The area formula $\boxed{n \le m}$

Lemma 1 : Suppose $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear $\boxed{n \le m}$ then

$$\mathcal{H}^n(L(A)) = [\![L]\!]\,\mathcal{H}^n(A) \qquad \forall A \subset \mathbb{R}^n$$

$\boxed{f : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ Lipschitz}}$ & $\boxed{n \le m}$

Lemma 2 : $A \subset \mathbb{R}^n$ $\mathcal{H}^n$ - measurable - Then

(i) $f(A)$ is $\mathcal{H}^n$ measurable

(ii) the mapping $y \longmapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is $\mathcal{H}^n$ measurable on $\mathbb{R}^m$

↑ multiplicity function

(iii) $$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n(y) \le (\operatorname{Lip} f)^n\, \mathcal{H}^n(A)$$

Lemma 3: Let $t > 1$ and $B = \{x : Df(x) \text{ exists } \& \ Jf(x) > 0\}$ there exists $\{E_k\}_{k=1}^{\infty}$ Borel subsets of $\mathbb{R}^n$ s.t

i) $B = \bigcup_{k=1}^{\infty} E_k$

ii) $f|_{E_k}$ is one-to-one

iii) for each $k \in \mathbb{N}$, there exists a symmetric automorphism $T_k : \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$\mathrm{Lip}\,(f|_{E_k} \circ T_k^{-1}) \leq t \quad , \quad \mathrm{Lip}\,(T_k \circ (f|_{E_k})^{-1}) \leq t$$

$$t^{-n} |\det T_k| \leq Jf|_{E_k} \leq t^n |\det T_k|$$

Question: What is the content of Lemma 3?

Theorem (Area formula): Let $f: \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz $\boxed{n \le m}$. For each $\mathcal{H}^n$ measurable set $A \subset \mathbb{R}^n$

(1) $$\int_A Jf\, dx = \int_{\mathbb{R}^m} \mathcal{H}^0(\underbrace{f^{-1}(y) \cap A}_{1})\, d\mathcal{H}^n(y) = \mathcal{H}^n(f(A))$$ (f 1-1)

Theorem (Change of variables) Let $f: \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz $\boxed{n \le m}$ if $g: \mathbb{R}^n \to \mathbb{R}$ $\mathcal{H}^n$-measurable

(2) $$\int_{\mathbb{R}^n} g(x)\, Jf(x)\, dx = \int_{\mathbb{R}^m} \Big(\sum_{x \in f^{-1}(\{y\})} g(x) \Big)\, d\mathcal{H}^n(y)$$

Pf: • (1) is (2) $g = \chi_A$

• approx. meas. functions by simple functions

Pf of the area formula, wlog $Df(x)$ & $Jf(x)$ exists $\forall x \in A$ wlog $\mathcal{H}^n(A) < \infty$.

Case 1 : $A \subset \{Jf > 0\}$, fix $t > 1$ choose $\{E_j\}$ in Lem 3.

$\mathcal{D}_k$ dyadic cubes of length 2^{-k}

$$F_j^i = E_j \cap Q_i \cap A \qquad Q_i \in \mathcal{D}_k$$

disjoint sets

$$\bigcup_{j,i} F_j^i = A$$

Claim 1

(MCT)

$$\lim_{k \to \infty} \sum_{i,j=1}^{\infty} \mathcal{H}^n(f(F_j^i)) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^m(y)$$

$$\mathcal{H}^n(f(F_j^i)) = \mathcal{H}^n(\underbrace{f|_{E_j} \circ T_j^{-1}} \circ T_j(F_j^i)) \le \mathrm{Lip}(f|_{E_j} \circ T_j^{-1})^n \mathcal{H}^n(T_j(F_j^i))$$

$$\le t^n \mathcal{H}^n(T_j(F_j^i))$$

$$\mathcal{H}^n(f(F_j^i)) \le t^n\, \mathcal{H}^n(T_j(F_j^i)) \le t^{2n}\, \mathcal{H}^n(f(F_j^i)) \qquad (*)$$

$$\mathcal{H}^n(T_j(F_j^i)) = \mathcal{H}^n\big(\underbrace{T_j \circ (f|_{E_j})^{-1}} \circ (f|_{E_j})(F_j^i)\big) \le t^n\, \mathcal{H}^n(f(F_j^i))$$

$$t^{-2n}\, \mathcal{H}^n(f(F_j^i)) \le t^{-n}\, \mathcal{H}^n(T_j(F_j^i)) = \underbrace{t^{-n}\,|\det T_j|}\, \mathcal{H}^n(F_j^i)$$

Lem 3

$$\le \int_{F_j^i} Jf\, dx \le t^n\, |\det T_j|\, \mathcal{H}^n(F_j^i)$$

LL

$$\le t^n\, \mathcal{H}^n(T_j(F_j^i))$$

$$\le t^{2n}\, \mathcal{H}^n(f(F_j^i)) \quad \text{by } (*)$$

$$\sum_{i,j} t^{-2n}\, \mathcal{H}^n(f(F_j^i)) \le \underbrace{\sum_{i,j} \int_{F_j^i} Jf\, dx} \le \sum_{i,j} t^{2n}\, \mathcal{H}^n(f(F_j^i))$$

$k \to \infty$ Claim 1

$$t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^m \le \int_A Jf\, dx \le t^{2n} \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^m$$

$t \to 1$

Claim 2 : $A \subset \{Jf = 0\}$ fix $0 < \varepsilon \le 1$

$$f = p \circ g : \quad g : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n \qquad p : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$$

$$g(x) = (f(x), \varepsilon x)$$

Claim 2 : $0 < Jg(x) \le C\varepsilon$ for $x \in A$ (Binet - Cauchy)

$$\mathcal{H}^n(f(A)) = \mathcal{H}^n(p(g(A))) \le \mathcal{H}^n(g(A)) \le \underbrace{\int_{\mathbb{R}^{n+m}} \mathcal{H}^0(A \cap g^{-1}\{y,z\})\, d\mathcal{H}^n(y,t))}_{\underbrace{\int_A Jg(x)\,dx}_{< C\varepsilon\, \mathcal{H}^n(A)}}$$

$$\mathcal{H}^n(f(A)) \le C\varepsilon\, \mathcal{H}^n(A)$$

$$\forall \varepsilon > 0 \quad \Rightarrow \quad \mathcal{H}^n(f(A)) = 0$$

$$\operatorname{spt} \mathcal{H}^0(A \cap f^{-1}\{y\}) \subset f(A)$$

$$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n(y) = 0 = \int_A Jf\, dx$$

Applications

① length of a curve $f : \mathbb{R} \to \mathbb{R}^m$ Lipschitz & 1-1

$$Jf = |\dot{f}| \qquad \gamma = f[a,b] \subset \mathbb{R}^m$$

$$\mathcal{H}^1(\gamma) = \int_a^b |\dot{f}(t)|\,dt \qquad \text{length of } \gamma.$$

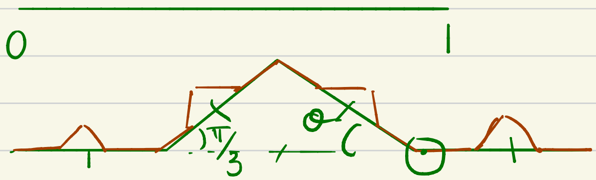
② surface area of a graph $\Gamma = \{(x, g(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$

$g : \mathbb{R}^n \to \mathbb{R}$ Lipschitz

$$f(x) = (x, g(x)) \qquad Df(x) = \begin{array}{c} \overbrace{}^{n} \\ \left(\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{array}\right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \ddots \\ 1\end{array}}\right\} n \\ \vphantom{\frac{\partial g}{\partial x_1}} \end{array}$$

$$[\![Df(x)]\!]^2 = 1 + |Dg|^2 = \text{sum of squares of } n \times n \text{ subdeterminants}.$$

$$\Gamma_B = \{(x, g(x)) : x \in B\}; \quad \mathcal{H}^n(\Gamma_B) = \int_B \sqrt{1 + |Dg|^2}\,dx$$



$\downarrow \varphi$ continuous

$n \to \infty$

$\uparrow$ # of iterations

$\mathcal{H}^1(\Gamma_\theta) = +\infty$

$\exists\, s =$ depends on θ

$0 < \mathcal{H}^s(F_\theta) < \infty$

(3) Let $M \subset \mathbb{R}^m$ Lipschitz n-dimensional embedded submanifold.

$U \subset \mathbb{R}^n$ $\quad f : U \longrightarrow \mathbb{R}^m$ chart for M ; Lipschitz

$A \subset f(U)$ Borel

$$g_{ij} = \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle \quad i,j = 1 \dots n \quad \text{then} \quad (Df)^* \circ Df = (g_{ij})_{ij}$$

$$g = \det g_{ij} \qquad Jf = \sqrt{g}$$

$$\mathcal{H}^n(A) = \text{volume of } A \text{ in } M = \int_{f^{-1}(A)} \sqrt{g}\, dx$$

The co-area formula $\boxed{n \geq m}$

Lemma 4: Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear $A \subset \mathbb{R}^n$ $\mathcal{H}^n$-meas

Then: (i) the mapping $y \longmapsto \mathcal{H}^{n-m}(A \cap L^{-1}\{y\})$ is $\mathcal{H}^m$-meas.

(ii) $$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}\{y\})\,dy = [\![L]\!]\, \mathcal{H}^n(A)$$

$$\boxed{f : \mathbb{R}^n \to \mathbb{R}^m \text{ Lipschitz}} \quad \& \quad \boxed{n \geq m}$$

Lemma 5: Let $A \subset \mathbb{R}^n$ be $\mathcal{H}^n$-measurable. Then

(i) $A \cap f^{-1}\{y\}$ is $\mathcal{H}^{n-m}$ measurable for $\mathcal{H}^m$ a.e y

(ii) The mapping $y \to \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})$ is $\mathcal{H}^m$ measurable

(iii) $$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\,dy \leq \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\mathrm{Lip}\, f)^m\, \mathcal{H}^n(A)$$

Lemma 6: Let $t > 1$ assume $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is Lipschitz and set

$$B = \{ x : Dh(x) \text{ exists } \& \ Jh(x) > 0 \}$$

then there exists a countable collection $\{ D_k \}_{k=1}^{\infty}$ of Borel sets of $\mathbb{R}^n$

s.t.

(i) $\mathcal{H}^n \left(B \setminus \bigcup_{k=1}^{\infty} D_k \right) = 0$

(ii) $h|_{D_k}$ is one-to-one

(iii) For each k $\exists S_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ symmetric automorphism s.t.

$$\operatorname{Lip}\left(S_k^{-1} \circ h|_{D_k} \right) \leq t \ , \quad \operatorname{Lip}\left(h|_{D_k}^{-1} \circ S_k \right) \leq t$$

$$t^{-n} |\det S_k| \leq Jh|_{D_k} \leq t^n |\det S_k|$$

Contrast : Lemma 3 & Lemma 6

Lemma 3 $\quad f : \mathbb{R}^n \to \mathbb{R}^m \quad n \leq m$

iii) for each $k \in \mathbb{N}$, there exists a symmetric automorphism $T_k : \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$\operatorname{Lip}\left(f|_{E_k} \circ T_k^{-1}\right) \leq t, \quad \operatorname{Lip}\left(T_k \circ (f|_{E_k})^{-1}\right) \leq t$$

Lemma 6 : $h : \mathbb{R}^n \to \mathbb{R}^n$

(iii) For each k $\exists S_k : \mathbb{R}^n \to \mathbb{R}^n$ symmetric automorphism s.t.

$$\operatorname{Lip}\left(S_k^{-1} \circ h|_{D_k}\right) \leq t, \quad \operatorname{Lip}\left(h^{-1}_{|D_k} \circ S_k\right) \leq t$$

Theorem (co-area formula). Let $f : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz with $\boxed{m \leq n}$. For each $\mathcal{H}^n$-measurable set $A \subset \mathbb{R}^n$

$$\int_A Jf(x)\, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{H}^m(y)$$

Remarks: ① $f : \mathbb{R}^n \to \mathbb{R}$ $\quad f(x) = \|x\| \quad Df = \frac{x}{\|x\|} \quad Jf = 1$

$$\mathcal{H}^n(A) = \int_A 1\, d\mathcal{H}^n(x) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(A \cap f^{-1}(y))\, dy = \int_0^\infty \mathcal{H}^{n-1}(A \cap \partial B_r)\, dr$$

② $A = \{Jf = 0\}$ then for $\mathcal{H}^m$ a.e y

$$\mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) = 0$$

Sard's theorem says that if $f \in C^k(\mathbb{R}^n, \mathbb{R}^m)$ $\quad k \geq 1+n-m$

$$\{Jf = 0\} \cap f^{-1}\{y\} = \phi \quad \text{a.e } y$$

Q: how do we use LG ?

Case 1 $\quad A \subset \{Jf > 0\}$ $\quad f : \mathbb{R}^n \to \mathbb{R}^m$

$$h_\lambda(x) = (f(x), P_\lambda(x))$$

$P_\lambda : \mathbb{R}^n \to \mathbb{R}^{m-n}$

$h_\lambda : \mathbb{R}^n \to \mathbb{R}^n$

use LG to decompose h_λ

$$A_\lambda = \{x \in A : \det Dh_\lambda \neq 0\} \quad \dots$$

Case 2 $\quad A \subset \{Jf = 0\}$ $\quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$

$g(x,y) = f(x) + \varepsilon y$ (apply case 1 to g).

Theorem (Integration over level sets)

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ Lipschitz $\boxed{n \geq m}$, let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be $\mathcal{H}^n$ - measurable

(i) $g|_{f^{-1}(y)}$ is $\mathcal{H}^{n-m}$ summable $\mathcal{H}^m$ a.e y

(ii) $$\int_{\mathbb{R}^n} g \, Jf(x) \, d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \left(\int_{f^{-1}(y)} g \, d\mathcal{H}^{n-m} \right) d\mathcal{H}^m(y)$$

Remarks: the co-area formula is this theorem for $g = \chi_A$

Applications

1) Polar coordinates : $g : \mathbb{R}^n \to \mathbb{R}$ $\mathcal{H}^n$- measurable

• $$\int_{\mathbb{R}^n} g\, dx = \int_0^\infty \left(\int_{\partial B_s} g \, d\mathcal{H}^{n-1} \right) ds$$

in particular $$\underbrace{\frac{d}{dr} \int_{B_r} g}_{} = \int_{\partial B_r} g \, d\mathcal{H}^{n-1} \quad \text{a.e } r$$

Pf $f(x) = |x|$ $Jf = 1$ apply $(\cdot)$ to $g \chi_{B_r}$

$$\int_{B_r} g\, dx = \int_0^r \left(\int_{\partial B_s} g \, d\mathcal{H}^{n-1} \right) ds$$

2) Integration over level sets

Assume $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ Lipschitz then

$(Jf = |Df|$

i)
$$\int_{\mathbb{R}^n} |Df| = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}\{f = t\}\, dt$$

ii) if $\operatorname{ess\,inf} |Df| > 0$; $g: \mathbb{R}^n \to \mathbb{R}$ $\mathcal{H}^n$ summable

$$\int_{\{f>t\}} g\, dx = \int_t^{\infty} \left(\int_{\{f=s\}} g/_{|Df|}\, d\mathcal{H}^{n-1} \right) ds$$

&
$$\frac{d}{dt} \int_{\{f>t\}} g\, dx = - \int_{\{f=t\}} g/_{|Df|}\, d\mathcal{H}^{n-1} \quad \mathcal{L}^1 \text{ a.e } t.$$

Pf: apply co-area $g\chi_{\{f>t\}}$