Theorem : (Poincaré & Sobolev inequalities for BV functions)

(i) There exists a constant $C_1 > 0$ s.t.

$$\|f\|_{L^{n/n-1}(\mathbb{R}^n)} \leq C_1 \|Df\|(\mathbb{R}^n)$$

for all $f \in BV(\mathbb{R}^n)$

(ii) There exists a constant $C_2 > 0$ s.t.

$$\|f - f_{x,r}\|_{L^{n/n-1}(B(x,r))} \leq C_2 \|Df\|(\mathring{B}(x,r))$$

for all $B(x,r) \subset \mathbb{R}^n$ and $f \in BV_{loc}(\mathbb{R}^n)$ where

$$f_{x,r} = \fint_{B(x,r)} f(y)\, dy$$

Pf 1) if $f \in C_c^1(\mathbb{R}^n)$ ✓ $\exists \{f_k\}_k \subset C_c^1(\mathbb{R}^n)$ $f_k \to f$ in $L^1(\mathbb{R}^n)$

$\|Df_k\|(\mathbb{R}^n) \longrightarrow \|Df\|(\mathbb{R}^n)$

$$\|f_k\|_{L^{n/n-1}(\mathbb{R}^n)} \le C_1 \|Df_k\|(\mathbb{R}^n)$$

$$\downarrow k \to \infty$$

$$C_1 \|Df\|(\mathbb{R}^n)$$

Claim $\|f_k\|_{L^{n/n-1}(\mathbb{R}^n)} \longrightarrow \|f\|_{L^{n/n-1}(\mathbb{R}^n)}$

$\exists \{f_{n_k}\} \subset \{f_k\}$ $f_{n_k} \longrightarrow f$ a.e

$|f_{n_k}|^{n/n-1} \longrightarrow |f|^{n/n-1}$

Fatou

$$\left(\int |f|^{n/n-1}\,dx\right)^{n-1/n} \le \left(\liminf \int |f_{n_k}|^{n/n-1}\right)^{n-1/n} \le C_1 \|Df\|(\mathbb{R}^n)$$

(3) For each $\alpha \in (0,1]$, there exists $C_3(\alpha) > 0$ s.t

$$\|f\|_{L^{n/n-1}(B(x,r))} \leq C_3(\alpha) \, \|Df\| (\mathring{B}(x,r))$$

for all $B(x,r) \subset \mathbb{R}^n$, $f \in BV_{loc}(\mathbb{R}^n)$ satisfying

$$\frac{L^n(B(x,r) \cap \{f=0\})}{L^n(B(x,r))} \geq \alpha$$

Pf 3) $\|f\|_{L^{n/n-1}(B(x,r))} \leq \|f - f_{x,r}\|_{L^{n/n-1}(B(x,r))} + \underbrace{\|f_{x,r}\|_{L^{n/n-1}(B(x,r))}}$

$$\leq C_2 \|Df\| (\mathring{B}(x,r)) +$$

$$\|f_{x,r}\|_{L^{n/n-1}(B(x,r))} = |f_{x,r}| \left(\int_{B(x,r)} 1 \, dy \right)^{\frac{n-1}{n}} \leq |f_{x,r}| (w_n r^n)^{\frac{n-1}{n}}$$

$$|f_{x,r}| = \frac{1}{w_n r^n} \left| \int_{B(x,r) \cap \{f \neq 0\}} f(y) \, dy \right| \leq \frac{1}{w_n r^n} \left(\int_{B(x,r)} |f|^{n/n-1} \right)^{n-1/n} \left(L^n(B(x,r) \cap \{f \neq 0\}) \right)^{1/n}$$

$$|f_{x,r}| \le \underbrace{\|f\|_{L^{n/n-1}(B(x,r))} \frac{\left(\mathcal{L}^n(B(x,r)\cap\{f\neq 0\})\right)^{1/n}}{\omega_n r^n}}$$

$1 - 1/n$

$$\|f\|_{L^{n/n-1}(B(x,r))} \le C_2 \|Df\|(\mathring{B}(x,r)) + (\omega_n r^n)^{n-1/n}$$

$$\le C_2 \|Df\|(\mathring{B}(x,r)) + \|f\|_{L^{n/n-1}(B(x,r))} \frac{\left(\mathcal{L}^n(B(x,r)\cap\{f\neq 0\})\right)^{1/n}}{(\omega_n r^n)^{1/n}}$$

$$\le C_2 \|Df\|(\mathring{B}(x,r_0)) + \|f\|_{L^{n/n-1}(B(x,r))}(1-\alpha)$$

$$\alpha \|f\|_{L^{n/n-1}(B(x,r))} \le C_2 \|Df\|(\mathring{B}(x,r))$$

Theorem (Isoperimetric inequalities)

Let $E \subset \mathbb{R}^n$ bounded set of finite perimeter. Then

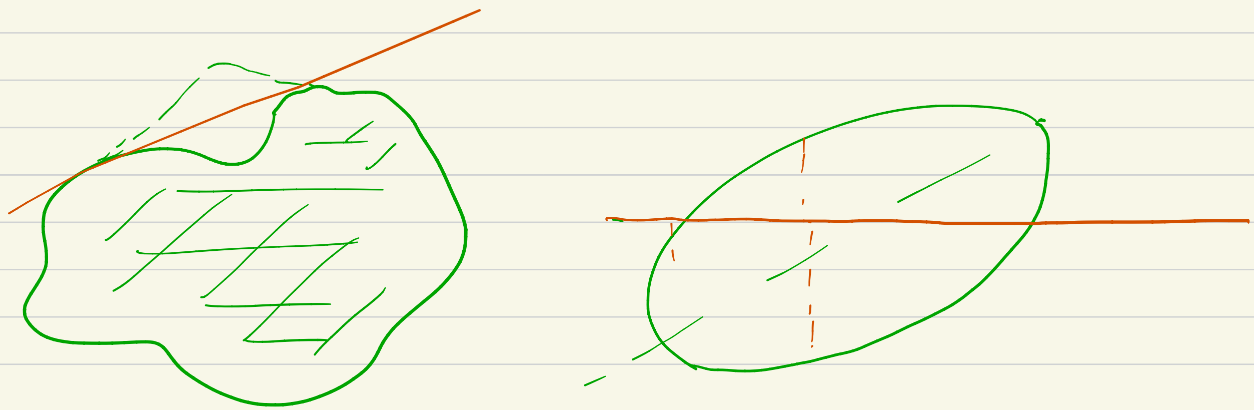
i) $\left(\mathcal{L}^n(E) \right)^{n-1/n} \leq C_1 \|\partial E\|(\mathbb{R}^n)$ (isoperimetric inequality)

Pf = χ_E + Sobolev

ii) For $B(x,r) \subset \mathbb{R}^n$ (relative isoperimetric inequality)

$$\min \left\{ \mathcal{L}^n(B(x,r) \cap E) ; \mathcal{L}^n(B(x,r) \setminus E) \right\}^{n-1/n} \leq 2C_2 \|\partial E\|(B(x,r))$$

i)

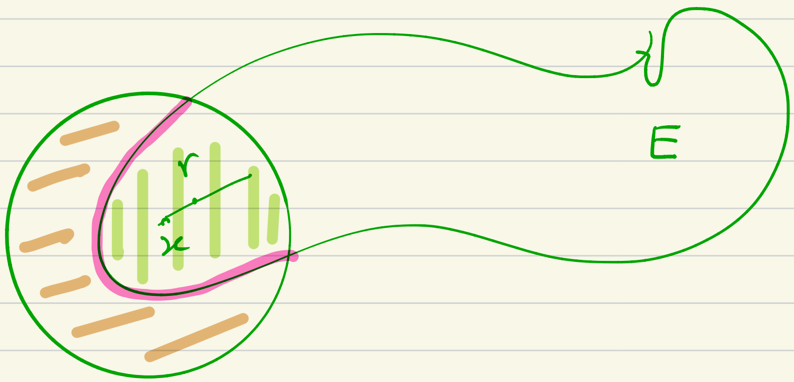


$$\left(C_1 = \frac{\pi^{1/2}}{2\pi} \right)$$

$$(\pi r^2)^{1/2} \leq C_1 2\pi r$$

Isoperimetric problem in $\mathbb{R}^2$: given A area of E find the set with smallest perimeter. $\Longleftrightarrow$ finding set of maximum area with given perimeter (minimizer convex) : minimizer / maximizer a disc

$$\min\left\{ \mathcal{L}^n(B(x,r)\cap E);\ \mathcal{L}^n(B(x,r)\setminus E)\right\}^{n-1/n} \leq 2C_2\, \|\partial E\|(B(x,r))$$



Pf : $f = \chi_{B(x,r)\cap E}$ $\qquad$ $f_{x,r} = \dfrac{\mathcal{L}^n(B(x,r)\cap E)}{\mathcal{L}^n(B(x,r))}$

$$\int_{B(x,r)} |f - f_{x,r}|^{n/n-1}\, dy = \mathcal{L}^n(E\cap B(x,r))\left(1 - \frac{\mathcal{L}^n(B(x,r)\cap E)}{\mathcal{L}^n(B(x,r))}\right)^{n/n-1}$$

$$+ \ \mathcal{L}^n(B(x,r)\setminus E)\left(\frac{\mathcal{L}^n(B(x,r)\cap E)}{\mathcal{L}^n(B(x,r))}\right)^{n/n-1}$$

$$\int_{B(x,r)} |f - f_{x,r}|^{n/n-1} = L^n(B(x,r) \cap E) \left(\frac{L^n(B(x,r) \setminus E)}{L^n(B(x,r))} \right)^{n/n-1}$$

$$+ \quad L^n(B(x,r) \setminus E) \left(\frac{L^n(B(x,r) \cap E)}{L^n(B(x,r))} \right)^{n/n-1}$$

$$\int_{B(x,r)} |f - f_{x,r}|^{n/n-1} \geq L^n(B(x,r) \cap E) \left(\frac{L^n(B(x,r) \setminus E)}{L^n(B(x,r))} \right)^{n/n-1}$$

$$\geq L^n(B(x,r) \setminus E) \left(\frac{L^n(B(x,r) \cap E)}{L^n(B(x,r))} \right)^{n/n-1}$$

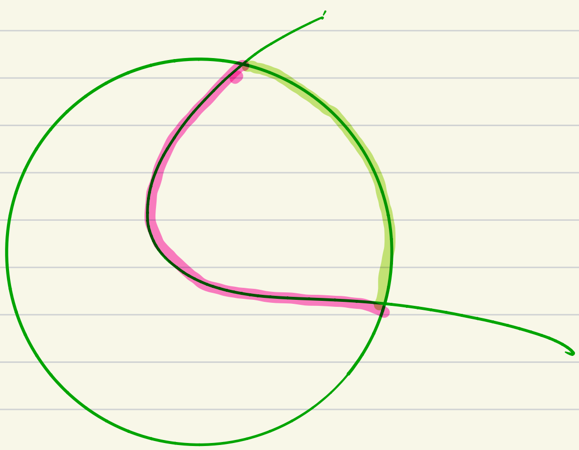
either $\quad \dfrac{L^n(B(x,r) \setminus E)}{L^n(B(x,r))} \geq 1/2 \quad$ or $\quad \dfrac{L^n(B(x,r) \cap E)}{L^n(B(x,r))} \geq 1/2$

$$\int_{B(x,r)} |f - f_{x,r}|^{n/n-1} \geq (1/2)^{n/n-1} L^n(B(x,r) \cap E) \quad \Big| \quad \geq (1/2)^{n/n-1} L^n(B(x,r) \setminus E)$$

$$\uparrow \quad \geq \left(\int_{B(x,r)} |f - f_{x,r}|^{n/n-1} \right)^{n-1/n} \geq \left[\left(\frac{1}{2}\right)^{n-1/n} \min \left\{ \mathcal{L}^n(B(x,r) \cap E) \, ; \, \mathcal{L}^n(B(x,r) \setminus E) \right\}^{-} \right]^{\frac{n-1}{n}}$$

$$C_2 \, \|Df\| \, (\mathring{B}(x,r)) = C_2 \|\partial E\| (\mathring{B}(x,r))$$

$$\underbrace{f = \chi_{E \cap B(x,r)}}$$



Recall $E \subset \mathbb{R}^n$ set l. f. p. $\exists \, \|\partial E\|$ Radon measure

$\nu_E : \mathbb{R}^n \to \mathbb{R}^n$ meas.

(i) $|\nu_E(x)| = 1$ $\|\partial E\|$ a.e $\forall \phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

(ii) $$\int_E \operatorname{div} \phi \, dx = \int \phi \cdot \nu_E \, d\|\partial E\|$$

A few remarks about sets of locally finite perimeter (l.f.p.)

(1) $E \subset \mathbb{R}^n$ set of locally finite perimeter iff $\forall \varphi \in C^1_c(\mathbb{R}^n)$

$$(*) \quad \int_E \nabla\varphi \, dx = \int_{\mathbb{R}^n} \varphi \, \nu_E \, d\|\partial E\|$$

$\Rightarrow \quad \phi = \varphi e_i \qquad \{e_1 \ldots e_n\}$ on of $\mathbb{R}^n \qquad \phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

$$\operatorname{div}\phi = \frac{\partial\varphi}{\partial x_i}$$

$$\int_{\mathbb{R}^n} \operatorname{div}\phi = \int \phi \cdot \nu_E \, d\|\partial E\|$$

$$\int_{\mathbb{R}} \frac{\partial\varphi}{\partial x_i} = \int \varphi e_i \cdot \nu_E \, d\|\partial E\|$$

if (*) satisfied

$\phi = (\phi^1 \ldots \phi^n)$

$$\int \frac{\partial\phi^j}{\partial x_j} = \int \phi^j e_j \cdot \nu_E \, d\|\partial E\|$$

$$\Downarrow$$

$$\int_E \operatorname{div}\phi = \int \phi \cdot \nu_E \, d\|\partial E\|$$

(2) What role does the topological boundary play?

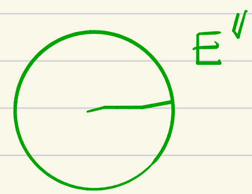
E set l.f.p $\qquad E' \subset \mathbb{R}^n \qquad E \Delta E' = E \setminus E' \cup E' \setminus E$

if $\mathcal{L}^n(E \Delta E') = |E \Delta E'| = 0 \qquad \phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_E \operatorname{div}\phi \, dx = \int_{E'} \operatorname{div}\phi \, dx \Rightarrow E' \text{ set of l.f.p.}$$

$$\nu_E \, d\|\partial E\| = \nu_{E'} \, d\|\partial E'\|$$

$E = B \qquad E' = B \cup \mathbb{Q}^n$



$|E \Delta E'| = 0$

$\partial E = S^{n-1} \qquad \partial E' = \mathbb{R}^n \setminus \operatorname{int} B$

$\uparrow$ top ∂

$\mathcal{L}^n(\partial E') = +\infty$

$\operatorname{spt}\|\partial E\| \subset \partial E$