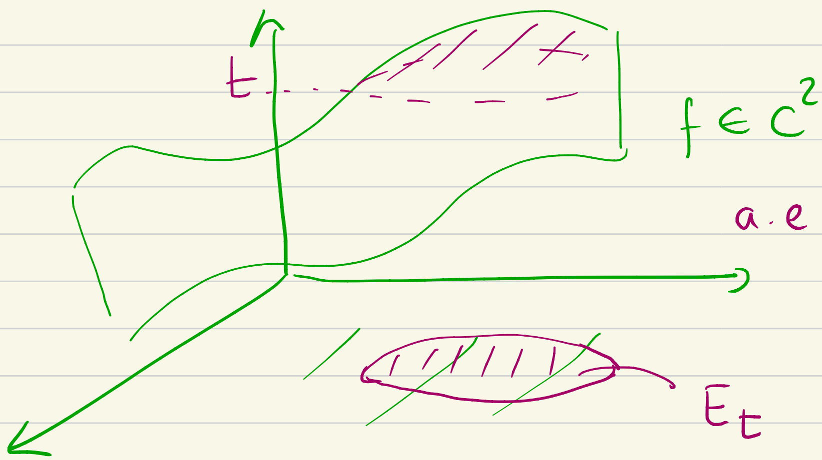
Co-area formula for BV functions

$f : U \to \mathbb{R}$ $\qquad E_t = \{x \in U : f(x) > t\}$ $\qquad t \in \mathbb{R}$

Lemma 1: If $f \in BV(U)$ the mapping $t \longmapsto \|\partial E_t\|(U)$ is $\mathcal{L}^1$ measurable.



a.e $\{f = t\}$ C^1 submanifold

Pf: $(x,t) \longrightarrow \chi_{E_t}(x)$ is $\mathcal{L}^n \times \mathcal{L}^1$ measurable

$\varphi \in C^1_c(U, \mathbb{R}^n)$ $\qquad t \longmapsto \int_{E_t} \operatorname{div}\phi = \int \chi_{E_t} \operatorname{div}\phi$ $\quad \mathcal{L}^1$ meas.

D count. dense set $C^1_c(U, \mathbb{R}^n)$

$$t \longmapsto \|\partial E_t\|(U) = \sup_{\phi \in D;\ |\phi| \le 1} \int_{E_t} \operatorname{div}\phi \qquad \mathcal{L}^1 \text{ meas.}$$

$f : \mathbb{R}^n \to \mathbb{R}$ Lipschutz co-area formula

$$\int_{-\infty}^{\infty} \mathcal{H}^{n-1}\{f = t\}\, dt = \int_{\mathbb{R}^n} \underbrace{|Df|\, dx}$$

f is compactly supported

f is Lipschutz $\|Df\|_{L^\infty} \le C$ $f \in L^1$, $Df \in L^1$

$\Rightarrow f \in BV(\mathbb{R}^n)$ $E_t = \{x : f(x) > t\}$

$$\mathcal{H}^{n-1}\{t = f\} = \|\partial E_t\|(\mathbb{R}^n)$$

$$\int_{-\infty}^{\infty} \|\partial E_t\|(\mathbb{R}^n)\, dt = \|Df\|(\mathbb{R}^n)$$

Theorem (Co-area formula for BV functions)

(i) If $f \in BV(U)$ then E_t has locally finite perimeter for $\mathcal{L}^1$ a.e $t \in \mathbb{R}$ and

$$\|Df\|(U) = \int_{-\infty}^{\infty} \|\partial E_t\|(U)\, dt$$

(ii) Conversely if $f \in L^1(U)$ and $\int_{-\infty}^{\infty} \|\partial E_t\|(U)\, dt < \infty$ then

$$f \in BV(U).$$

__Theorem__ (Gagliardo - Nirenberg - Sobolev inequality)

There exists a constant $C_1(n) = C_1 > 0$ s.t $\forall f \in C_c^1(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |f|^{n/n-1} \, dx \right)^{n-1/n} \leq C_1 \int_{\mathbb{R}^n} |Df|$$

__Theorem__ (Poincaré's inequality)

There exists a constant $C_2(n) = C_2 > 0$ s.t $\forall f \in C_c^1(\mathbb{R}^n)$

$$\left(⨍_{B(x,r)} |f - f_{x,r}|^{n/n-1} \, dy \right)^{n-1/n} \leq C_2 r ⨍_{B(x,r)} |Df| \, dy$$

$$f_{x,r} = ⨍_{B(x,r)} f \, dy = \frac{1}{L^n(B(x,r))} \int_{B(x,r)} f \, dy .$$

Pf $\quad i = 1, \dots n$

$$f(x_1, \dots x_i, \dots x_n) = \int_{-\infty}^{x_i} \frac{\partial f}{\partial x_i}(x_1 \dots, t_i, \dots x_n)\, dt_i$$

$$|f(x)|^{n/n-1} \leq \left(\prod_{i=1}^{n} \int_{-\infty}^{\infty} |Df|(x_1 \dots t_i, \dots x_n)\, dt_i \right)^{1/n-1}$$

$$\int_{-\infty}^{\infty} |f(x)|^{n/n-1} dx_1 \leq \left(\int_{-\infty}^{\infty} |Df|(t_1, x_2 \dots x_n)\, dt_1 \right)^{1/n-1} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Df|\, dt_i \right)^{1/n-1} dx_1$$

$$\leq \left(\int_{-\infty}^{\infty} |Df|\, dt_1 \right)^{1/n-1} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Df|\, dt_i\, dx_1 \right)^{1/n-1}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x)|^{n/n-1} dx_1\, dx_2 \leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |Df|\, dt_1 \right)^{1/n-1} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Df|\, dt_i\, dx_1 \right)^{1/n-1} dx_2$$

$$\vdots$$

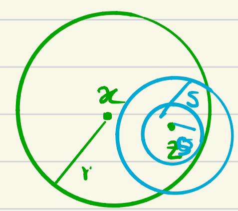
$$\int_{\mathbb{R}^n} |f(x)|^{n/n-1} dx \leq \left(\int_{\mathbb{R}^n} |Df|\, dx \right)^{n/n-1}$$

The proof requires the following lemma:

<u>Lemma</u>: There exists $C_3 = C_3(n) > 0$ s.t. $\forall f \in C^1_c(\mathbb{R}^n)$ & $z \in B(x,r)$

$$\int_{B(x,r)} |f(y) - f(z)|\, dy \leq C r^n \int_{B(x,r)} |Df|\, |y-z|^{1-n}\, dy$$

Pf: $|f(y) - f(z)| = \left| \int_0^1 \frac{d}{dt} f(z + t(y-z))\, dt \right|$



$$\leq \int_0^1 |\langle Df(z + t(y-z)); y - z \rangle|\, dt$$

$$\leq |y-z| \int_0^1 |Df(z + t(y-z))|\, dt \leftarrow$$

$s > 0$

$$\int_{B(x,r) \cap \partial B(z,s)} |f(y) - f(z)|\, d\mathcal{H}^{n-1}(y) \leq s \int_0^1 \int_{B(x,r) \cap \partial B(z,s)} |Df(z + \overbrace{t(y-z)}^{\omega})|\, d\mathcal{H}^{n-1}(y)\, dt$$

$$\int_{B(x,r)\cap\partial B(z,s)} |f(y)-f(z)|\,d\mathcal{H}^{n-1}(y) \le s\int_0^1 \int_{B(x,r)\cap\partial B(z,s)} |Df(\overbrace{z+t(y-z)}^{w})|\,d\mathcal{H}^{n-1}(y)\,dt$$

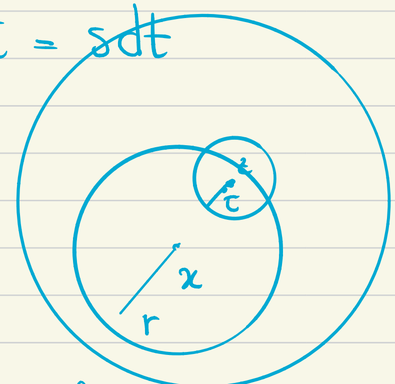
$$\le s\int_0^1 \frac{s^{n-1}}{s^{n-1}t^{n-1}}\int_{B(x,r)\cap\partial B(z,ts)} |Df(w)|\,d\mathcal{H}^{n-1}(w)\,dt$$

$$\le s^n \int_0^1 \frac{1}{(st)^{n-1}}\int_{B(x,r)\cap \partial B(z,ts)} |Df(w)|\,d\mathcal{H}^{n-1}(w)\,dt$$

$$\tau = ts$$

$$d\tau = s\,dt$$

$$\le s^n\int_0^1\int_{B(x,r)\cap\partial B(z,ts)} \frac{|Df(w)|}{|w-z|^{n-1}}\,d\mathcal{H}^{n-1}\,dt$$



$$\le s^{n-1}\int_0^s \int_{B(x,r)\cap\partial B(z,\tau)} |Df(w)|/|w-z|^{n-1}\,d\mathcal{H}^{n-1}(w)\,d\tau$$

$$\le s^{n-1}\int_0^s\int_{\partial B(z,\tau)} \chi_{B(x,r)}\,|Df(w)|/|w-z|^{n-1}\,d\mathcal{H}^{n-1}(w)\,d\tau \le s^{n-1}\int_{B(z,s)} \chi_{B(x,r)}\frac{|Df(w)|}{|z-w|^{n-1}}$$

$$\int_{B(x,r)\cap \partial B(z,s)} |f(y)-f(z)|\, d\gamma^{n-1}(y) \le s^{n-1} \int_{B(x,r)\cap B(z,s)} \frac{|Df(w)|}{|z-w|^{n-1}}\, dw \le s^{n-1} \int_{B(x,r)}$$

integrate s between 0 and $2r$

$$\int_{B(x,r)} |f(y)-f(z)|\, d\gamma^{n-1}(y) \le C r^n \int_{B(x,r)} \frac{|Df(w)|\, dw}{|z-w|^{n-1}}$$

Pf of Poincare

$$\fint_{B(x,r)} |f(y) - f_{x,r}|\, dy = \fint_{B(x,r)} \left| f(y) - \fint_{B(x,r)} f(z)\, dz \right| dy$$

$$= \fint_{B(x,r)} \left| \fint_{B(x,r)} \big(f(y) - f(z)\big)\, dz \right| dy$$

$$\leq \fint_{B(x,r)} \left(\fint_{B(x,r)} |f(y) - f(z)|\, dy \right) dz$$

$$\leq C \fint_{B(x,r)} \int_{B(x,r)} \frac{|Df(z)|}{|z-y|^{n-1}}\, dz\, dy$$

$$\leq C \fint_{B(x,r)} |Df(z)| \left(\int_{B(x,r)} \frac{dy}{|z-y|^{n-1}} \right) dz$$

$$\leq Cr \fint_{B(x,r)} |Df(z)|\, dz$$

$$\int_{B(x,r)} \frac{dy}{|z-y|^{n-1}} \leq \int_{B(z,2r)} \frac{dy}{|z-y|^{n-1}} \leq Cr$$

$$\fint_{B(x,r)} |f - f_{x,r}|\,dy \le Cr \fint_{B(x,r)} |Df(z)|\,dz$$

Claim: $\exists\, C(n) > 0 \quad \forall g \in C^1(B_r(x))$

$$\left(\fint_{B(x,r)} |g|^{n/n-1}\,dy \right)^{n-1/n} \le C(n) \left(r \fint_{B(x,r)} |Dg| + \fint_{B(x,r)} |g| \right)$$

assume claim $\quad g = f - f_{x,r}$

$$\left(\fint_{B(x,r)} |f - f_{x,r}|^{n/n-1}\,dy \right)^{n-1/n} \le C(n) \left(r \fint_{B(x,r)} |Df|\,dy + \fint_{B(x,r)} |f - f_{x,r}| \right)$$

$$\le Cr \fint_{B(x,r)} |Df|$$

To show the claim $\bar{g} : \mathbb{R}^n \to \mathbb{R}$ $\bar{g} \in C^1_c$

$$\left(\int_{\mathbb{R}} |\bar{g}|^{n/n-1} dy \right)^{n/n-1} \leq C \int_{\mathbb{R}^n} |D\bar{g}| dy$$

To show the claim we may assume $r = 1$ if not replace g by $h(y) = \frac{g(ry)}{r}$ and show claim for h on B_1

$$\left(⨍_{B_1} |h|^{n/n-1} dy \right)^{n-1/n} \leq C(n) \left(⨍_{B_1} |Dh| + ⨍_{B_1} |h| \right)$$

$\exists\, \bar{h} \in C^1_c(B_2)$ $\bar{h} = h$ on B_1 &

$$\int_{\mathbb{R}^n} |\bar{h}| + \int_{\mathbb{R}^n} |D\bar{h}| \leq C \left(\int_{B_1} |h| + \int_{B_1} |Dh| \right)$$