

④ Why am I asking these questions?