Functions of bounded variation & Sets of finite perimeter

Definitions: $U \subset \mathbb{R}^n$ open.

i) A function $f \in L^1(U)$ has bounded variation in U if

$$\sup\left\{ \int_U f \operatorname{div}\phi \, dx : \phi \in C^1_c(U, \mathbb{R}^n) : |\phi| \le 1 \right\} < \infty$$

we write $f \in BV(U)$

ii) $E \subset \mathbb{R}^n$ measurable has finite perimeter in U if $\chi_E \in BV(U)$

iii) $f \in L^1_{loc}(U)$ has locally bounded variation if for each open set $V \subset\subset U$ (V compactly contained in U)

$$\sup\left\{ \int_V f \operatorname{div}\phi \, dx : \phi \in C^1_c(V, \mathbb{R}^n) : |\phi| \le 1 \right\} < \infty$$

$$f \in BV_{loc}(U)$$

iv) $E \subset \mathbb{R}^n$ measurable has locally finite perimeter in U if $\chi_E \in BV_{loc}(U)$.

<u>Theorem</u> (Structure theorem for BV_{loc} functions)

Assume $f \in BV_{loc}(U)$, there exist a Radon measure μ on U and $\sigma : U \to \mathbb{R}^n$ μ-measurable function s.t.

(i) $|\sigma(x)| = 1$ μ a.e

(ii) $\forall \phi \in C_c^1(U, \mathbb{R}^n)$

$$\int_U f \operatorname{div} \phi \, dx = - \int_U \phi \cdot \sigma \, d\mu$$

<u>Remark</u>: The structure theorem asserts that the weak first partial derivatives of BV functions are Radon measures.

Notation : ① If $f \in BV_{loc}(U)$, $\mu = \|Df\|$ & $[Df] = \|Df\| \llcorner \sigma$ i.e.
$\phi \in C^1_c(U, \mathbb{R}^n)$

variation measure

$$\int_U f \,\mathrm{div}\,\phi \,dx = -\int_U \phi \cdot \sigma \, d\|Df\| = -\int_U \phi \cdot d[Df]$$

② $E \subset \mathbb{R}^n$ set of locally finite perimeter if $\chi_E \in BV_{loc}(U)$
$\|D\chi_E\| = \|\partial E\|$ & $\sigma = -\nu_E$ thus $\forall \phi \in C^1_c(U, \mathbb{R}^n)$

$$\int_E \mathrm{div}\,\phi \, dx = \int_E \phi \cdot \nu_E \, d\|\partial E\|$$

$\|\partial E\|(U)$ perimeter of E in U

$\mu_E = \nu_E \, d\|\partial E\|$ Gauss - Green measure

$|\mu_E| = \|\partial E\|$

(4) Recall from Riesz Representation Theorem; if $f, \chi_E \in BV_{loc}(U)$

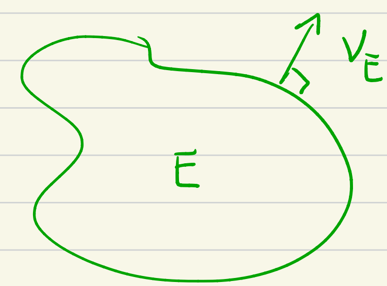
$V \subset\subset U$, V open

$$\|Df\|(V) = \sup\left\{ \int_V f \operatorname{div}\phi : \phi \in C_c^1(V, \mathbb{R}^n) : |\phi| \le 1 \right\}$$

$$|\mu_E|(V) = \sup\left\{ \int_E \operatorname{div}\phi : \phi \in C_c^1(V, \mathbb{R}^n) : |\phi| \le 1 \right\}$$

Example: 1. C^1 domain with $\mathcal{H}^{n-1}(\partial E) < \infty$ set of finite perimeter

$$\int_E \operatorname{div}\phi = \int_{\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{n-1}$$



$$\mu_E = \nu_E \, d\mathcal{H}^{n-1} \llcorner \partial E$$

Theorem (Isoperimetric inequalities)

Let $E \subset \mathbb{R}^n$ bounded set of finite perimeter. Then

i) $\left(\mathcal{L}^n(E) \right)^{n-1/n} \leq C_1 |\mu_E| (\mathbb{R}^n)$ (isoperimetric inequality)

ii) For $B(x,r) \subset \mathbb{R}^n$ (relative isoperimetric inequality)

$$\min \{ \mathcal{L}^n(B(x,r) \cap E) ; \mathcal{L}^n(B(x,r) \setminus E) \}^{n-1/n} \leq 2 C_2 |\mu_E| (B(x,r))$$

Remark:

$E \subset \mathbb{R}^n$ set of locally finite perimeter iff $\forall \varphi \in C^1_c(\mathbb{R}^n)$

$$\int_E \nabla \varphi \, dx = \int_{\mathbb{R}^n} \varphi \, d\mu_E$$

Theorem (Lower semi - continuity of perimeter)

Suppose $\{E_k\}$ seq of sets of l.f.p & $E \subset \mathbb{R}^n$ s.t for each compact set $K \subset\subset \mathbb{R}^n$

$$\lim_{j \to \infty} \underline{|(E_j \Delta E) \cap K|} = 0 \iff \underline{\int_K |\chi_{E_j} - \chi_E| \, dx} \underset{j \to \infty}{\longrightarrow} 0$$

$$\limsup_{j \to \infty} |\mu_{E_j}|(K) < \infty \qquad \Downarrow$$

$$\chi_{E_j} \longrightarrow \chi_E \text{ in } L^1_{loc}$$

then E is a set of l. f. p. in $\mathbb{R}^n$

$$\mu_{E_j} \longrightarrow \mu_E \qquad \&$$

for every open set $U \subset \mathbb{R}^n$

$$|\mu_E|(U) \leq \liminf_{j \to \infty} |\mu_{E_j}|(U)$$

<u>Proposition</u> : If $E \subset \mathbb{R}^n$ set of l.f.p. in $\mathbb{R}^n$ then

$$\operatorname{sup} |\mu_E| = \{ x \in \mathbb{R}^n : 0 < |E \cap B(x,r)| < \omega_n r^n , \forall r > 0 \} \subset \partial E$$

Moreover there exists a Borel set F s.t

$$|E \Delta F| = 0 \quad \& \quad \operatorname{spt} |\mu_F| = \partial F$$

<u>Theorem</u> : If $R>0$ & $\{E_k\}$ sets of finite perimeter in $\mathbb{R}^n$ s.t

(i) $\sup_{k\geq 1} |\mu_{E_k}|(\mathbb{R}^n) < \infty$ & (ii) $E_k \subset B_R$

then there exist a subsequence $\{E_{k'}\}$ and a set E of f.p. s.t.

$E_{k'} \to E$ in the sense that $\chi_{E_{k'}} \to \chi_E$ in L^1 ; $E \subset B_R$

and

$$\mu_{E_{k'}} \rightharpoonup \mu_E$$

<u>Notation</u> $P(E) = |\mu_E|(\mathbb{R}^n)$ perimeter of E

Existence of minimizers in geometric variational problems

DIRECT METHOD OF CALCULUS OF VARIATIONS (in the class of sets of locally finite perimeter)

1. Prove compactness of an arbitrary minimizing sequence.
2. Show minimality of the limit via lower semi-continuity.

Plateau type problems:

Classical Plateau problem: minimize area among surfaces with fixed boundary: given an open set $A \subset \mathbb{R}^n$ and $E_0 \subset \mathbb{R}^n$ set of finite perimeter, the Plateau problem in A with boundary data E_0 consists in minimizing $P(E)$ among all sets of f.p. that coincide with E_0 outside A

Proposition: (Existence of minimizers for the Plateau type problem)

Let $A \subset \mathbb{R}^n$ open bounded set & E_0 a set of finite perimeter in $\mathbb{R}^n$.
There exists a set of finite perimeter E s.t $E \setminus A = E_0 \setminus A$
and

$$P(E) \leq P(F) \quad \text{for every } F \text{ s.t } F \setminus A = E_0 \setminus A$$

E is a minimizer for the variational problem

$$\gamma(A, E_0) = \inf \{ P(F) : F \setminus A = E_0 \setminus A \}$$

Relative isoperimetric problems

Given an open set $A \subset \mathbb{R}^n$ the relative isoperimetric problem in A amounts to the volume constrained minimization of the relative perimeter in A

(*) $\alpha(A,m) = \inf \{ |\mu_E|(A) = P(E;A) : E \subset A : |E| = m \}$

where $m \in (0, |A|)$.

A minimizer in (*) normalized to obtain $\mathrm{spt}\,|\mu_E| = \partial E$ is relative isoperimetric set in A.

Proposition (Existence of isoperimetric sets)

If A is an open bounded set of finite perimeter, $m \in (0, |A|)$ then there exists a set of finite perimeter $E \subset A$ such that $P(E; A) = \alpha(A, m)$ & $|E| = m$. In particular, E is a minimizer in the variational problem (*).

end of review

Proposition : (Local perimeter bound on volume)

If $n \geq 2$, $t \in (0,1)$, $x \in \mathbb{R}^n$ and $r > 0$ then there exists a constant $c(n,t) > 0$ such that

$$P(E ; B(x,r)) \geq c(n,t) \, |E \cap B(x,r)|^{n-1/n}$$

for every set of locally finite perimeter E such that

$$|E \cap B(x,r)| \leq t \, |B(x,r)|$$

Pf : Recall the relative isoperimetric inequality

$$P(E ; B(x,r)) \geq c(n) \min \{ |E \cap B(x,r)| ; |B(x,r) \setminus E| \}^{\frac{n-1}{n}}$$

Since $|E \cap B(x,r)| \leq t \, |B(x,r)|$

$$|B(x,r) \setminus E| \geq (1-t) \, |B(x,r)| \geq \frac{1-t}{t} |E \cap B(x,r)|$$

$$P(E ; B(x,r)) \geq c(n) \min \{ 1, {}^{1-t}/_{t} \}^{n-1/n} \, |E \cap B(x,r)|^{n-1/n}$$