Regularization : $E \subset \mathbb{R}^n$ Lebesgue measurable set $\chi_E \in L^1_{loc}(\mathbb{R}^n)$

consider

$$\chi_E * \eta_\varepsilon \qquad \text{regularization of } \chi_E$$

$\uparrow$ mollifier

$$\eta_\varepsilon(z) = \varepsilon^{-n}\,\eta(z/\varepsilon)$$

$$u_\varepsilon(x)\ \chi_E * \eta_\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\,\chi_E(y)\,dy = \int_{E\cap B(x,\varepsilon)} \eta_\varepsilon(x-y)\,dy$$

$$0 \leq \chi_E * \eta_\varepsilon \leq 1$$

$$u_\varepsilon(x) = \begin{cases} 1 & |E\cap B_\varepsilon(x)| = \omega_n r^n \\ 0 & |E\cap B_\varepsilon(x)| = 0 \end{cases}$$

$$\nabla u_\varepsilon \sim -\frac{1}{\varepsilon}\,\nu_E$$

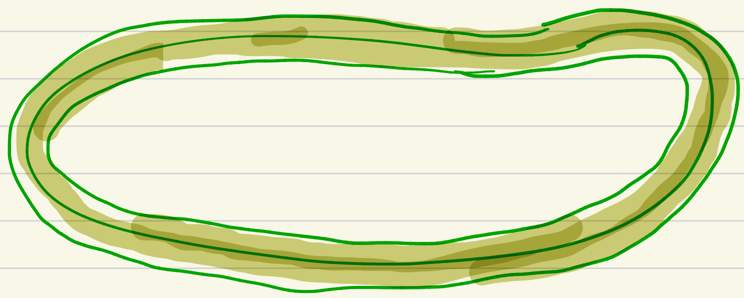
on ε ngbd ∂E

0 elsewhere

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon| \sim \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \chi_{\{y:\, d(y,\partial E) < \varepsilon\}}(y) \sim \frac{\mathcal{H}^n\{y:\, \mathrm{dist}(y,\partial E) < \varepsilon\}}{\varepsilon}$$

$$\Big\downarrow \varepsilon$$

$$\mathcal{H}^{n-1}(\partial E)$$



Proposition : If E set of l.f.p in $\mathbb{R}^n$ then

$$(\nu_E \|\partial E\|)_\varepsilon = \nu_E \|\partial E\| * \eta_\varepsilon = -\nabla(\underbrace{\chi_E * \eta_\varepsilon}_{u_\varepsilon})\, d\mathcal{H}^n \quad \forall \varepsilon > 0$$

$$-\nabla(\chi_E * \eta_\varepsilon)\,\mathcal{H}^n \longrightarrow \nu_E \|\partial E\| \quad \text{as } \varepsilon \to 0$$

$$|\nabla(\chi_E * \eta_\varepsilon)|\,\mathcal{H}^n \longrightarrow \|\partial E\| \quad \text{as } \varepsilon \to 0$$

Conversely if $E \subset \mathbb{R}^n$ Lebesgue measurable s.t

$$(*) \qquad \limsup_{\varepsilon \to 0} \int_K |\nabla(\chi_E * \eta_\varepsilon)|\, dx < \infty$$

for every compact set $K \subset\subset \mathbb{R}^n$ then E set of l.f.p.

By Morse - Sard for a.e $t > 0$ the level set

$\{\chi_E * \eta_\varepsilon > t\}$ is an open set with smooth boundary

Wild set of finite perimeter

Claim: Given $n \geq 2$ $\forall \varepsilon > 0$ $\exists E_\varepsilon \subset B_1$ set of finite perimeter s.t

$$|E| = \mathcal{H}^n(E) < \varepsilon \quad \& \quad \mathcal{H}^n(\operatorname{spt}\|\partial E\|) \geq \omega_n - \varepsilon$$

$\partial E = B \setminus E$

$\operatorname{spt}\|\partial E\| \subset \partial E$

$\{x_i\} = \mathbb{Q}^n \cap B_1$ $\quad \underbrace{|B_{r_i}(x_i) \cap B|}_{\omega_n r_i^n} < \varepsilon/2^i$

$E = \bigcup B_{r_i}(x_i)$ $\quad \boxed{B_{r_i}(x_i) \subset B}$ $\quad |E| < \varepsilon$

open

$|\operatorname{spt}\|\partial E\|| \geq$ ✓

$x \in \partial E$ $\quad r > 0$ $\quad B(x,r) \cap E \ni y$ $\quad B_\rho(y) \subset E \cap B_r(x)$

$\Rightarrow |E \cap B_r(x)| > 0$ $\qquad \partial E = \operatorname{spt}\|\partial E\| \cup A_1$

$A_1 = \{x \in \partial E : \exists r > 0 \quad |E \cap B_r(x)| = \omega_n r_n\}$

Use Besicovitch $\quad A_1 \subset \bigcup_{i=1}^{N} \bigcup_{B_i \in \mathcal{G}_i} B$ $\qquad |A_1| \leq N \sum^{\infty} \omega_n r_i^n$

$\leq N|E|$

$$E = \bigcup_{i=1}^{\infty} B_{r_i}(x_i) \quad \text{is a set of finite perimeter}$$

$$B_{r_i}(x_i) \subset B \qquad \sum \omega_n r_i^n \le \varepsilon$$

$$E_N = \bigcup_{i=1}^{N} B_{r_i}(x_i) \quad \text{set of finite perimeter}$$

$$\chi_{E_N} \longrightarrow \chi_E \quad \text{in } L^1 \qquad P(E_N) = \|\partial E_N\|(B_1)$$

$$P(E) \le \liminf_{N \to \infty} P(E_N) \qquad \sigma_{n-1} = |S^{n-1}| = \omega_n n$$

$$P(E_N) \le \sum_{i=1}^{N} P(B_{r_i}(x_i)) = \sum_{i=1}^{N} \sigma_{n-1} r_i^{n-1}$$

$$\le n\omega_n \sum_{i=1}^{N} r_i^{n-1} \le n\omega_n \sum_{i=1}^{\infty} r_i^{n-1} < \infty$$

$$\uparrow$$

$$?$$

$$\omega_n r_i^n < \varepsilon/2^i \qquad r_i < \left(\frac{\varepsilon/2^i}{\omega_n}\right)^{1/n}$$

$$\sum_{i=1}^{\infty} r_i^{n-1} \le \sum_{i=1}^{\infty} \left(\frac{\varepsilon}{\omega_n}\right)^{n-1/n} 2^{-i\, n/n-1} = C_{n,\varepsilon} \sum_{i=1}^{\infty} \left(2^{-n/n-1}\right)^i < \infty$$

Theorem: If $R>0$ & $\{E_k\}$ sets of finite perimeter in $\mathbb{R}^n$ s.t

(i) $\sup_{k\geq 1} \underbrace{\|\partial E_k\|(\mathbb{R}^n)}_{M} < \infty$ & (ii) $E_k \subset B_R$

then there exist a subsequence $\{E_{k'}\}$ and a set E of f.p. s.t.

$E_{k'} \to E$ in the sense that $\chi_{E_{k'}} \xrightarrow[\text{in } L^1]{} \chi_E$; $E \subset B_R$

and

$$\nu_{E_{k'}} \|\partial E_{k'}\| \to \nu_E \|\partial E\|.$$

Notation $P(E) = \|\partial E\|(\mathbb{R}^n)$ perimeter of E

Pf $f_k = \chi_{E_k}$ $\|f_k\|_{BV(\mathbb{R}^n)} = \|f_k\|_{L^1(\mathbb{R}^n)} + \|Df_k\|(\mathbb{R}^n)$

$\leq \omega_n R^n + M$

$f_{k'} \to f$ in L^1 & $f_{k''} \to f$ in L^1 & $f_{k''} \to \underset{\uparrow \chi_E}{f}$ a.e

Existence of minimizers in geometric variational problems

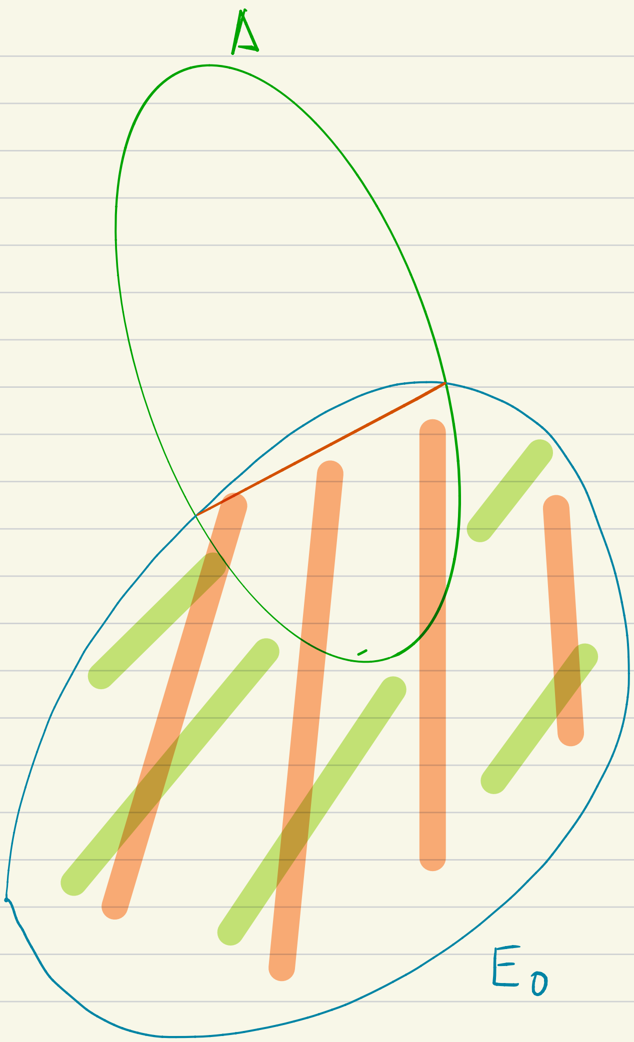
DIRECT METHOD OF CALCULUS OF VARIATIONS (in the class of sets of locally finite perimeter)

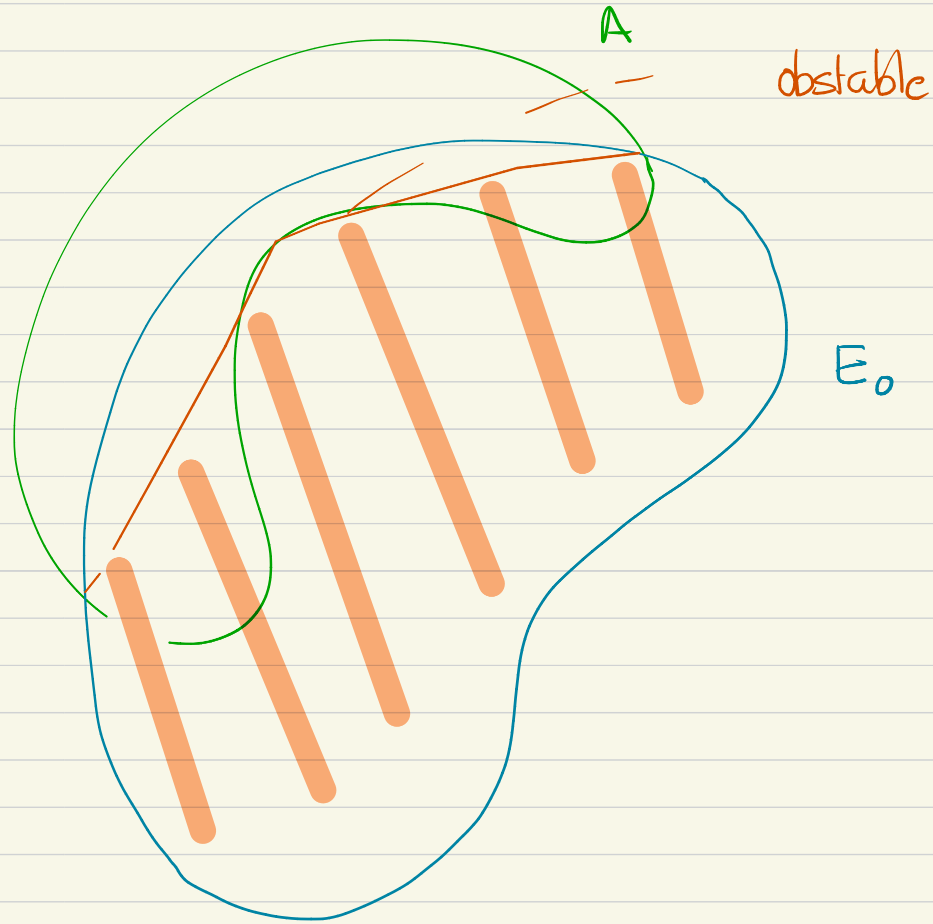
(1) Prove compactness of an arbitrary minimizing sequence.

(2) Show minimality of the limit via lower semi-continuity.

Plateau type problems:

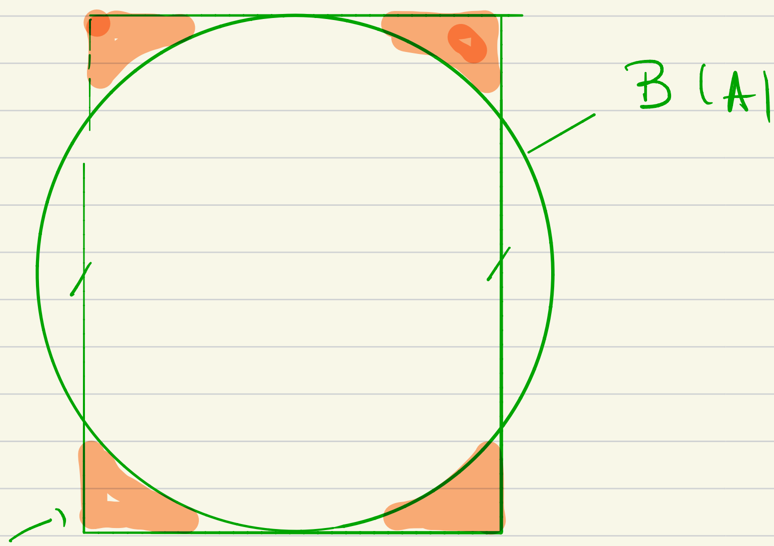
Classical Plateau problem: minimize area among surfaces with fixed boundary: given an open set $A \subset \mathbb{R}^n$ and $E_0 \subset \mathbb{R}^n$ set of finite perimeter, the Plateau problem in A with boundary data E_0 consists in minimizing $P(E)$ among all sets of f.p. that coincide with E_0 outside A

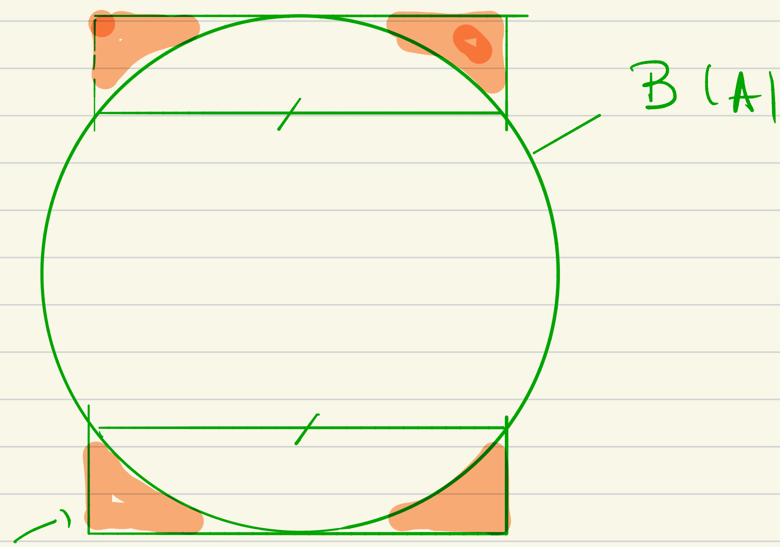




$$\gamma(A, E_0) = \inf \{ P(F) : E_0 \setminus A = F \setminus A \}$$

In general no uniqueness of minimizers





$\gamma(B_1; E_0)$

$E_0 = \{ x \in \mathbb{R}^2 : |x_2| \le 1 \quad |x_1| \le 1/\sqrt{2} \}$

$E_1 = E_0 \cap \{ x \in \mathbb{R}^2 : \quad 1 > |x_2| > \frac{1}{\sqrt{2}} \}$

<u>Proposition</u>: (Existence of minimizers for the Plateau type problem)

Let $A \subset \mathbb{R}^n$ open bounded set & E_0 a set of finite perimeter in $\mathbb{R}^n$.
There exists a set of finite perimeter E s.t $E \setminus A = E_0 \setminus A$
and

$$P(E) \leq P(F) \quad \text{for every } F \text{ s.t } F \setminus A = E_0 \setminus A$$

E is a minimizer for the variational problem

$$\gamma(A, E_0) = \inf \{ P(F) : F \setminus A = E_0 \setminus A \}$$