Remarks & Notation

① If E is a set of l.f.p so is $\mathbb{R}^n \setminus E$: $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

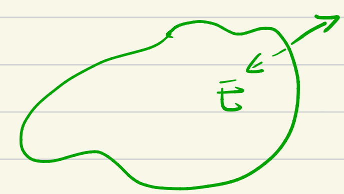
$$\int_{\mathbb{R}^n} \operatorname{div} \varphi \, dx = 0 \implies \int_E \operatorname{div} \varphi + \int_{\mathbb{R}^n \setminus E} \operatorname{div} \varphi = 0$$

$$\int_E \operatorname{div} \varphi = - \int_{\mathbb{R}^n \setminus E} \operatorname{div} \varphi \implies E \text{ l.f.p} \implies \mathbb{R}^n \setminus E \text{ l.f.p}$$

$$\nu_{\mathbb{R}^n \setminus E} \, \|\partial(\mathbb{R}^n \setminus E)\| = - \nu_E \, \|\partial E\|$$

② E set of l.f.p.

$\nu_E \, d\|\partial E\|$ = Gauss - Green measure



$\|\partial E\|$ = perimeter measure

③ if μ Radon measure on $\mathbb{R}^n$

closed set.

$$\operatorname{spt} \mu = \{ x \in \mathbb{R}^n : \mu(B(x,r)) > 0 \quad \forall r > 0 \}$$

Theorem (Lower semi - continuity of perimeter)

Suppose $\{E_k\}$ seq of sets of l.f.p & $E \subset \mathbb{R}^n$ s.t for each compact set $K \subset\subset \mathbb{R}^n$

$$\lim_{j \to \infty} |(E_j \Delta E) \cap K| = 0 \iff \int_K |\chi_{E_j} - \chi_E| \, dx \xrightarrow[j \to \infty]{} 0$$

$$\Downarrow$$

$$\limsup_{j \to \infty} \|\partial E_j\|(K) < \infty \qquad \chi_{E_j} \longrightarrow \chi_E \text{ in } L^1_{loc}$$

then E is a set of l.f.p. in $\mathbb{R}^n$

$$\nu_{E_j} \|\partial E_j\| \longrightarrow \nu_E \|\partial E\| \quad \&$$

for every open set $U \subset \mathbb{R}^n$

$$\|\partial E\|(U) \leq \liminf_{j \to \infty} \|\partial E_j\|(U)$$

$\chi_{E_j} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$ $\qquad \phi \in C^1_c(U, \mathbb{R}^n)$ $\quad U$ open $\quad \|\phi\|_\infty \le 1$

$$\int_E \operatorname{div} \phi = \int \chi_E \operatorname{div} \phi = \lim_{j\to\infty} \int \chi_{E_j} \operatorname{div} \phi$$

$$= \lim_{j\to\infty} \int_{E_j} \operatorname{div} \phi = \liminf_{j\to\infty} \int \nu_{E_j} \cdot \phi \, d\|\partial E_j\|$$

$\downarrow \sup \phi$

$$\|\partial E\|(U) \le \liminf_{j\to\infty} \|\partial E_j\|(U) \qquad E \text{ s.l.f.p.}$$

$$\int_E \operatorname{div} \phi = \int_E \nu_E \cdot \phi \, d\|\partial E\| = \lim_{j\to\infty} \int \nu_{E_j} \cdot \phi \, d\|\partial E_j\|$$

$$\Downarrow$$

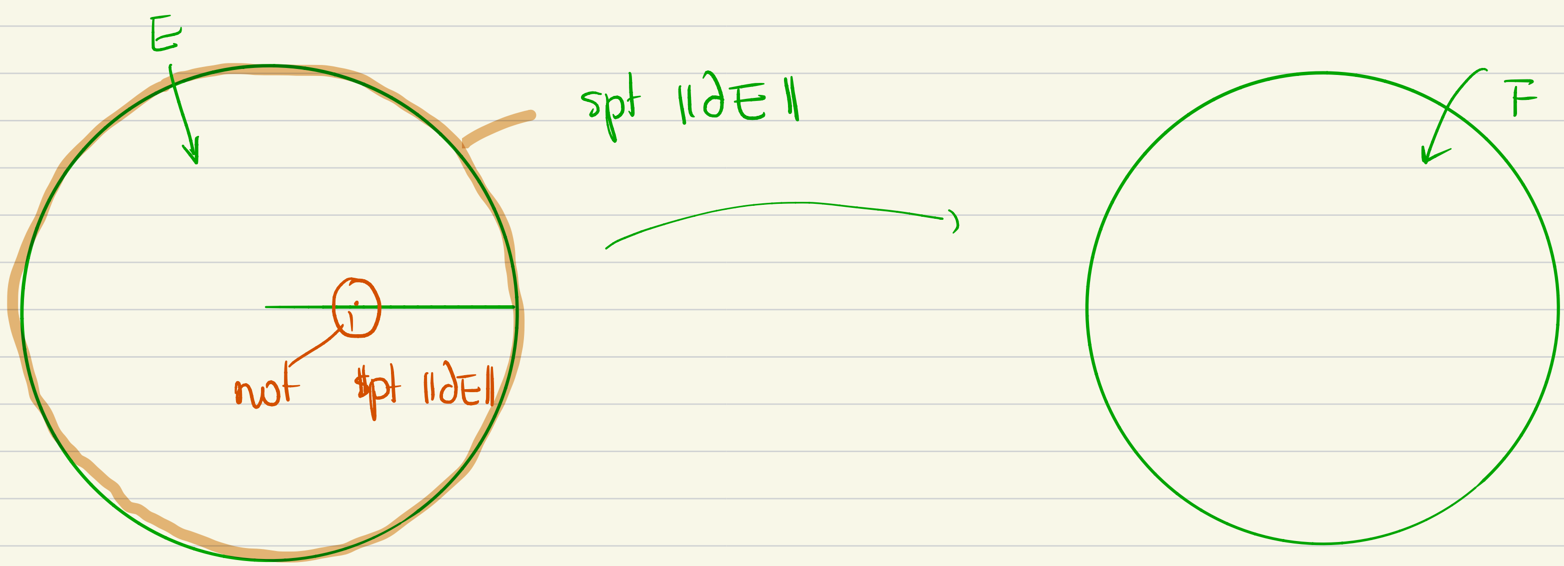
$$\nu_{E_j} \|\partial E_j\| \rightharpoonup \nu_E \|\partial E\|$$

Proposition: If $E \subset \mathbb{R}^n$ set of l.f.p. in $\mathbb{R}^n$ then

$$\operatorname{spt} \|\partial E\| = \{ x \in \mathbb{R}^n : 0 < \mathcal{H}^n(E \cap B_r(x)) < \omega_n r^n, \ \forall r > 0 \} \subset \partial E$$

Moreover there exists a Borel set F s.t

$$\mathcal{H}^n(E \Delta F) = 0 \quad \& \quad \operatorname{spt} \|\partial F\| = \partial F$$



$$\{ x \in \mathbb{R}^n : \quad 0 < |E \cap B_r(x)| < \omega_n r^n \} \subset \partial E$$

Pf : Step 1 : i) show $\operatorname{spt}\|\partial E\| \subset \{x : 0 < |E \cap B_r(x)| < \omega_n r^n \ \forall r>0\}$

$x \in \mathbb{R}^n \quad r>0 \quad \varphi \in C^1_c(B_r(x); \mathbb{R})$

$$\underbrace{\int_{E \cap B(x,r)} \nabla \varphi}_{\substack{\| \\ 0}} = \int_{B(x,r)} \varphi \, \nu_E \, d\|\partial E\| \qquad \text{assume } |E \cap B_r(x)| = 0$$

$\Downarrow$

$\|\partial E\|(B(x,r)) = 0 \Rightarrow x \notin \operatorname{spt}\|\partial E\|$

if $\exists r>0 \quad |E \cap B_r(x)| = 0 \Rightarrow x \notin \operatorname{spt}\|\partial E\|$

if $\exists r>0 \quad |E \cap B_r(x)| = \omega_n r^n \Rightarrow |(\mathbb{R}^n \setminus E) \cap B_r(x)| = 0$

$\Rightarrow x \notin \operatorname{spt}\|\partial(\mathbb{R}^n \setminus E)\| = \operatorname{spt}\|\partial E\|$

ii) $\{x : 0 < |B_r(x) \cap E| < \omega_n r^n \ \forall r>0\} \subset \operatorname{spt}\|\partial E\|$

if $x \notin \operatorname{spt}\|\partial E\| \quad \exists r>0 \quad \|\partial E\|(B_r(x)) = 0 \quad \forall \varphi \in C^1_c(B_r(x))$

$\Rightarrow \|\partial E\|(B_\rho(x)) = 0$

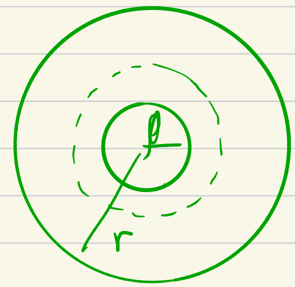
$$0 = \int \varphi \, \nu_E \, d\|\partial E\| = \int_E \nabla \varphi = \int_{\mathbb{R}^n} \chi_E \nabla \varphi \qquad \forall \rho < r$$

(this guides the proof

$$|E \cap B_r(x)| = \int_{B_r(x)} \chi_E \, dy \underset{\text{want}}{=} \begin{cases} 0 \\ \omega_n r^n \end{cases} \iff \chi_E \text{ is constant a.e on } B_r(x)$$)

η mollifier $\qquad \eta_\varepsilon(x) = \varepsilon^{-n} \eta(x/\varepsilon)$

$$(\chi_E)_\varepsilon = \chi_E * \underbrace{\eta_\varepsilon}_{\text{smooth}} \qquad 0 < \rho < r/2 \qquad \varepsilon < r - \rho$$



$y \in B_\rho(x)$

$$\nabla (\chi_E)_\varepsilon (y) = \int \chi_E(z) \nabla \eta_\varepsilon (z-y) \, dz = 0$$

$(\chi_E)_\varepsilon$ smooth 0 gradient in $B_\rho(x)$ thus $(\chi_E)_\varepsilon = \underbrace{c_\varepsilon}_{\text{constant}}$ on $B_\rho(x)$

$$(\chi_E)_\varepsilon \longrightarrow \chi_E \text{ in } L^1(B_\rho(x))$$

$$\begin{array}{l} \| \\ c_\varepsilon \longrightarrow c \quad \text{constant a.e } B_\rho(x) \subset B_r(x) \end{array}$$

Step 2 : Construction of F , assume E is Borel construct F Borel $|E \Delta F| = 0$ and

$$\partial F \underset{\uparrow}{=} \{x \in \mathbb{R}^n : \ 0 < |F \cap B_r(x)| < \omega_n r^n \ \ \forall r > 0\} = \operatorname{spt} \|\partial F\|$$

want this

$A_0 = \{x \in \mathbb{R}^n : \exists r > 0 \ \ |E \cap B_r(x)| = 0\}$ — open set

$A_1 = \{x : \exists r > 0 \ \ |E \cap B_r(x)| = \omega_n r^n\}$ — open

Consider $\{x_j\} \subset A_0$; $\exists r_j > 0$ $|E \cap B_{r_j}(x_j)| = 0$ & $A_0 \subset \bigcup_j B(x_j, r_j)$

$$\Rightarrow |E \cap A_0| \leq \sum_j |E \cap B(x_j, r_j)| = \boxed{0 = |E \cap A_0|}$$

Similarly $|(\mathbb{R}^n \setminus E) \cap A_1| = 0 \Rightarrow \boxed{|A_1 \setminus E| = 0}$

$F = A_1 \cup E \setminus A_0$ F Borel

$$|F \setminus E| \leq |(A_1 \cup E) \cap E^c| = |A_1 \cap E^c| = 0$$

$$|E \setminus F| = |E \cap F^c| = \left| E \cap \left[(A_1 \cup E)^c \cup A_0 \right] \right|$$

$$= \left| E \cap \left((A_1^c \cap E^c) \cup A_0 \right) \right| = |E \cap A_0| = 0$$

$$|E \Delta F| = 0$$

$$\mathbb{R}^n \setminus (A_1 \cup A_0) = \operatorname{spt} \|\partial E\| = \operatorname{spt} \|\partial F\| \subset \partial F$$

step 1

if $x \notin \mathbb{R}^n \setminus (A_1 \cup A_0) \Rightarrow x \in A_1$ or $x \in A_0$

if $x \in A_1$ $\exists r > 0$ $|E \cap B_r(x)| = \omega_n r^n$ $\forall y \in B_{r/2}(x)$

$$|E \cap B_{r/2}(y)| = \omega_n \left(\frac{r}{2}\right)^n \Rightarrow y \in A_1 \text{ and } B_{r/2}(x) \subset F$$

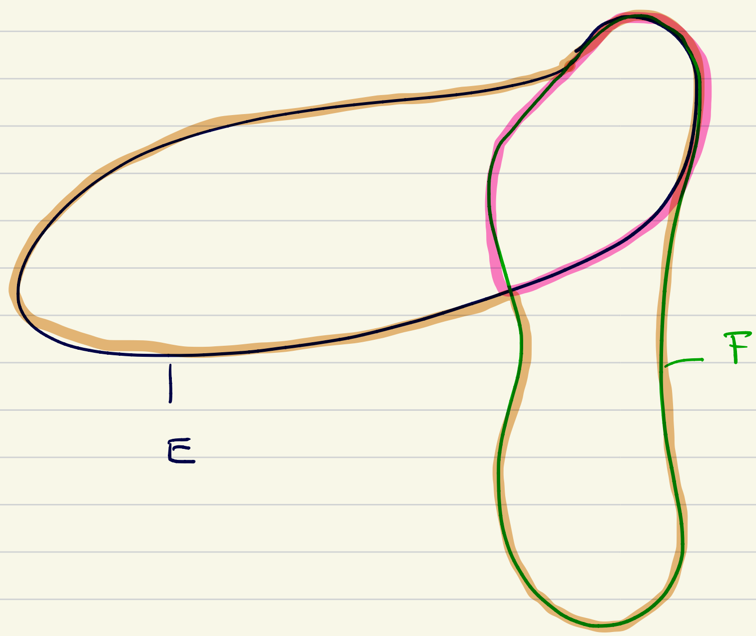
$A_1 \subset \operatorname{int} F$ $\quad F = A_1 \cup E \setminus A_0$

A_0 open $\quad \overline{F} \subset \mathbb{R}^n \setminus A_0$

$$\partial F = \overline{F} \setminus \operatorname{int} F \subset (\mathbb{R}^n \setminus A_0) \setminus A_1 = \mathbb{R}^n \setminus (A_0 \cup A_1)$$

Lemma: If $E, F \subset \mathbb{R}^n$ sets of l.f.p. then $E \cup F$ & $E \cap F$ are sets of l.f.p and for $U \subset \mathbb{R}^n$ open

$$\|\partial(E \cup F)\|(U) + \|\partial(E \cap F)\|(U) \leq \|\partial E\|(U) + \|\partial F\|(U)$$



Wild set of finite perimeter

Claim: Given $n \geq 2$ $\forall \varepsilon > 0$ $\exists E_\varepsilon \subset B_1$ set of finite perimeter s.t

$$\mathcal{H}^n(E) < \varepsilon \quad \& \quad \mathcal{H}^n(\operatorname{spt} \|\partial E\|) \geq \omega_n - \varepsilon$$

$$\Downarrow$$

$$\mathcal{H}^{n-1}(\operatorname{spt} \|\partial E\|) = +\infty$$

$$\|\partial E\|(B_1) < \infty$$