Problem 1:
Definition: Let $S \subset \mathbb{R}^n$, $m \leq n-1$, and $\epsilon \in (0, \frac{1}{4})$. Assume that $0 \in S$. We say that S has the weak ϵ- approximation property in $B_1(0)$ if $\forall \rho \in (0,1]$ and for each $Q \in S \cap B_1(0)$ there exists an m plane $L(\rho, Q)$ containing Q and such that

$$S \cap B_\rho(Q) \subset (\epsilon\rho) - \text{neighborhood of } L(\rho, Q) \cap B_\rho(Q).$$

Prove that there is a function $\beta : (0,\infty) \to (0,\infty)$ with $\lim_{t\to 0} \beta(t) = 0$ such that if S satisfies the weak ϵ- approximation property in $B_1(0)$ then

$$\mathcal{H}^{m+\beta(\epsilon)}(S \cap B_1(0)) = 0.$$

Here $\mathcal{H}^s$ denotes the s dimensional Hausdorff measure.

Problem 2: Let μ be a Borel measure on $\mathbb{R}^n$, and let $E \subset \mathbb{R}^n$ be a μ-measurable set with $0 < \mu(E) < \infty$. Show that for $s > 0$

- if
$$\limsup_{r\to 0} \frac{\mu(B(x,r) \cap E)}{r^s} < c < \infty \quad \forall x \in E,$$
then $\mathcal{H}^s(E) > 0$,
- if
$$\limsup_{r\to 0} \frac{\mu(B(x,r) \cap E)}{r^s} > c > 0 \quad \forall x \in E,$$
then $\mathcal{H}^s(E) < \infty$.

Problem 3: Let $E \subset \mathbb{R}^n$ satisfy $0 < \mathcal{H}^s(E) < \infty$, for $0 < s < 1$. Show that the density

$$\theta^s(E,x) = \lim_{r\to 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{\omega_s r^s}$$

fails to exit at almost every point of E (i.e. $\theta^s(E,x)$ exists at most in a subset of E of $\mathcal{H}^s$ measure 0).

Remark: Marstrand proved this result in 1954. Later on he showed that if $s > 0$, and $\theta^s(E,x)$ exists on a subset $F \subset E$ with $\mathcal{H}^s(F) > 0$, then s must be an integer.

Problem 4: Let μ_j, μ be Radon measures on a metric space X. Assume that for each $x \in X$, and each $j = 1, 2, ...$

$$\theta(\mu_j, x, r) = \frac{\mu_j(B_r(x))}{\omega_n r^n}, \text{ and } \theta(\mu, x, r) = \frac{\mu(B_r(x))}{\omega_n r^n},$$

are non-decreasing functions of r. Assume also that μ_j converges weakly to μ, and that $x_j \to x$ as $j \to \infty$. Prove that

$$\limsup_{j\to\infty} \theta(\mu_j, x_j) \leq \theta(\mu, x).$$

Here $\theta(\mu_j, x) = \lim_{r\to 0} \theta(\mu_j, x, r)$, and $\theta(\mu, x) = \lim_{r\to 0} \theta(\mu, x, r)$.

Remark: Note that in particular if $\mu_j = \mu$ for each j and $\theta(\mu, x, r)$ is a non-decreasing function of r, then the result above proves the upper semi-continuity of the density.

Problem 5: Let $M \subset \mathbb{R}^m$, $0 < n < m$, and $\mu = \mathcal{H}^n \llcorner M$. Assume that μ is a Radon measure, and that for each $x \in M$ $\theta(\mu, x, r) = \frac{\mu(B_r(x))}{\omega_n r^n}$ is a non-decreasing function of r. Let $\lambda_j > 0$ be a sequence converging to 0 as $j \to \infty$. For $x \in M$, let

$$M_j = \frac{1}{\lambda_j}(M - x) = \{y = \frac{1}{\lambda_j}(z - x) : z \in M\},$$

and

$$\mu_j = \mathcal{H}^n \llcorner (M_j \cap B_1(0)).$$

Show that for each j, μ_j is a Radon measure. Prove that there exists a subsequence μ_{j_k} of μ_j that converges weakly to a Radon measure ν, and that

$$\theta(\mu, x) = \theta(\nu, 0). \qquad (*)$$

Note that in particular $(*)$ asserts that $\lim_{r\to 0} \theta(\nu, 0, r)$ exits.

Remark: The situation described in Problem 5 occurs when M is a minimal n-dimensional submanifold of $\mathbb{R}^m$. In that case $\nu = \mathcal{H}^n \llcorner C$, where C is a cone of vertex 0. C is a tangent cone of M at x. As defined this cone depends on the subsequence λ_{j_k}. One of the big open questions in the subject is whether there is a unique tangent cone. Moreover the set $\{x \in M : \theta(\mu, x) = 1\}$ is open and smooth. The set $\{x \in M : \theta(\mu, x) > 1\}$ is a closed set of Hausdorff dimension at most $n - 1$.