Proposition: (Existence of minimizers for the Plateau type problem)

Let $A \subset \mathbb{R}^n$ open bounded set & $E_0$ a set of finite perimeter in $\mathbb{R}^n$.
There exists a set of finite perimeter $E$ s.t $E \setminus A = E_0 \setminus A$
and

$$P(E) \leq P(F) \quad \text{for every } F \text{ s.t } F \setminus A = E_0 \setminus A$$

$E$ is a minimizer for the variational problem

$$\gamma(A, E_0) = \inf \{ P(F) : F \setminus A = E_0 \setminus A \}$$

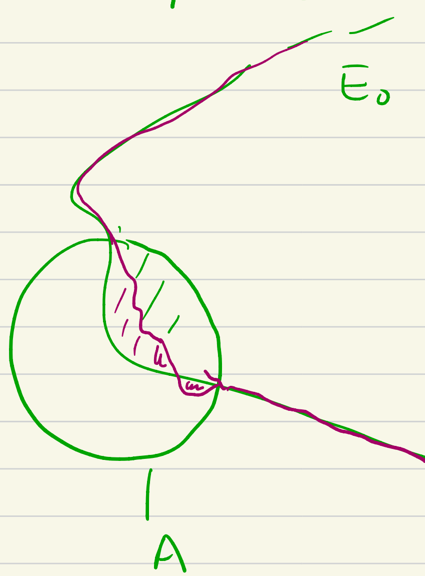
Pf: ① $E_0$ is admissible i.e $P(E_0) < \infty$ $E_0 \setminus A = E_0 \setminus A$

$$\gamma(A, E_0) \leq P(E_0) < \infty$$

Take $\{E_k\}_k$ $E_k \setminus A = E_0 \setminus A$

$$P(E_k) \longrightarrow \gamma(A, E_0) = \gamma < \infty$$

$$P(E_k) \leq P(E_0)$$



$$M_k = E_k \Delta E_0 = E_k \setminus E_0 \cup E_0 \setminus E_k = E_k \cap E_0^c \cup E_0 \cap E_k^c$$

↑ finite perimeter

$$M_k \subset A \implies |M_k| \le |A| < \infty$$

$$P(M_k) \le P(E_k \cap E_0^c) + P(E_0 \cap E_k^c)$$

$$\le P(E_k) + P(E_0^c) + P(E_0) + P(E_k^c) = 2P(E_0) + 2P(E_k)$$

$$\lesssim 4P(E_0)$$

$$\sup_k \left( |M_k| + P(M_k) \right) = \sup_k \| \chi_{M_k} \|_{BV} \le |A| + 4P(E_0) < \infty$$

$\exists \{M_{k'}\}$

$\chi_{M_{k'}} \longrightarrow \chi_M$ in $L^1(\mathbb{R}^n)$ & $M$ is set of finite perimeter

Claim

$$E_k = E_0 \cup M_k \setminus (E_0 \cap M_k) \qquad (M_k = E_k \Delta E_0 = E_k \cap E_0^c \cup E_0 \cap E_k^c)$$

$$E_0 \cup M_k \setminus (E_0 \cap M_k) = E_k \cup (E_0 \setminus E_k) \cap (E_0 \setminus E_k)^c$$

$$= E_k \cap (E_0^c \cup E_k) = E_k$$

$$E_k = E_0 \cup M_k \setminus (E_0 \cap M_k)$$

$$\chi_{E_k} = \chi_{E_0 \cup M_k} - \chi_{E_0 \cap M_k} \qquad \chi_{M_k} \longrightarrow \chi_M \text{ in } L^1$$

$$= \chi_{E_0} + \chi_{M_k} - \chi_{E_0}\chi_{M_k} - \chi_{E_0}\chi_{M_k}$$

$$k \to \infty \downarrow$$

$$= \chi_{E_0} + \chi_M - \chi_{E_0}\chi_M - \chi_{E_0}\chi_M$$

$$= \chi_{E_0 \cup M} - \chi_{E_0 \cap M} = \chi_{E_0 \cup M \setminus (E_0 \cap M)}$$

$$E = E_0 \cup M \setminus (E_0 \cap M) \qquad \chi_{E_k} \longrightarrow \chi_E$$

l.s.c $\quad P(E) \le \liminf_{k \to \infty} P(E_k) = \gamma \qquad M \subset A$

We need to show $E \setminus A = E_0 \setminus A \Longrightarrow \boxed{P(E) = \gamma} \qquad A^c \subset M^c$

$$E \setminus A = E_0 \cup M \setminus E_0 \cap M \setminus A = (E_0 \cup M) \cap (E_0 \cap M)^c \cap A^c$$

$$= E_0 \cap A^c \cap (E_0^c \cup M^c) \cup M \cap (E_0^c \cup M^c) \cap A^c$$

$$= \underbrace{E_0 \cancel{\cap A^c} \cap E_0^c}_{\phi} \cup \underbrace{E_0 \cap A^c}_{E_0 \cap A^c = E_0 \setminus A} \cap M^c \cup \underbrace{M \cap \cancel{E_0^c} \cap A^c}_{\phi} \cup \underbrace{M \cap \cancel{M^c} \cap A^c}_{\phi}$$

# Relative isoperimetric problems

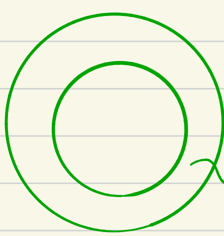
Given an open set $A \subset \mathbb{R}^n$ the relative isoperimetric problem in $A$ amounts to the volume constrained minimization of the relative perimeter in $A$

$$(*) \quad \alpha(A,m) = \inf \{ \|\partial E\|(A) = P(E;A) : E \subset A : |E| = m \}$$

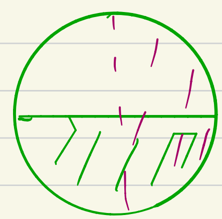
where $m \in (0, |A|)$.

Example: $A$ unit ball

$m = \frac{|A|}{2}$



not minimizer

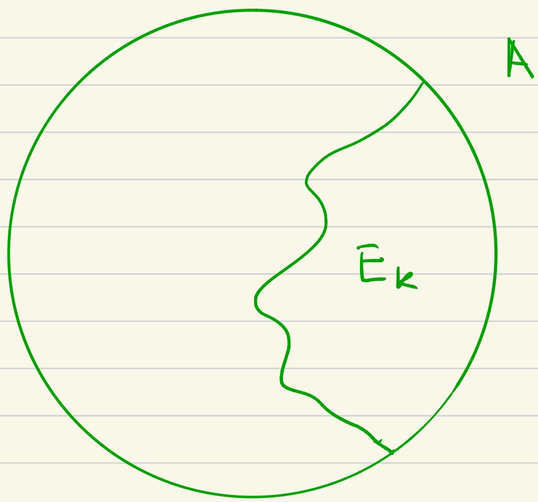


A minimizer in $(*)$ normalized to obtain $\operatorname{spt}\|\partial E\| = \partial E$ is relative isoperimetric set in $A$.

**Proposition** (Existence of isoperimetric sets)

If $A$ is an open bounded set of finite perimeter, $m \in (0, |A|)$ then there exists a set of finite perimeter $E \subset A$ such that $P(E; A) = \alpha(A, m)$ & $|E| = m$. In particular, $E$ is a minimizer in the variational problem $(*)$.

**Pf** there exists a competitor




$E_t = A \cap \{x : x_1 < t\}$ $t \in \mathbb{R}$

By continuity argument $\exists\, t \in \mathbb{R}$ s.t. $|E_t| = m$

$$\alpha \le P(E_t : A) \le P(A) + \operatorname{diam} A$$

$E_k \subset A$ finite perimeter $P(E_k; A) \longrightarrow \alpha$ $|E_k| = m$

$$P(E_k) \le P(E_k : A) + P(A) \quad k \text{ large enough} \quad \le \alpha + 1 + P(A)$$

$\sup_k (P(E_k) + |E_k|) < \infty \qquad E_k \subset A \qquad \{E_{k'}\} \subset \{E_k\}$

$\chi_{E_{k'}} \longrightarrow \chi_E$ in $L^1 \qquad |E_{k'}| = m \longrightarrow |E| = m$

$\Rightarrow E \subset A$

l.s.c

$$\alpha \leq P(E : A) \leq \liminf_{k \to \infty} P(E_k, A) = \alpha.$$