PROBLEMS WEEK 2

Problem 1: (Lebesgue points for Radon measures) Show that if μ is a Radon measure in $\mathbb{R}^n$ and $f \in L^p_{loc}(\mathbb{R}^n)$ for some $1 \leq p < \infty$ then for μ-a.e. $x \in \mathbb{R}^n$

$$\lim_{r\to 0} \fint_{B(x,r)} |f - f_{x,r}|^p \, d\mu = 0.$$

Problem 2: (Campanato's criterion) Let $B = B(0,1) \subset \mathbb{R}^n$ be the unit ball with center 0. Let $1 \leq p < \infty$ and $\gamma \in (0,1)$. Show that if $u \in L^p(B)$ (i.e. $\int_B |u(y)|^p \, dy < \infty$) and u satisfies

$$\left(\fint_{B(x,r)} |u(y) - u_{x,r}|^p \, dy \right)^{\frac{1}{p}} \leq \kappa r^\gamma,$$

for some $\kappa > 0$ where $B(x,r) \subset B$ and $u_{x,r} = \fint_{B(x,r)} f(u) \, dy$, then there exists $\overline{u} : B \to]\mathbb{R}$ such that $\overline{u} = u$ a.e. on B and for $x, y \in \frac{3}{4}B$,

$$|\overline{u}(x) - \overline{u}(y)| \leq C\kappa |x-y|^\gamma.$$

Hint: Show that $\{u_{x,2^{-j}r}\}$ for $x \in \frac{3}{4}B$ and $B(x,r) \subset B$ is a Cauchy sequence.

Problem 3: Let μ be a Radon measure in $\mathbb{R}^n$. Set for $x \in \mathbb{R}^n$,

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu,$$

if f is a μ-measurable function, and

$$M_\mu \nu(x) = \sup_{r>0} \frac{\nu(B(x,r))}{\mu(B(x,r))},$$

if ν is a Radon measure in $\mathbb{R}^n$.

1. Show that there exists a constant $C < \infty$ depending only on n, with the following property: if μ and ν are Radon measures in $\mathbb{R}^n$, then

$$\mu\left(\{x \in \mathbb{R}^n : M_\mu \nu(x) > t\}\right) \leq C t^{-1} \nu(\mathbb{R}^n).$$

2. Show that for $1 < p < \infty$ there exists a constant $C_p < \infty$, depending only on n and p with the following property: if μ is a Radon measure in $\mathbb{R}^n$, then

$$\int (M_\mu f)^p \, d\mu \leq C_p \int |f|^p \, d\mu,$$

for all μ-measurable functions f.