PROBLEMS WEEK 4

We define $\mathbb{R}^m$**-valued Radon measures on** $\mathbb{R}^n$ as bounded linear functionals on $C_c(\mathbb{R}^n,\mathbb{R}^m)$ (in the sense of the Riesz Representation Theorem). Notation: for $f \in C_c(\mathbb{R}^n,\mathbb{R}^m)$

$$\langle \vec{\mu}, f\rangle = \int_{\mathbb{R}^n} f \cdot d\vec{\mu} = \int_{\mathbb{R}^n} f \cdot \sigma \, d|\vec{\mu}| = \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu.$$

Here for an open set $U \subset \mathbb{R}^n$

$$\mu(U) = \sup\{\langle \vec{\mu}, f\rangle : \ f \in C_c(U,\mathbb{R}^m), \ |f| \le 1\}$$

and for $A \subset \mathbb{R}^n$

$$\mu(A) = \inf\{\mu(U) : \ A \subset U, \ U \text{ open}\}.$$

$\sigma : \mathbb{R}^n \to \mathbb{R}^m$ is μ measurable and $|\sigma| = 1$ μ-a.e.

Let $\{\vec{\mu_k}\}_k$ and $\vec{\mu}$ be $\mathbb{R}^m$-valued Radon measures. We say that $\vec{\mu_k}$ converges weakly to $\vec{\mu}$, $\vec{\mu_k} \rightharpoonup \vec{\mu}$ if for all $f \in C_c(\mathbb{R}^n,\mathbb{R}^m)$

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f \cdot d\vec{\mu_k} = \int_{\mathbb{R}^n} f \cdot d\vec{\mu}.$$

Problem 1: Show that if $\{\vec{\mu_k}\}_k$ and $\vec{\mu}$ are $\mathbb{R}^m$-valued Radon measures on $\mathbb{R}^n$ with $\vec{\mu_k} \rightharpoonup \vec{\mu}$ then for every open set $U \subset \mathbb{R}^n$,

$$\mu(U) \le \liminf_{k\to\infty} \mu_k(U).$$

Problem 2: Assume $\{\vec{\mu_k}\}_k$ are $\mathbb{R}^m$-valued Radon measures on $\mathbb{R}^n$.

1. Show that if $\vec{\mu_k} \rightharpoonup \vec{\mu}$ and $\mu_k \rightharpoonup \nu$ then for every Borel set $E \subset \mathbb{R}^n$

   $$\mu(U) \le \nu(E).$$

   Furthermore if E is a bounded Borel set with $\nu(\partial E) = 0$ then

   $$\vec{\mu}(E) = \lim_{k\to\infty} \vec{\mu_k}(E).$$

2. Show that if $\vec{\mu_k} \rightharpoonup \vec{\mu}$, $\mu_k(\mathbb{R}^n) \to \mu(\mathbb{R}^n)$, and $\mu(\mathbb{R}^n) < \infty$, then $\mu_k \rightharpoonup \mu$.

Problem 3: Let B be the unit ball centered at 0, $\rho \in C_c^\infty(B)$, $\rho \geq 0$, $\rho(-x) = \rho(x)$ for every $x \in \mathbb{R}^n$, and $\displaystyle\int_B \rho(x)\,dx = 1$. For $\varepsilon \in (0,1)$, and $x \in \mathbb{R}^n$, let

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$$

If $\vec{\mu}$ is an $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$, we define $\vec{\mu} * \rho_\varepsilon : \mathbb{R}^n \rho \mathbb{R}^m$ as

$$\vec{\mu} * \rho_\varepsilon(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\,d\vec{\mu}(y)$$

Check that $\vec{\mu} * \rho_\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and

$$\nabla(\vec{\mu} * \rho_\varepsilon)(x) = \vec{\mu} * \nabla\rho_\varepsilon(x) = \int_{\mathbb{R}^n} \nabla\rho_\varepsilon(x-y)\,d\vec{\mu}(y)$$

The ε-**regularization** $\vec{\mu_\varepsilon}$ of $\vec{\mu}$ is the $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$

$$\langle \vec{\mu_\varepsilon}, f\rangle = \int_{\mathbb{R}^n} f(x)(\vec{\mu} * \rho_\varepsilon)(x)\,dx, \qquad f \in C_c(\mathbb{R}^n, \mathbb{R}^m).$$

1. Show that if $\vec{\mu}$ is a $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$, then as $\varepsilon \to 0$,

$$\vec{\mu_\varepsilon} \rightharpoonup \vec{\mu}, \qquad \mu_\varepsilon \rightharpoonup \mu.$$

2. Moreover. if $I_\varepsilon(E) = \{x \in \mathbb{R}^n : \operatorname{dist}(x,E) < \varepsilon\}$, then for every Borel set $E \subset \mathbb{R}^n$

$$\mu_\varepsilon(E) \leq \mu(I_\varepsilon(E)).$$