De Giorgi's structure theorem

Assume E is a set of locally finite perimeter in $\mathbb{R}^n$, then

(i) $\partial^* E \subset \bigcup_{k=1}^{\infty} K_k \cup N$ where $|\mu_E|(N) = 0$ &

K_k is a compact subset of a C^1 hypersurface S_k

(ii) For $x \in \partial^* E \cap K_k$ $\nu_E(x) = \nu_{S_k}(x)$

(iii) $|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E \Rightarrow \mathcal{H}^{n-1}(N) = 0$

Pf : wlog E of finite perimeter

Recall that for $x \in \partial^* E$

$$\boxed{\lim_{r \to 0} \frac{|B(x,r) \cap E \cap H_x^+|}{r^n} = 0 = \lim_{r \to 0} \frac{|B(x,r) \setminus E \cap H_x^-|}{r^n}} \quad (1)$$

By Egoroff's theorem & Lusin's theorem

$$(2)\quad \begin{cases} \partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N & |\mu_E|(N) = 0 \\ \text{the convergence in (1) is uniform on } K_k \\ \nu_E|_{K_k} \text{ is continuous for each } k \in \mathbb{N} \end{cases}$$

For $\delta > 0$ define

$$f_k(\delta) = \sup \left\{ \frac{\langle \nu_E(x) ; y - x \rangle}{|y-x|} : 0 < |x-y| < \delta \ ; \ x, y \in K_k \right\}$$

<u>Claim</u>: for each $k \in \mathbb{N}$, $f_k(\delta) \to 0$ as $\delta \to 0$

Whitney's extension theorem

Let $C \subset \mathbb{R}^n$ be a closed set $f : C \longrightarrow \mathbb{R}$, $d : C \longrightarrow \mathbb{R}^n$

(i) $$R(x,y) = \frac{f(y) - f(x) - \langle d(x), y-x \rangle}{|y-x|} \qquad x, y \in C \quad x \neq y$$

(ii) Let $K \subset C$ compact and $\delta > 0$ s.t

$$\rho_K(\delta) = \sup \{ |R(x,y)| : 0 < |x-y| < \delta \; ; \; x, y \in K \}$$

Theorem: Assume f, d are continuous, and for each compact set $K \subset C$; $\rho_K(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$ then there exists $\bar{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ s.t

(i) $\bar{f}$ is C^1

(ii) $\bar{f} = f$ and $D\bar{f} = d$ on C

Apply Whitney's extension theorem with $f = 0$ $d = \nu_E$ on K_k.
(hypotheses satisfied by claim) then there exist C^1 functions
$\bar{f}_k : \mathbb{R}^n \to \mathbb{R}$ s.t. $\bar{f}_k = 0$ on K_k Q: f_k ?

$$D\bar{f}_k = \nu_E \text{ on } K_k$$

Let $S_k = \{x \in \mathbb{R}^n : \bar{f}_k = 0 \ \& \ |D\bar{f}_k| \geq 1/2\}$ by the implicit function theorem S_k is an $(n-1)$ dimensional submanifold on $\mathbb{R}^n$, $K_k \subset S_k$ & (i) & (ii) in theorem are proved

To show (iii) $|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$ recall that if $B \subset \partial^* E$
Borel

$$\mathcal{H}^{n-1}(B) \leq c(n) \, |\mu_E|(B) \Rightarrow \underbrace{\mathcal{H}^{n-1} \llcorner \partial^* E}_{\text{Radon}} \ll |\mu_E| \text{ measure}$$

Let $x \in K_k \subset S_k$ then $\displaystyle\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(B(x,r) \cap S_k)}{\omega_{n-1} r^{n-1}} = 1$

$x \in \partial^* E$ $\downarrow$

$$\lim_{r \to 0} \frac{|\mu_E|(B(x,r))}{\omega_{n-1} r^{n-1}} = 1$$

$$\Longrightarrow \lim_{r\to 0} \frac{\mathcal{H}^{n-1}(S_k \cap B(x,r))}{|\mu_E|(B(x,r))} = 1 \quad \text{for} \quad x \in K_k$$

$\uparrow$

Radon - Nykodym for $\mathcal{H}^{n-1}$ a.e $x \in K_k$

$$\lim_{r\to 0} \frac{\mathcal{H}^{n-1}(K_k \cap B(x,r))}{\mathcal{H}^{n-1}(S_k \cap B(x,r))} = 1$$

because of μ Radon $B \subset A$ μ-meas for μ a.e

$$\lim_{r\to 0} \frac{\mu(B \cap B(x,r))}{\mu(A \cap B(x,r))} = 1$$

and for $\mathcal{H}^{n-1}$ a.e $x \in K_k$

$$\lim_{r\to 0} \frac{\mathcal{H}^{n-1}(K_k \cap B(x,r))}{\mathcal{H}^{n-1}(\partial^* E \cap B(x,r))} = 1$$

$$\Rightarrow \lim_{r\to 0} \frac{|\mu_E|(B(x,r))}{\mathcal{H}^{n-1}(\partial^* E \cap B(x,r))} = 1 \Longrightarrow |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$$

Definition: let $x \in \mathbb{R}^n$, $x \in \partial_* E$ the measure theoretic boundary of E if

$$\limsup_{r \to 0} \frac{|E \cap B(x,r)|}{r^n} > 0$$

&

$$\limsup_{r \to 0} \frac{|B(x,r) \setminus E|}{r^n} > 0$$

Lemma: (i) $\partial^* E \subset \partial_* E$

(ii) $\mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0$

Pf: ii) $\alpha \in (0,1)$ and $r_j \to 0$ s.t (check one can choose the same sequence)

$$|E \cap B(x, r_j)| > \alpha \omega_n r_j^n \quad \& \quad |B(x, r_j) \setminus E| > \alpha \omega_n r_j^n$$

by the relative isoperimetric inequality

$$\min\{|E \cap B(x, r_j)|, |B(x, r_j) \setminus E|\}^{n-1/n} \leq c(n) |\mu_E|(B(x, r_j))$$

$$\Rightarrow \quad |\mu_E|(B(x, r_j)) \geq c(n) \alpha^{n-1/n} r_j^{n-1}$$

$$\limsup_{r \to 0} \frac{|\mu_E|(B(x,r))}{r^{n-1}} > 0 \quad \Longrightarrow$$

covering argument $\Rightarrow$

$$\mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) \le c(n)\, |\mu_E|(\partial_* E \setminus \partial^* E)$$
$$\le c(n)\, |\mu_E|(\mathbb{R}^n \setminus \partial^* E) = 0.$$

Theorem: (Gauss – Green theorem)

suppose $E \subset \mathbb{R}^n$ s.l.f.p then

i) $\mathcal{H}^{n-1}(\partial_* E \cap K) < \infty \quad \forall\, K \subset\subset \mathbb{R}^n$ compact

ii) For $\mathcal{H}^{n-1}$ a.e $x \in \partial_* E$ there exists a unique measure theoretic unit normal $\nu_E(x)$ s.t

$$\int_E \operatorname{div} \phi = \int_{\partial_* E} \langle \phi, \nu_E(x) \rangle \, d\mathcal{H}^{n-1} \qquad \forall \phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$$

Theorem (Federer): Let $E \subset \mathbb{R}^n$ measurable: E s.l.f.p iff

$$\mathcal{H}^{n-1}(\partial_* E \cap K) < \infty \quad \forall\, K \subset\subset \mathbb{R}^n \text{ compact.}$$