Theorem: (Local approximation by smooth functions)

Assume $f \in BV(U)$ then there exist functions $\{f_k\}_{k=1}^{\infty} \in BV(U) \cap C^{\infty}(U)$

i) $f_k \to f$ in $L^1(U)$

$\to$ (ii) $\|Df_k\|(U) \longrightarrow \|Df\|(U)$ as $k \to \infty$

(iii) $Df_k \, dx \longrightarrow d[Df]$ weakly as vector valued Radon measures in U; i.e. if for B Borel

$$\mu_k(B) = \int_{B\cap U} Df_k \, dx \quad \& \quad \mu(B) = \int_{B \cap U} d[Df]$$

then $\mu_k \longrightarrow \mu$

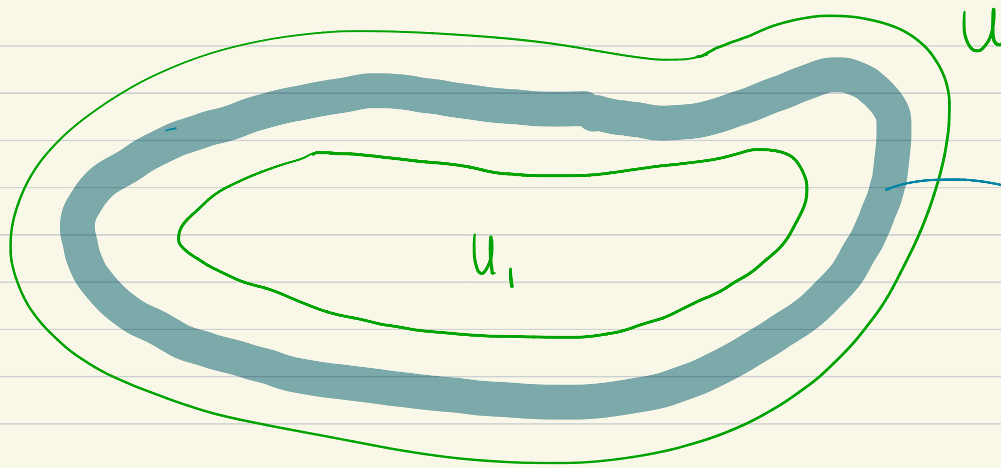
Pf: Fix $\varepsilon > 0$, $m, k \in \mathbb{N}$: $U_k = \{x \in U : \operatorname{dist}(x, \partial U) > \frac{1}{m+k}\} \cap \underset{\text{open}}{B_{m+k}}$

Since $\|Df\|(U) < \infty$ choose m large enough so

$$\|Df\|(U \setminus U_1) < \varepsilon \; ; \quad U_0 = \phi \quad U_{-1} = \phi$$

$$V_k = U_{k+1} \setminus \overline{U_{k-1}} \qquad V_0 = U_1 \qquad V_1 = U_2$$

$U = \bigcup_k V_k$



$V_k = U_{k+1} \setminus \overline{U_{k-1}}$

$\xi_k \in C_c^\infty(V_k)$

$0 \le \xi_k \le 1$

$\sum_{k=1}^\infty \xi_k = 1$ in U

η mollifier, for each k select $\varepsilon_k > 0$ s.t

$$\begin{cases} \operatorname{spt}\left(\eta_{\varepsilon_k} * (f\xi_k)\right) \subset V_k \\ \int_U \left| \eta_{\varepsilon_k} * (f\xi_k) - f\xi_k \right| dx < \varepsilon/2^k \\ \int_U \left| \eta_{\varepsilon_k} * (f D\xi_k) - f D\xi_k \right| dx < \varepsilon/2^k \end{cases}$$

$$f_\varepsilon = \sum_{k=1}^\infty \eta_{\varepsilon_k} * (f\xi_k) \in C^\infty(U) \qquad f = \sum_{k=1}^\infty f\xi_k$$

$\|f_\varepsilon - f\|_{L^1(U)} \le \varepsilon$ by l.s.c $\|Df\|(U) \le \liminf_{\varepsilon \to \infty} \|Df_\varepsilon\|(U)$

Let $\phi \in C_c^1(U, \mathbb{R}^n)$ $\quad |\phi| \le 1$ then

$$\int_U f_\varepsilon \operatorname{div}\phi \, dx = \sum_{k=1}^{\infty} \int_U \underbrace{\eta_{\varepsilon_k}}\, * (\xi_k f) \operatorname{div}\phi \, dx$$

$\xi_k |\eta_{\varepsilon_k} * \phi| \le 1$

$$= \sum_{k=1}^{\infty} \int_U f \xi_k \underbrace{\operatorname{div}(\phi * \eta_{\varepsilon_k})}_{\operatorname{div}(\xi_k(\eta_{\varepsilon_k} * \phi)) - D\xi_k \cdot \eta_{\varepsilon_k} * \phi} dx = \underbrace{\sum_{k=1}^{\infty} \int_U f \operatorname{div}(\xi_k (\eta_{\varepsilon_k} * \phi))}_{\mathrm{I}} - \underbrace{\int_U f D\xi_k \cdot \eta_k * \phi}_{\mathrm{II}}$$

$$|\mathrm{I}| = \left|\sum_{k=1}^{\infty} \int_U f \operatorname{div}(\xi_k (\eta_{\varepsilon_k} * \phi))\right| \le \left|\int_U f \operatorname{div}(\xi_1(\eta_{\varepsilon_1} * \phi))\right| + \left|\sum_{k=2}^{\infty} \int f \operatorname{div}(\xi (\eta_{\underbrace{\varepsilon_k}} * \phi))\right|$$

$$\le \|Df\|(U) + \sum_{k=2}^{\infty} \|Df\|(V_k) = \|Df\|(U) + 3\underbrace{\|Df\|(U \setminus U_1)}_{\varepsilon}$$

$$|\mathrm{II}| = \left|\sum_{k=1}^{\infty} \left(\int_U f D\xi_k \cdot \eta_{\varepsilon_k} * \phi - \int_U f D\xi_k \cdot \phi \right)\right| = \left|\sum_{k=1}^{\infty} \int_U \left((f D\xi_k) * \eta_{\varepsilon_k} - f D\xi_k \right) \cdot \phi \right|$$

$$f \sum_{k=1}^{\infty} D\xi_k = \sum_{k=1}^{\infty} f D\xi_k = 0$$

$$\le \sum_{k=1}^{\infty} \int_U |\phi| \underbrace{|f D\xi_{\varepsilon_k} * \eta_{\varepsilon_k} - (f D\xi_k)|}_{< \varepsilon / 2^k} \le \varepsilon$$

$$\left| \int f_\varepsilon \operatorname{div}\phi \, dx \right| \le |\mathrm{I}| + |\mathrm{II}| \le \|Df\|(U) + 3\varepsilon + \varepsilon$$

sup on ϕ

$$\|Df_\varepsilon\|(U) \le \|Df\|(U) + 4\varepsilon$$

$$\limsup_{\varepsilon \to 0} \|Df_\varepsilon\|(U) \le \|Df\|(U)$$

Pf of iii) $\xi \in C_c^1(\mathbb{R}^n)$ $\quad 0 \le \xi \le 1 \quad \xi = 1$ in U_1, $\quad \operatorname{spt}\xi \subset U$

$\forall \phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ we want $\underbrace{\int \phi \cdot d\mu_k} \longrightarrow \int \phi \cdot d\mu$

$\mu_k = Df_k \, dx \qquad \mu = [Df]$

$$\underbrace{\int_{\mathbb{R}^n} \phi \, d\mu_k} = \int_U \langle \phi, Df_k \rangle \, dx = \int_U \langle \phi\xi, Df_k \rangle + \int_U \langle \phi(1-\xi), Df_k \rangle$$

$$= -\int_U \operatorname{div}(\xi\phi) f_k + \int_U (1-\xi)\langle \phi, Df_k \rangle \qquad f_k \to f \text{ in } L^1(U)$$

$$\downarrow \qquad\qquad\qquad \uparrow$$

$$-\int_U \operatorname{div}(\xi\phi) f \qquad\qquad \le \|Df_k\|(U \setminus \bar{U}_1)$$

$$\|$$

$$\int_U \xi\phi \, d[Df] \qquad \left| \int \phi \, d\mu_k - \int \phi \, d[Df] \right| \le \|Df\|(U \setminus U_1) + \|Df_k\|(U \setminus \bar{U}_1)$$

$$\| \qquad\qquad |\ \ | \le \|Df\|(U \setminus U_1)$$

$$\int_U \xi\phi \, d[Df] = \underbrace{\int_U \phi \, d[Df]} + \int_U (\xi - 1)\phi \, d[Df]$$

$$\left| \underbrace{\int \phi \, d\mu_k - \int \phi \, d[Df]} \right| \le \|Df\|(U \setminus U_1) + \|Df_k\|(U \setminus \bar{U}_1)$$

$$\limsup_{k \to \infty} \left| \quad \downarrow \quad \right| \le \|Df\|(U \setminus U_1) + \limsup_{l \to \infty} \|Df_k\|(\underbrace{U \setminus \bar{U}_1}_{\text{open set}})$$

by (c) $\downarrow$

$$\le 2\|Df\|(U \setminus U_1) \le 2\varepsilon$$

Theorem : (Compactness for BV functions)

Let $U \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Assume $\{f_k\}_k \subset BV(U)$ s.t

$$\sup_k \|f_k\|_{BV(U)} < \infty$$

Then there exists $\{f_{k_j}\} \subset \{f_k\}$ & $f \in BV(U)$ s.t

$$f_{k_j} \longrightarrow f \quad \text{in } L^1(U) \quad \text{as } j \to \infty$$

Pf : Assume $\{f_k\} \subset BV(\mathbb{R}^n)$ $\quad f_k = 0$ on B_M^C

By previous theorem $g_k \in C^\infty(\mathbb{R}^n)$ $\quad g_k = 0$ on B_{2M}^C

and $\displaystyle\int |f_k - g_k| < \frac{1}{k}$ $\quad L = \sup_k \displaystyle\int |Dg_k| < \infty$

Claim : $\exists \{g_{k_i}\} \subset \{g_k\}$ $\quad g_{k_i} \longrightarrow f$ in L^1

$f \in BV(\mathbb{R}^n)$

$\Downarrow$

$f_{k_i} \longrightarrow f$ in L^1

$$g_k^\varepsilon = \eta_\varepsilon * g_k$$

$$|g_k^\varepsilon(x) - g_k(x)| \leq \int_{B_1} \eta(z)\, | g_k(x - \varepsilon z) - g_k(x) |\, dz$$

$$\leq \int_{B_1} \eta(z) \left| \int_0^1 \frac{d}{dt} g_k(x - t\varepsilon z)\, dt \right| dz$$

$$\leq \varepsilon \int_{B_1} \eta(z) \int_0^1 |Dg_k(x - \varepsilon t z)|\, dt\, dz$$

$$\leq \varepsilon \sup_k \int |Dg_k| = \varepsilon L$$

$$\| \underbrace{g_k^\varepsilon}_{} - g_k \|_{L^1(\mathbb{R}^n)} \leq C\varepsilon$$

Claim : For $\varepsilon > 0$ (fixed) $\{ g_k^\varepsilon \}$ is equicontinuous & uniformly bounded support $g_k^\varepsilon \subset B(0, 3M)$]

$$|Dg_k^\varepsilon(x)| \le \varepsilon^{-n-1} \int |g_k|\,dx \le C\varepsilon^{-n-1}$$

$$|g_k^\varepsilon(x)| \le C\varepsilon^{-n}$$

Claim: for each $\delta > 0$ $\{f_{k_j}\} \subset \{f_k\}$ s.t

$$\limsup_{i,j\to\infty} \|f_{k_j} - f_{k_i}\|_{L^1(\mathbb{R}^n)} \le \delta \qquad (*)$$

$\delta = \frac{1}{2^m}$ $\quad \{f^1_{k_j}\}$ $\quad \underbrace{\{f^2_k\}}_{\substack{(*) \\ \frac{1}{2^2}}} \subset \underbrace{\{f^1_k\}}_{\substack{(*) \\ 1/2}}$ $\quad \underbrace{\{f^i_k\}}_{\frac{1}{2^i}}$ satisfies $(*)$

use a diagonal argument to show that you can extract a Cauchy subseq of $\{f_k\}$ $\Rightarrow$ Cauchy subseq. converges in L^1

$\{g^\varepsilon_k\}$ equicon. + unif bounded by AA there is a subsequence

$\{g^\varepsilon_{k_i}\}$ which conv. in $\mathbb{R}^n$ (0 outside B_{3M})

$$\|f_{k_i} - f_{k_j}\|_{L^1} \le \|f_{k_i} - g_{k_i}\|_{L^1} + \|g_{k_i} - g^\varepsilon_{k_i}\|_{L^1} + \|g^\varepsilon_{k_i} - g^\varepsilon_{k_j}\|_{L^1}$$

$$+ \|g^\varepsilon_{k_j} - g_{k_j}\|_{L^1} + \|g_{k_j} - f_{k_j}\|_{L^1}$$

$$\le \frac{1}{k_i} + \frac{1}{k_j} + C\varepsilon + \|g^\varepsilon_{k_i} - g^\varepsilon_{k_j}\|_{L^1}$$

$\{g^\varepsilon_{k_i}\}$ conv in L^1 thus cauchy

Choose ε s.t $C\varepsilon < \delta/2$

$$\limsup_{j,i \to \infty} \|f_{k_i} - f_{k_j}\|_{L^1} \le \delta/2$$