The isoperimetric problem

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Mathematics Sin Fronteras

The isoperimetric inequality

Theorem: Given a planar figure of area A and perimeter P

$$4\pi A \leq P^2$$

Equality occurs if and only if the figure is a disc.

Theorem (Wirtinger inequality): Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π (i.e. $f(\theta + 2\pi) = f(\theta)$).
Let $\overline{f}$ denote the mean value of f

$$\overline{f} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta.$$

Then

$$\int_0^{2\pi} [f(\theta) - \overline{f}]^2\, d\theta \leq \int_0^{2\pi} [f'(\theta)]^2\, d\theta.$$

Equality holds if and only if

$$f(\theta) = \overline{f} + a\cos\theta + b\sin\theta$$

for some constants a, b.

Fourier series

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π, the numbers a_n, b_n in (1) and c_n in (2) are called the Fourier coefficients of f. The corresponding series

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} \qquad \text{or} \qquad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

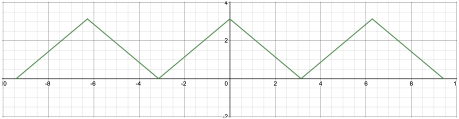
is called the Fourier series of f. Here

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\zeta)\cos n\zeta \, d\zeta \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\zeta)\sin n\zeta \, d\zeta \tag{1}$$

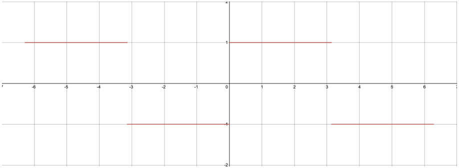
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\zeta) e^{in\zeta} \, d\zeta \tag{2}$$

Examples

$$f(\theta) = \begin{cases} \pi - \theta & 0 \leq \theta \leq \pi \\ \pi + \theta & -\pi \leq \theta < 0 \end{cases}$$



$$f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & -\pi < \theta < 0 \end{cases}$$



Does the Fourier series of a periodic function f converge to f?

For $N \in \mathbb{N}$ let

$$S_N^f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos n\theta + b_n \sin n\theta) = \sum_{-N}^{N} c_n e^{in\theta} \tag{3}$$

Theorem: If $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π, and S_N^f is defined as in (3) with a_n, b_n and c_n defined as in (1) and (2), then

$$\lim_{N\to\infty} S_N^f(\theta) = \frac{1}{2}[f(\theta-) + f(\theta+)]$$

for all θ. In particular,

$$\lim_{N\to\infty} S_N^f(\theta) = f(\theta)$$

for every θ at which f is continuous.

Wirtinger inequality

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π,

$$\overline{f} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta.$$

Then

$$\int_0^{2\pi} [f(\theta) - \overline{f}]^2\, d\theta \le \int_0^{2\pi} [f'(\theta)]^2\, d\theta.$$

Equality holds if and only if

$$f(\theta) = \overline{f} + a\cos\theta + b\sin\theta$$

for some constants a, b.

Proof: Let

$$f(\theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

where $a_0 = 2\overline{f}$ and

$$\begin{aligned} \int_0^{2\pi} [f(\theta) - \overline{f}]^2\, d\theta &= \int_0^{2\pi} \left[\sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\right]^2 d\theta \\ &= \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \end{aligned}$$

$$\begin{aligned} f(\theta) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) \\ f'(\theta) &= \sum_{n=1}^{\infty}(-na_n \sin n\theta + nb_n \cos n\theta) \\ \int_0^{2\pi} [f'(\theta)]^2 \, d\theta &= \pi \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2) \qquad \text{(Parseval's equation)} \end{aligned}$$

$$\int_0^{2\pi} [f'(\theta)]^2 \, d\theta - \int_0^{2\pi} [f(\theta) - \overline{f}]^2 \, d\theta = \pi \sum_{n=1}^{\infty}(n^2 - 1)(a_n^2 + b_n^2) \geq 0.$$

Equality occurs if

$$(n^2 - 1)(a_n^2 + b_n^2) = 0 \text{ either } n = 1 \text{ or } a_n = b_n = 0 \text{ for } n \geq 2$$

In this case

$$f(\theta) = \overline{f} + a_1 \cos\theta + b_1 \sin\theta. \quad \square$$

Second approach to the isoperimetric problem

The Minkowski Addition of 2 sets A, $B \subset \mathbb{R}^n$ is defined by

$$A \boxplus B := \{a + b : a \in A \text{ and } b \in B\}$$

Warm up:

1. Find $[0,3] \times [0,2] \boxplus [0,2] \times [0,1]$
2. Find $A \boxplus B$ where A is a triangle and B a rectangle.
3. For a set $S \subset \mathbb{R}^2$ and $\rho \in \mathbb{R}$, $\rho > 0$ let $\rho S = \{\rho x : x \in S\}$. Let $\rho \in (0, \frac{1}{2})$, and $B = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ and $Q = [0,1] \times [0,1]$. Find $B \boxplus \rho B$ and $Q \boxplus \rho B$.
4. Find the area and the perimeter of $B \boxplus \rho B$ and $Q \boxplus \rho B$.

Steiner's Inequality

Note that if $\Omega \subset \mathbb{R}^n$ and $\rho \geq 0$

$$\Omega_\rho = \Omega \boxplus \rho B = \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \Omega) \leq \rho\}$$

Theorem: Let $\Omega \subset \mathbb{R}^2$ be a closed and bounded set with piecewise C^1 boundary whose area is A and whose boundary has length L. Let $\rho \geq 0$. Then

$$\begin{aligned} \mathrm{Area}(\Omega_\rho) &\leq A + L\rho + \pi\rho^2 \\ L(\partial\Omega_\rho) &\leq L + 2\pi\rho. \end{aligned}$$

If Ω is convex then the inequalities are equalities.

Questions:

- Verify the equalities for a convex polygon.
- Sketch the proof for a convex bounded set.

Brunn's inequality

Let A and B be bounded measurable sets in the plane

$$\sqrt{\text{Area}(A \boxplus B)} \geq \sqrt{\text{Area}(A)} + \sqrt{\text{Area}(B)}.$$

Minkowski proved that equality holds if and only if $A = rB + x$ for some $r > 0$ and $x \in \mathbb{R}^2$ (i.e. A and B are homothetic).

Hadwiger's proof using Steiner's Inequality

Given a compact set $\Omega \subset \mathbb{R}^2$ we define:

- inradius

$$r_I = \sup\{r \geq 0 : \text{ there is } x \in \mathbb{R}^2 \text{ such that } x \boxplus rB \subset \Omega\}$$

- incenter is any x_I so that the incircle $x_I \boxplus r_I B \subset \Omega$

Isoperimetric Inequality of Hadwiger Suppose $\Omega \subset \mathbb{R}^2$ convex with piecewise C^1 boundary, area $\mathcal{A}$ and boundary length $\mathcal{L}$. Let M be a line through the incenter of Ω and a be the length of the chord passing through the incenter. Then

$$\mathcal{L}^2 - 4\pi\mathcal{A} \geq \frac{\pi^2}{4}(a - 2r_I)^2$$