The Reduced Boundary & De Giorgi's structure theorem

Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter in $\mathbb{R}^n$

Definition : Let $x \in \mathbb{R}^n$, $x \in \partial^* E$ the reduced boundary of E if

i) $|\mu_E|(B(x,r)) > 0 \quad \forall r > 0$ (ie $x \in \operatorname{spt} |\mu_E|$)

ii) $\lim_{r \to 0} \fint_{B(x,r)} \nu_E \, d|\mu_E| = \nu_E(x)$ (x Lebesgue point for ν_E)

&

iii) $|\nu_E(x)| = 1$

By Lebesgue-Besicovitch differentiation theorem

$$|\mu_E|(\mathbb{R}^n \setminus \partial^* E) = 0$$

Remarks ① If $E \subset \mathbb{R}^n$ open set with C^1 boundary by the divergence theorem

$$\int_E \operatorname{div} \varphi \, dx = \int_{\partial E} \varphi \cdot \nu_E \, d\mathcal{H}^{n-1} \quad \forall \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$$

then

$$\nu_E \in C, \quad \partial^* E = \partial E, \quad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial E$$

We study the extent to which this is true.

② If E set of locally finite perimeter & F s.t $|E \Delta F| = 0$ then F set of l.f.p. Since $\mu_E = \mu_F$

$$\partial^* E = \partial^* F \quad \text{(check)}$$

closed set

③ By definition $\partial^* E \subset \operatorname{spt} |\mu_E| = \partial E$

good representative

$$\Rightarrow \overline{\partial^* E} \subset \operatorname{spt} |\mu_E|$$

If $y \in \operatorname{spt} |\mu_E| \setminus \overline{\partial^* E}$ $\exists \rho > 0$ $B(y,\rho) \cap \overline{\partial^* E} = \emptyset$

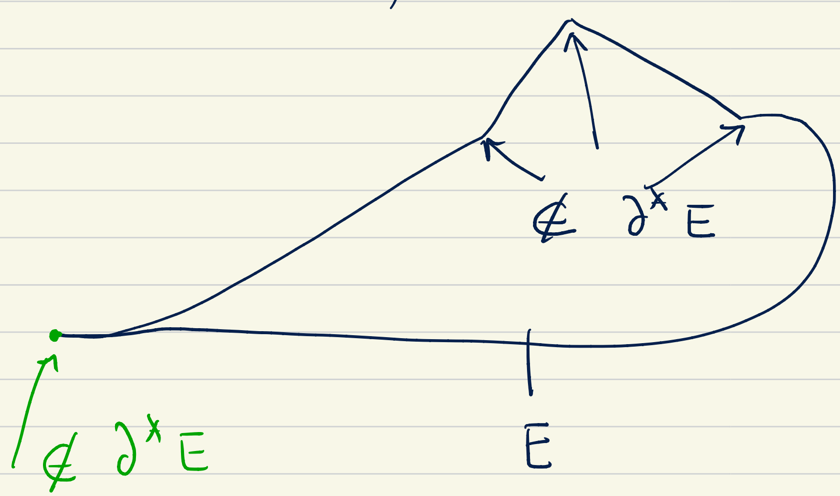
& $|\mu_E| (B(y,s)) > 0$ $\forall s \leq \rho$

since $|\mu_E| (\mathbb{R}^n \setminus \partial^* E) = 0$ then

$$0 < |\mu_E| (B(y,s)) = |\mu_E| (\partial^* E \cap B(y,s))$$

$$\Rightarrow \partial^* E \cap B(y,s) \neq \emptyset \qquad \times$$

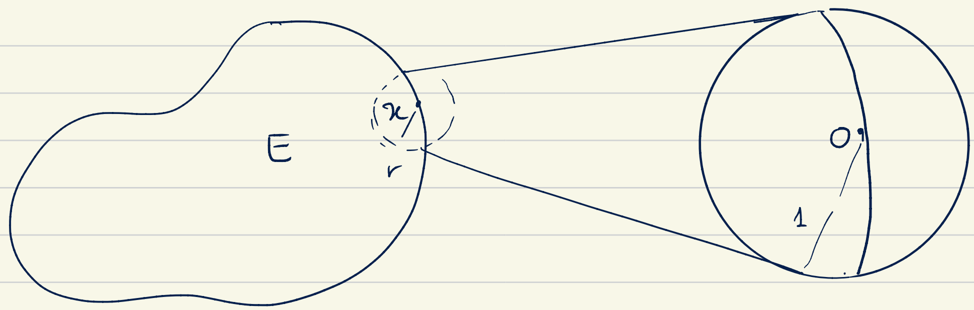
thus $\operatorname{spt} |\mu_E| = \overline{\partial^* E} = \partial E$ (good representative)



Local properties of sets of l.f.p : use blow-ups

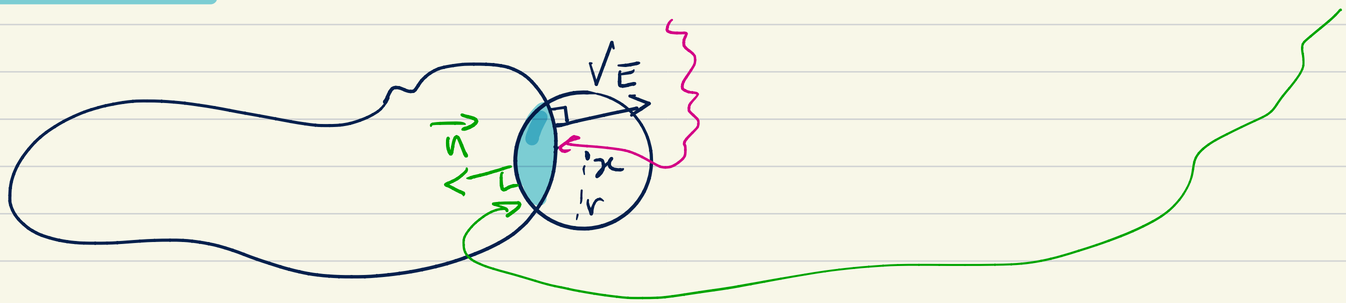
$E \subset \mathbb{R}^n$ set of l.f.p $x \in \mathbb{R}^n$ $r > 0$

$$E_{x,r} = \frac{1}{r}(E - x) = \Phi_{x,r}(E) = \left\{ y : \exists z \in E \quad y = \frac{z-x}{r} \right\}$$



Lemma : If $\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, for $\mathcal{L}^1$ a.e $r > 0$

$$\int_{E \cap B(x,r)} \operatorname{div} \phi \, dy = \int_{B(x,r)} \phi \, \nu_E \, d|\mu_E| + \int_{E \cap \partial B(x,r)} \phi \cdot \vec{n} \, d\mathcal{H}^{n-1}$$



Pf : Assume $h : \mathbb{R}^n \to \mathbb{R}$ smooth then

$$\int_E \operatorname{div}(h\phi)\,dx = \int_E h \operatorname{div}\phi\,dx + \int_E \langle \nabla h, \phi\rangle\,dx$$

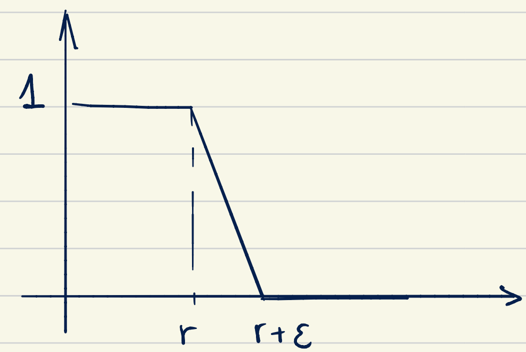
$$(\ast) \qquad \| $$

$$\int_{\mathbb{R}^n} \langle h\phi, \nu_E \rangle\, d|\mu_E|$$

idea: approximate $\chi_{B(x,r)}$ by smooth functions (h)

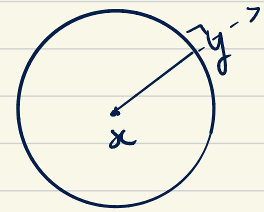
$(\ast)$ holds for $h_\varepsilon(y) = g_\varepsilon(|x-y|)$

$$g_\varepsilon(s) = \begin{cases} 1 & 0 \le s \le r \\ \dfrac{r-s+\varepsilon}{\varepsilon} & r \le s \le r+\varepsilon \\ 0 & s > r+\varepsilon \end{cases}$$



$$g_\varepsilon'(s) = \begin{cases} -1/\varepsilon & r < s < r+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\nabla h_\varepsilon(y) = \begin{cases} -\dfrac{1}{\varepsilon}\dfrac{y-x}{|y-x|} & r < |x-y| < r+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$



let $h = h_\varepsilon$ in $(\bigstar)$

$$\int_{\mathbb{R}^n} h_\varepsilon \langle \phi, \nu_E \rangle \, d|\mu_E| = \int_E h_\varepsilon \operatorname{div} \phi - \frac{1}{\varepsilon} \int_{\{r < |x-y| < r+\varepsilon\} \cap E} \left\langle \frac{y-x}{|x-y|} ; \phi \right\rangle dx$$

$\varepsilon \to 0$ a.e $r > 0$

$$\int_{B(x,r)} \langle \phi, \nu_E \rangle \, d|\mu_E| = \int_{B(x,r) \cap E} \operatorname{div} \phi - \int_{\partial B(x,r) \cap E} \langle \phi, \vec{n} \rangle \, d\mathcal{H}^{n-1} \qquad \blacksquare$$

Lemma: There exist constants $A_1, \dots, A_5$ depending only on n s.t if E s.l.f.p and $x \in \partial^* E$

(i) $\liminf_{r \to 0} \frac{|E \cap B(x,r)|}{r^n} \geq A_1 > 0$

(ii) $\liminf_{r \to 0} \frac{|B(x,r) \setminus E|}{r^n} \geq A_2 > 0$

?

(iii) $\liminf_{r \to 0} \frac{|\mu_E|(B(x,r))}{r^{n-1}} \geq A_3 > 0$

(iv) $\limsup_{r \to 0} \frac{|\mu_E|(B(x,r))}{r^{n-1}} \leq A_4 < \infty$

lower and upper density bounds for $|\mu_E|$ at points in $\partial^* E$

(v) $\limsup_{r \to 0} \frac{|\mu_{E \cap B(x,r)}|(\mathbb{R}^n)}{r^{n-1}} \leq A_5 < \infty$

Pf : For $x \in \partial^* E$ by previous lemma

$$\int_{B(x,r)} \langle \phi, \nu_E \rangle \, d|\mu_E| = \int_{B(x,r)\cap E} \operatorname{div} \phi - \int_{\partial B(x,r)\cap E} \langle \phi, \vec{n} \rangle \, d\mathcal{H}^{n-1} \qquad (1)$$

Taking sup $\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ $\|\phi\|_\infty \le 1$ we have

$$(2) \quad \boxed{|\mu_{E\cap B(x,r)}|(\mathbb{R}^n) \le |\mu_E|(B(x,r)) + \mathcal{H}^{n-1}(E \cap \partial B(x,r))}$$

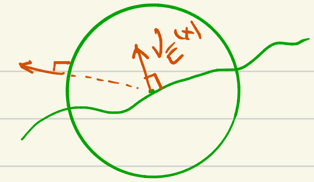
Choose $\phi(y) = \nu_E(x)$ (constant) in $B(x,r)$, (1) becomes

$$\int_{B(x,r)} \langle \nu_E(x), \nu_E(y) \rangle \, d|\mu_E|(y) = 0 - \int_{\partial B(x,r)\cap E} \langle \nu_E(x), \vec{n} \rangle \, d\mathcal{H}^{n-1}$$

since $x \in \partial^* E$; $\displaystyle\lim_{r\to 0} ⨍_{B(x,r)} \nu_E(y) \, d|\mu_E|(y) = \nu_E(x)$ &

$$\nu_E(x) \cdot ⨍_{B(x,r)} \nu_E(y) \, d|\mu_E|(y) = -\frac{1}{|\mu_E|(B(x,r))} \int_{\partial B(x,r)\cap E} \langle \nu_E(x), \vec{n} \rangle \, d\mathcal{H}^{n-1}$$

implies $\exists\, r_0 > 0$ s.t $0 < r < r_0$



$$\frac{1}{2} \leq \frac{-1}{|\mu_E|(B(x,r))} \int_{\partial B(x,r) \cap E} \langle \nu_E(x), \vec{n} \rangle \, d\mathcal{H}^{n-1} \leq 3/2$$

$$\frac{1}{2} |\mu_E| (B(x,r)) \leq - \int_{\partial B(x,r) \cap E} \langle \nu_E(x), \vec{n} \rangle \, d\mathcal{H}^{n-1} \leq \frac{3}{2} |\mu_E| (B(x,r))$$

(iv) $$\limsup_{r \to 0} \frac{|\mu_E|(B(x,r))}{r^{n-1}} \leq 2 \frac{\mathcal{H}^{n-1}(\partial B(x,r) \cap E)}{r^{n-1}} \leq 2\sigma_{n-1} \qquad (3)$$

(2) & (3) $\Rightarrow$ (v) $$|\mu_{E \cap B(x,r)}|(\mathbb{R}^n) \leq 3\mathcal{H}^{n-1}(\partial B(x,r) \cap E) \leq 3\sigma_{n-1} r^{n-1} \qquad (4)$$

Let $$g(r) = |E \cap B(x,r)| = \int_0^r \mathcal{H}^{n-1}(E \cap \partial B(x,s))\, ds$$

by absolute continuity for a.e $r > 0$

$$g'(r) = \mathcal{H}^{n-1}(E \cap \partial B(x,r))$$

Applying the isoperimetric inequality

$$g(r)^{n-1/n} = \underbrace{|E \cap B(x,r)|}_{\text{set of l.f.p}}{}^{n/n-1} \leq c \cdot |\mu_{E \cap B(x,r)}|(\mathbb{R}^n)$$

by (4) $\leq c\, \mathcal{H}^{n-1}(E \cap \partial B(x,r)) = c\, g'(r)$ a.e $r < r_0$

$$\frac{1}{c} \leq g'(r) / g(r)^{n-1/n} = g'(r)\, g(r)^{-(1-1/n)} = n\,(g^{1/n}(r))'$$

since $g(0) = 0 \Rightarrow g^{1/n}(r) \geq c(n)\, r \Rightarrow g(r) \geq c'(n)\, r^n$

for $r < r_0$ $\quad |E \cap B(x,r)| \geq c(n)\, r^n \xRightarrow{\text{taking liminf}}$ (i) $\quad$ (5)

Since

$|\mu_E| = |\mu_{\mathbb{R}^n \setminus E}|$ (ii) follows as (i) did

By the relative isoperimetric inequality

$c \min\{|E \cap B(x,r)|, |B(x,r) \setminus E|\}^{\frac{n-1}{n}} \leq |\mu_E|(B(x,r))$ $\quad$ by (5) + analogue for $B(x,r) \setminus E$

for $r < r_0$

$|\mu_E|(B(x,r)) \geq c_n\, r^{n-1}$ $\quad$ i) & ii) $\Rightarrow$ (iii) ✓