Definition: The function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if there exists a linear mapping

$$L: \mathbb{R}^n \to \mathbb{R}^m \quad \text{s.t.}$$

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y-x)|}{|x-y|} = 0$$

or equiv.

$$f(y) = f(x) + L(y-x) + o(|x-y|)$$

Notation: If such linear map L exists it is unique & we write

$L = Df(x)$ the derivative of f at x.

Rademacher's theorem: Assume that $f: \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz function. Then f is differentiable $\mathcal{L}^n$ a.e.

assume $m=1$, f is Lipschitz

For $v \in S^{n-1}$ $|v|=1$, $v \in \mathbb{R}^n$, $x \in \mathbb{R}^n$

$$D_v f(x) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} \quad \text{provided the limit exist.}$$

✓ <u>Claim</u> 1 : $D_v f(x)$ exists $\mathcal{L}^n$ a.e $x \in \mathbb{R}^n$

$$\text{grad } f = \left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n} \right) \qquad \frac{\partial f}{\partial x_k} = D_{e_k} f$$

✓ <u>Claim</u> 2 : $D_v f = \langle \text{grad } f ; v \rangle$ a.e

<u>Claim</u> 3 : f is diff. $\mathcal{L}^n$ a.e. $x \in \mathbb{R}^n$

Let $\{v_k\}_k \subset S^{n-1}$ countable dense subset of S^{n-1}

$$A_k = \{ x \in \mathbb{R}^n : D_{v_k} f(x) \text{ \& grad } f(x) \text{ exist}$$
$$D_{v_k} f(x) = \langle \text{grad } f(x), v_k \rangle \}$$

let $A = \bigcap_{k=1}^{\infty} A_k$ $\qquad L^n(\mathbb{R}^n \setminus A) \leq \sum_{k=1}^{\infty} L^n\underbrace{(\mathbb{R}^n \setminus A_k)}_{0} = 0$

Claim: f is differentiable at every point of A

fix $x \in A$ $\quad v \in S^{n-1}$ $\quad t \in \mathbb{R}$ $\quad t \neq 0$

$$Q(x,v,t) = \frac{f(x+tv) - f(x)}{t} - \langle v, \operatorname{grad} f(x) \rangle$$

$v' \in S^{n-1}$

$$|Q(x,v,t) - Q(x,v',t)| \leq \left| \frac{f(x+tv) - f(x+tv')}{t} \right| + |\langle v - v', \operatorname{grad} f(x) \rangle|$$

$$\leq \operatorname{Lip} f \, \frac{|tv - tv'|}{t} + |v - v'| \underbrace{|\operatorname{grad} f(x)|}$$

$$\leq \operatorname{Lip} f \, |v - v'| + \sqrt{n} \operatorname{Lip} f \, |v - v'|$$

$$|Q(x,v,t) - Q(x,v',t)| \leq (1+\sqrt{n}) \operatorname{Lip} f \, |v - v'|$$

$$|Q(x,v,t) - Q(x,v',t)| \le (1+\sqrt{n})\,\mathrm{Lip} f\, |v - v'| \qquad (*)$$

Fix $\varepsilon > 0$ choose N large enough so $\{v_1, \ldots, v_N\}$ form $\dfrac{\varepsilon}{2(1+\sqrt{n})\,\mathrm{Lip} f}$ net in S^{n-1} (i.e $\displaystyle |v_i - v_j| \ge \frac{\varepsilon}{2(1+\sqrt{n})\,\mathrm{Lip} f}$, $i \ne j$)

$$S^{n-1} \subset \bigcup_{i=1}^{N} B_{\varepsilon/2(1+\sqrt{n})\mathrm{Lip} f}(v_i)$$

$v \in S^{n-1}$ $\exists k = 1, \ldots, N$ $\displaystyle |v - v_k| \le \frac{\varepsilon}{2(1+\sqrt{n})\,\mathrm{Lip} f}$

since $\displaystyle \lim_{t \to 0} Q(x, v_j, t) = 0$ $j = 1 \ldots N$ $\exists \delta > 0$ s.t

$0 < |t| < \delta$ $|Q(x, v_j, t)| < \varepsilon/2$ $j = k$

$$|Q(x,v,t)| \le |Q(x,v,t) - Q(x,v_k,t)| + |Q(x,v_k,t)| \qquad |t| < \delta$$

$$\overset{(*)}{\le} (1+\sqrt{n})\,\mathrm{Lip} f\, |v - v_k| + \varepsilon/2 \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Choose $y \in \mathbb{R}^n$, $x \neq y$ $v = \dfrac{y-x}{|y-x|}$ $y = x + tv$

$t = |y-x|$

$$Q(x, v, t) = \frac{f(y) - f(x)}{|y-x|} - \left\langle \operatorname{grad} f(x), \frac{y-x}{|x-y|} \right\rangle$$

$$= \frac{f(y) - f(x) - \langle \operatorname{grad} f(x); y-x \rangle}{|x-y|}$$ ✓

$$Df(x) = \operatorname{grad} f(x)$$

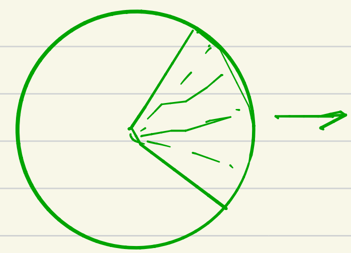
Corollary : (i) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ locally Lipschutz $\&$

$Z = \{ x \in \mathbb{R}^n : f(x) = 0 \}$ then $Df(x) = 0$ a.e $x \in Z$

(ii) $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ locally Lipschutz

$Df(x) \neq 0$

$Y = \{ x \in \mathbb{R}^n : g(f(x)) = x \}$



then $Dg(f(x))\, Df(x) = Id$ a.e $x \in Y$.

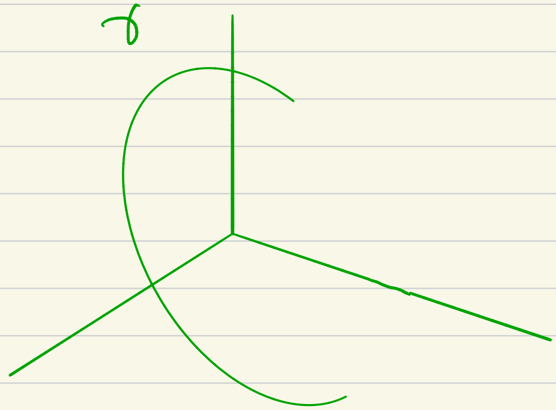
Questions : ① How do you compute the length of a C^1 curve?

② How do you compute the area of a C^1 graph?

③ What does integration using spherical wordinates tell us?

④ Why am I asking these questions?

① $f : [0,1] \subset \mathbb{R}^1 \longrightarrow \mathbb{R}^3$ $\quad C^1 \quad$ $1-1$

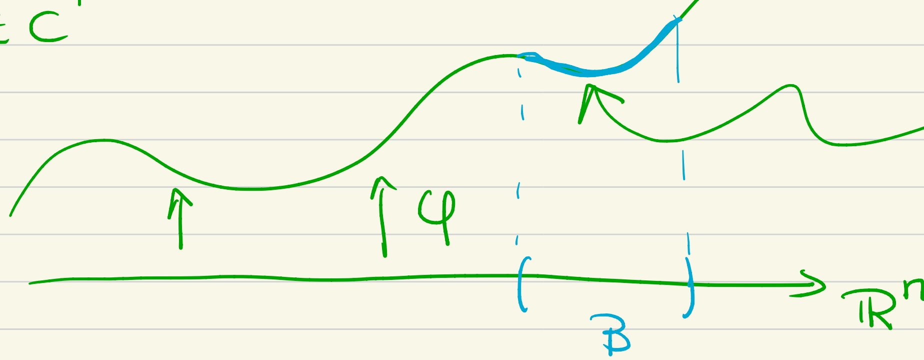


$$L(\gamma) = \mathcal{H}^1(\gamma) = \underbrace{\int_0^1 |\dot f(t)|\,dt}_{\mathcal{H}^1(f[0,1])}$$

$\gamma = f[0,1]$

② $\varphi \in C^1$

graph $\varphi = \{(\bar x, \varphi(\bar x))\}$



$$\int_B \sqrt{1 + |D\varphi|^2}\, d\bar x$$

$$\|$$

$$\mathcal{H}^n(\psi(B))$$

$\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$

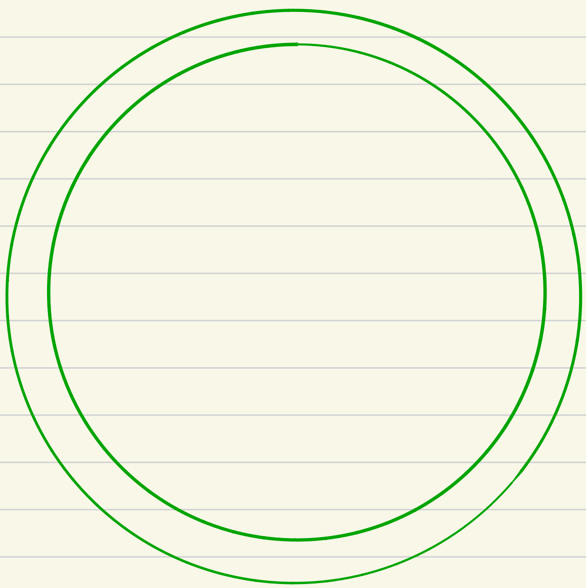
$\bar x \longrightarrow (\bar x, \varphi(\bar x))$

③ $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^+ \qquad C^1_c(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi(x)\, dx \underset{\substack{\uparrow \\ \text{spherical} \\ \text{coordinates}}}{=} \int_0^\infty \int_{S^{n-1}} r^{n-1} \varphi(r\omega)\, d\omega\, dr \quad \leftarrow\!\!\sim$$

$x = r\omega \qquad \omega \in S^{n-1}$

$f(x) = |x|$



Linear algebra intermission

Goal : Define the Jacobian of a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$

Definitions : i) A linear map $O : \mathbb{R}^n \to \mathbb{R}^m$ is orthogonal if

$$\langle Ox, Oy \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^n$$

ii) A linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric if

$$\langle x, Sy \rangle = \langle Sx, y \rangle \quad \forall x, y \in \mathbb{R}^n$$

iii) A linear map $D : \mathbb{R}^n \to \mathbb{R}^n$ is diagonal if $\exists d_1, \dots d_n \in \mathbb{R}$ s.t.

$$Dx = (d_1 x_1, \dots, d_n x_n) \quad \forall x \in \mathbb{R}^n$$

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

iv) Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear. The adjoint of A is a linear map

$A^* : \mathbb{R}^m \to \mathbb{R}^n$ defined by $\langle x, A^* y \rangle = \langle Ax, y \rangle$ $\forall x \in \mathbb{R}^n$
$\forall y \in \mathbb{R}^m$

Theorem : i) $A^{**} = A$ ii) $(A \circ B)^* = B^* \circ A^*$

iii) $O^* = O^{-1}$ if O is orthogonal

iv) $S^* = S$ if S is symmetric

v) If S symmetric $S : \mathbb{R}^n \to \mathbb{R}^n$, there exist
$O : \mathbb{R}^n \to \mathbb{R}^n$ orthogonal & $D : \mathbb{R}^n \to \mathbb{R}^n$ diagonal st

$$S = O \circ D \circ O^{-1} \quad (S \text{ diagonalizable})$$

(vi) If $O : \mathbb{R}^n \to \mathbb{R}^m$ orthogonal, then $\boxed{n \leq m}$ and

$O^* \circ O = I$ on $\mathbb{R}^n$

$O \circ O^* = I$ on $O(\mathbb{R}^n)$

Theorem: (Polar decomposition)

Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping

(i) if $\boxed{n \le m}$ $\exists S: \mathbb{R}^n \to \mathbb{R}^n$ symmetric &

$\exists O: \mathbb{R}^n \to \mathbb{R}^m$ orthogonal s.t

$$L = O \circ S$$

(ii) if $\boxed{n \ge m}$ $\exists S: \mathbb{R}^m \to \mathbb{R}^m$ symmetric

$\exists O: \mathbb{R}^m \to \mathbb{R}^n$ orthogonal

$$L = S \circ O^*$$

Definitions: Assume $L: \mathbb{R}^n \to \mathbb{R}^m$ linear

(i) if $n \le m$ $L = O \circ S$ as above, the Jacobian of L is

$$[\![L]\!] = |\det S|$$

(ii) if $n \ge m$ $L = S \circ O^*$ as above, the Jacobian of L is

$$[\![L]\!] = |\det S|$$

Note: $[\![L]\!] = [\![L^*]\!]$

<u>Theorem</u> (Jacobians & adjoints) $L : \mathbb{R}^n \to \mathbb{R}^m$ linear

(i) if $n \le m$ $[\![L]\!]^2 = \det (L^* \circ L)$

(ii) if $n \ge m$ $[\![L]\!]^2 = \det (L \circ L^*)$

<u>Theorem</u>: (Binet - Cauchy formula)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, by Rademacher's theorem f is differentiable a.e. and $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$ exists and is a linear mapping

$$Df = \begin{pmatrix} \partial f^1/\partial x_1 & \cdots & \partial f^1/\partial x_n \\ \vdots & & \\ \partial f^m/\partial x_1 & \cdots & \partial f^m/\partial x_n \end{pmatrix}$$

the Jacobian of f is

$$Jf(x) = [\![Df(x)]\!]$$

The area formula $\boxed{n \leq m}$

Lemma 1: Suppose $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear $\boxed{n \leq m}$ then

$$\mathcal{H}^n(L(A)) = [\![L]\!]\, \mathcal{H}^n(A) \qquad \forall A \subset \mathbb{R}^n$$

$\boxed{L : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ Lipschitz}}$ & $\boxed{n \leq m}$

Lemma 2: $A \subset \mathbb{R}^n$ $\mathcal{H}^n$-measurable. Then

(i) $f(A)$ is $\mathcal{H}^n$ measurable

(ii) the mapping $y \longmapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is $\mathcal{H}^n$ measurable on $\mathbb{R}^m$

multiplicity function

(iii) $$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n(y) \leq (\operatorname{Lip} f)^n\, \mathcal{H}^n(A)$$

Lemma 1 (proof) $\quad L:\mathbb{R}^n \to \mathbb{R}^m \quad n\leq m \quad L = O\circ S$

① $[\![L]\!]=0 \Rightarrow \det S = 0 \qquad \dim S(\mathbb{R}^n) \leq n-1 \Rightarrow \dim L(\mathbb{R}^n) \leq n-1$

$\mathcal{H}^n(L(\mathbb{R}^n)) = 0 \qquad A\subset\mathbb{R}^n \qquad \mathcal{H}^n(L(A)) = 0$

② $[\![L]\!]>0 \qquad \nu(A) = \mathcal{H}^n(L(A)) \qquad \nu$ Radon measure

$\nu \ll \mathcal{H}^n \qquad\qquad \underbrace{O^*\circ O}\circ S$

$$\frac{\nu(B(x,r))}{\mathcal{H}^n(B(x,r))} = \frac{\mathcal{H}^n(L(B(x,r)))}{\mathcal{H}^n(B(x,r))} = \frac{\mathcal{H}^n(O^*\circ L(B(x,r)))}{\mathcal{H}^n(B(x,r))}$$

$$= \frac{\mathcal{H}^n(S(B(x,r)))}{\mathcal{H}^n(B(x,r))} = |\det S| = [\![L]\!] = D_{\mathcal{H}^n}\nu(x)$$

$$\mathcal{H}^n(L(A)) = \int_A D_{\mathcal{H}^n}\nu(x)\,dx = [\![L]\!]\,\mathcal{H}^n(A)$$

Lemma 2 (proof) i) $f(A)$ measurable A bounded

approximate A by "interior" compact sets $K \subset\subset A$ $\Rightarrow f(K)$ compact

K compact, $f(K)$ measurable

$$\mathcal{H}^n(f(A)) \leq (\mathrm{Lip}\, f)^n \mathcal{H}^n(A)$$

$$\mathcal{H}^n(f(A \setminus K)) \leq (\mathrm{Lip}\, f)^n \mathcal{H}^n(A \setminus K)$$

ii) Decompose $\mathbb{R}^n$ into dyadic cubes of side length 2^{-k} (half open.)

$\mathcal{D}_k$

$$g_k = \sum_{Q \in \mathcal{D}_k} \chi_{f(A \cap Q)} \quad \mathcal{H}^n \text{ measurable.}$$

$g_k(y) = \#$ cubes $Q \in \mathcal{D}_k$ s.t. $y \in f(A \cap Q)$

$f^{-1}(y) \cap (A \cap Q) \neq \emptyset$

$\{g_k\}$

$g_k \leq g_{k+1}$ monotone seq

$$g_k(y) \xrightarrow[k \to \infty]{} \mathcal{H}^0(A \cap f^{-1}(y)) \quad \mathcal{H}^n \text{ measurable.}$$

(iii)

$$\int_{\mathbb{R}^m} g_k \, d\mathcal{H}^n = \int_{\mathbb{R}^m} \sum_{Q \in \mathcal{D}_k} \chi_{f(A \cap Q)} \, d\mathcal{H}^n$$

MCT $\downarrow$

$$= \sum_{Q \in \mathcal{D}_k} \mathcal{H}^n (f(A \cap Q))$$

$$\int_{\mathbb{R}^m} \mathcal{H}^0 (f^{-1}(y) \cap A) \, d\mathcal{H}^n(y)$$

$$\leq (\mathrm{Lip}\, f)^n \sum_{Q \in \mathcal{D}_k} \mathcal{H}^n (A \cap Q) =$$

$$\leq (\mathrm{Lip}\, f)^n \, \mathcal{H}^n(A)$$