Functions of bounded variation & Sets of finite perimeter

Definitions: $U \subset \mathbb{R}^n$ open.

i) A function $f \in L^1(U)$ has bounded variation in U if

$$\sup \left\{ \int_U f \operatorname{div} \phi \, dx : \phi \in C_c^1(U, \mathbb{R}^n) : |\phi| \le 1 \right\} < \infty$$

we write $f \in BV(U)$

ii) $E \subset \mathbb{R}^n$ measurable has finite perimeter in U if $\chi_E \in BV(U)$

iii) $f \in L^1_{loc}(U)$ has locally bounded variation if for each open set $V \subset\subset U$ (V compactly contained in U)

$$\sup \left\{ \int_V f \operatorname{div} \phi \, dx : \phi \in C_c^1(V, \mathbb{R}^n) : |\phi| \le 1 \right\} < \infty$$

$$f \in BV_{loc}(U)$$

iv) $E \subset \mathbb{R}^n$ measurable has locally finite perimeter in U if $\chi_E \in BV_{loc}(U)$.

? $f : B \to \mathbb{R}$ continuous + bounded $\quad \phi \in C_c^1(B, \mathbb{R}^n)$ $\quad \|\phi\|_\infty \leq 1$

$$\underbrace{\left| \int_B f \operatorname{div} \phi \, dx \right|} \leq \|f\|_\infty \int_B |\operatorname{div} \phi| \, dx$$

$f = \chi_B$

unit ball

unit normal

$$\left| \int_B \operatorname{div} \phi \, dx \right| \underset{\substack{\uparrow \\ \text{div. theorem}}}{=} \left| \int_{\partial B} \phi \cdot \nu_B \, d\sigma(x) \right|$$

$$\leq \|\phi\|_\infty \int_{\partial B} |\nu_B| \, d\sigma$$

$$\leq \|\phi\|_\infty \mathcal{H}^{n-1}(\partial B) \leq \mathcal{H}^{n-1}(\partial B)$$

$\chi_B \in BV(\mathbb{R}^n)$, B set of finite perimeter

Can generalize if E is s.t ∂E is C^1 & $\mathcal{H}^{n-1}(\partial E) < \infty$

Theorem (Structure theorem for BV_{loc} functions)

Assume $f \in BV_{loc}(U)$, there exist a Radon measure μ on U and $\sigma : U \to \mathbb{R}^n$ μ-measurable function s.t.

(i) $|\sigma(x)| = 1$ μ a.e

(ii) $\forall \phi \in C_c^1(U, \mathbb{R}^n)$

$$\int_U f \operatorname{div} \phi \, dx = - \int_U \phi \cdot \sigma \, d\mu$$

Remark: The structure theorem asserts that the weak first partial derivatives of BV functions are Radon measures.

Pf Define $L : C_c^1(U, \mathbb{R}^n) \to \mathbb{R}$ linear

$$L\phi = - \int_U f \operatorname{div} \phi \, dx$$

$|L\phi| \le C_V \|\phi\|_\infty$

$V \subset\subset U$ open

(A) $\sup \{ L\phi : \phi \in C_c^1(V, \mathbb{R}^n) ; \|\phi\|_\infty \le 1 \} = C_V < \infty$

$\phi \in C^1_c(V, \mathbb{R}^n)$ $\qquad \|\phi\|_\infty \le 1 \qquad |L\phi| \le C_V$

if $\|\phi\|_\infty = 1 \qquad |L\phi| \le C_V = C_V \|\phi\|_\infty$

if $\|\phi\|_\infty \neq 0$ & $\|\phi\|_\infty \neq 1 \qquad \psi = \dfrac{\phi}{\|\phi\|_\infty}$

$\psi \in C^1_c(V, \mathbb{R}^n) \qquad \|\psi\|_\infty = 1$

$\|L\psi\| \le C_V \qquad \left| L\dfrac{\phi}{\|\phi\|_\infty} \right| = |L\phi| \dfrac{1}{\|\phi\|_\infty} \le C_V$

goal extend L to an operator on $C_c(U, \mathbb{R}^n)$

$\phi \in C_c(U, \mathbb{R}^n) \qquad \operatorname{spt}\phi \subset K \subset V \subset\subset U$

$\exists \{\phi_k\} \subset C^1_c(V, \mathbb{R}^n) \qquad \phi_k \to \phi$ uniformly on V

define $\boxed{\bar{L}\phi = \lim_{k \to \infty} L\phi_k}$

$\bar{L}$ well defined

$\underbrace{\{\phi_k\}, \{\hat{\phi}_k\}}_{C^1_c(V, \mathbb{R}^n)} \qquad \phi_k \xrightarrow{\text{unif}} \phi, \quad \hat{\phi}_k \to \phi$

$$|L\phi_k - L\hat{\phi}_k| = |L(\phi_k - \hat{\phi}_k)| \le C_V \|\phi_k - \hat{\phi}_k\|_\infty$$

$$\le C_V\left(\|\phi_k - \phi\|_\infty + \|\phi - \hat{\phi}_k\|_\infty\right) \xrightarrow[k\to\infty]{} 0$$

$\bar{L} : C_c(U, \mathbb{R}^n) \to \mathbb{R}$ linear

$\forall\, V \subset\subset U$

$$\sup\{\bar{L}\phi : \phi \in C_c(V, \mathbb{R}^n) : |\phi| \le 1\} \le C_V < \infty$$

$|L\phi_k| \le C_V \|\phi_k\|_\infty \quad k \to \infty$

$|\bar{L}\phi| \le C_V \|\phi\|_\infty$

by RRT $\exists\, \sigma, \mu$ (as in theorem) s.t. $|\sigma| = 1$ μ a.e.

$$\bar{L}\phi = \int \phi \cdot \sigma \, d\mu \underset{\uparrow}{=} -\int f \operatorname{div}\phi \, dx$$

if $\phi \in C_c^1(U, \mathbb{R}^n)$

Notation: ① If $f \in BV_{loc}(U)$, $\mu = \|Df\|$ & $[Df] = \|Df\| \llcorner \sigma$ i.e.
$\phi \in C^1_c(U, \mathbb{R}^n)$

variation measure

$$\int_U f \operatorname{div} \phi \, dx = - \int_U \phi \cdot \sigma \, d\|Df\| = - \int_U \phi \cdot d[Df]$$

② $E \subset \mathbb{R}^n$ set of locally finite perimeter if $\chi_E \in BV_{loc}(U)$
$\|D\chi_E\| = \|\partial E\|$ & $\sigma = -\nu_E$ thus $\forall \phi \in C^1_c(U, \mathbb{R}^n)$

$$\int_E \operatorname{div} \phi \, dx = \int_E \phi \cdot \nu_E \, d\|\partial E\|$$

$\|\partial E\|(U)$ perimeter of E in U

③ $f \in BV_{loc}(U) \cap L^1(U)$ then $f \in BV(U)$ iff $\|Df\|(U) < \infty$ &

$$\|f\|_{BV(U)} = \|f\|_{L^1(U)} + \|Df\|(U)$$

④ Recall from Riesz Representation Theorem; if $f, \chi_E \in BV_{loc}(U)$

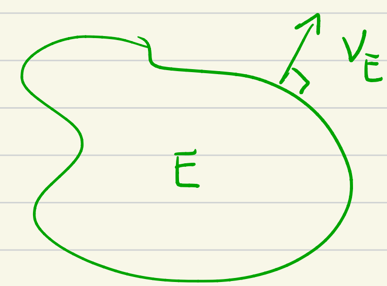
$V \subset\subset U$, V open

$$\|Df\|(V) = \sup\left\{ \int_V f \operatorname{div}\phi \;:\; \phi \in C^1_c(V,\mathbb{R}^n) \;:\; |\phi| \le 1 \right\}$$

$$\|\partial E\|(V) = \sup\left\{ \int_E \operatorname{div}\phi \;:\; \phi \in C^1_c(V,\mathbb{R}^n) \;:\; |\phi| \le 1 \right\}$$

<u>Examples</u>: 1. C^1 domain with $\mathcal{H}^{n-1}(\partial E) < \infty$ set of finite perimeter

$$\int_E \operatorname{div}\phi = \int_{\partial E} \phi \cdot \nu \, d\mathcal{H}^{n-1}$$

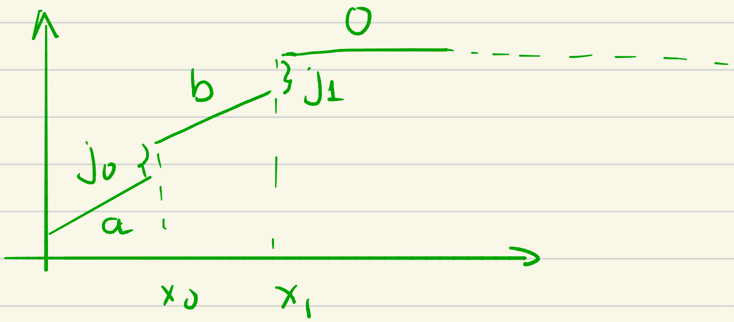


2. $f \in C^1_c(\mathbb{R}^n)$ $\quad \phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_{\mathbb{R}^n} f \operatorname{div}\phi = \int_{\mathbb{R}^n} \operatorname{div}(f\phi) - \langle Df, \phi\rangle = -\int_{\mathbb{R}^n} \langle Df, \phi \rangle \, dx$$

$$\|Df\| = \mathcal{L}^n \llcorner |Df| \qquad \sigma = \begin{cases} Df/|Df| & \text{if } Df \neq 0 \\ 0 & \text{if } Df = 0 \end{cases}$$

3.



$$[Df] = a\, \mathcal{L}^1|_{[0,x_0]} + b\, \mathcal{L}^1_{[x_0,x_1]} + j_0\, \delta_{x_0} + j_1\, \delta_{x_1}$$

if $f \in BV(\mathbb{R}^n)$

$$[Df] = \mathcal{L}^n \llcorner Df + \text{singular measure}$$

Approximation and Compactness

Theorem: (Lower semi-continuity of the variation measure)

Suppose $f_k \in BV(U)$ & $f_k \longrightarrow f$ in $L^1_{loc}(U)$ then

$$\|Df\|(U) \le \liminf_{k \to \infty} \|Df_k\|(U)$$

Pf: $\phi \in C^1_c(U, \mathbb{R}^n)$ $\quad |\phi| \le 1$

$$\int_U f \operatorname{div} \phi = \lim_{k \to \infty} \int_U f_k \operatorname{div} \phi = - \lim_{k \to \infty} \int_U \phi \cdot \sigma_k \, d\|Df_k\|$$

$$= - \liminf_{k \to \infty} \int_U \underset{\le 1}{\phi} \cdot \underset{\le 1}{\sigma_k} \, d\|Df_k\|$$

$$\le \liminf_{k \to \infty} \|Df_k\|(U)$$

Theorem : (Local approximation by smooth functions)

Assume $f \in BV(U)$ then there exist functions $\{f_k\}_{k=1}^{\infty} \in BV(U) \cap C^{\infty}(U)$

i) $f_k \to f$ in $L^1(U)$

(ii) $\|Df_k\|(U) \longrightarrow \|Df\|(U)$ as $k \to \infty$

(iii) $Df_k \, dx \rightharpoonup d[Df]$ weakly as vector valued Radon measures in U ; i.e. if for B Borel

$$\mu_k(B) = \int_{B \cap U} Df_k \, dx \quad \& \quad \mu(B) = \int_{B \cap U} d[Df]$$

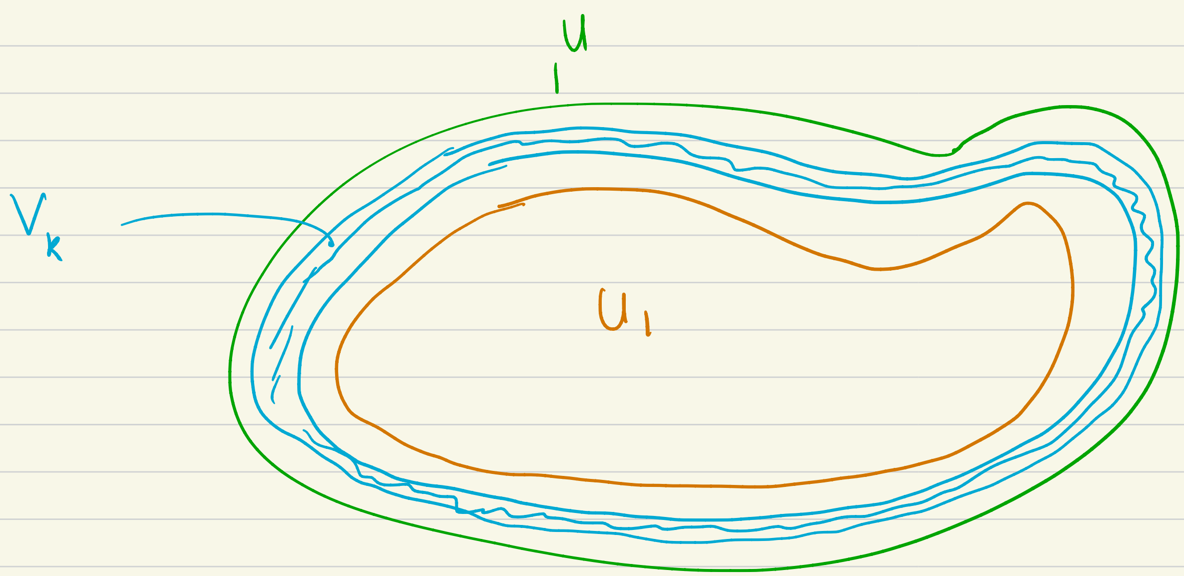
then $\mu_k \rightharpoonup \mu$

Remark : Warning it is not true $\|D(f_k - f)\| \to 0$!!!

Pf: fix $\varepsilon > 0$ $\quad m \in \mathbb{N}$ $\quad k \in \mathbb{N}$ $\quad \|Df\|(U) < \infty$

$$U_k = \{x \in U : \operatorname{dist}(x, \partial U) > \frac{1}{m+k}\} \cap B(0, m+k)$$

choose m large enough $\|Df\|(U \setminus U_1) < \varepsilon$ $\quad U_0 = \emptyset$



define

$V_k = U_{k+1} \setminus \overline{U_{k-1}}$ open

$\xi_k \in C_c^\infty(V_k)$ s.t

$0 \leq \xi_k \leq 1$

$$\sum_{k=1}^{\infty} \xi_k = 1 \text{ on } U_1$$

$\eta_\varepsilon(x) = \varepsilon^{-n} \eta\left(\frac{x}{\varepsilon}\right)$ $\quad \eta \geq 0$ $\quad \eta \in C_c^\infty(B_1)$ $\quad \int \eta \, dx = 1$

$\eta(x) = \eta(-x)$

for k select $\varepsilon_k > 0$ s.t $\quad \operatorname{spt}(\eta_{\varepsilon_k} * (f \xi_k)) \subset V_k$

$$\int_U |\eta_{\varepsilon_k} * (f\xi_k) - f\xi_k|\,dx < \varepsilon/2^k \qquad (*)$$

$$\int_U |\eta_{\varepsilon_k} * (f D\xi_k) - f D\xi_k| < \varepsilon/2^k$$

$$\underline{f_\varepsilon} = \sum_{k=1}^{\infty} \eta_{\varepsilon_k} * (f\xi_k) \in C^\infty(U) \qquad f = \sum_{k=1}^{\infty} f\xi_k$$

$$\|f_\varepsilon - f\|_{L^1(U)} \underset{(*)}{\leq} \sum_{k=1}^{\infty} \left| \int_U \eta_{\varepsilon_k} * (f\xi_k) - (f\xi_k)\right| \leq \varepsilon$$

thus $f_\varepsilon \longrightarrow f$ in $L^1(U)$ as $\varepsilon \to 0$

Theorem : (Compactness for BV functions)

Let $U \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary.

Assume $\{f_k\}_k \subset BV(U)$ s.t

$$\sup_k \|f_k\|_{BV(U)} < \infty$$

Then there exists $\{f_{k_j}\} \subset \{f_k\}$ & $f \in BV(U)$ s.t

$$f_{k_j} \longrightarrow f \quad \text{in } L^1(U) \quad \text{as } j \to \infty$$