Theorem: If $\{E_k\} \subset \mathcal{A}(C_A, r_0)$ with $0 \in \partial E_k$, $\forall k \geq 1$ there exists a subsequence $\{E_{n_k}\}$, a set E of l.f.p & a non-negative Radon measure μ such that

$$\chi_{E_{n_k}} \to \chi_E \quad L^1_{loc}(\mathbb{R}^n), \quad \mu_{E_{n_k}} \to \mu_E \quad \& \quad |\mu_{E_{n_k}}| \to \mu$$

1. μ is Ahlfors regular up to scale r_0 & constant C_A
2. $|\mu_E| \ll \mu$
3. If $x \in \partial E$ $\exists x_{n_k} \in \partial E_{n_k}$ s.t $x_{n_k} \to x$
4. If $x \in \operatorname{spt} \mu$ $\exists x_{n_k} \in \partial E_{n_k}$ s.t $x_{n_k} \to x$
5. If $x_{n_k} \in \partial E_{n_k}$ & $x_{n_k} \to x$ then $x \in \operatorname{spt} \mu$

(5) If $x_{n_k} \in \partial E_{n_k}$ & $x_{n_k} \longrightarrow x$; given $\varepsilon > 0$ and $s < r_0$ $\exists\ n_k = n_k(\varepsilon, s)$ s.t

$$B(x_{n_k}, (1-\varepsilon)s) \subset B(x, (1-\tfrac{\varepsilon}{2})s)$$

$$C_A^{-1}\ s^{n-1}(1-\varepsilon)^{n-1} \leq |\mu_{E_{n_k}}|(B(x_{n_k}, (1-\varepsilon)s))$$

$$\leq |\mu_{E_{n_k}}|(\overline{B(x, (1-\varepsilon/2)s})$$

$$\leq \limsup_{n_k \to \infty} |\mu_{E_{n_k}}|(\overline{B(x, (1-\varepsilon/2)s})$$

$$\leq \mu(\overline{B(x, (1-\varepsilon/2)s})$$

$$\leq \mu(B(x,s))$$

Letting $\varepsilon \to 0$

$(\star)$ $\quad c_A^{-1} s^{n-1} \le \mu(B(x,s)) \Rightarrow x \in \operatorname{spt}\mu$ $\quad$ 5 ✓

By 3 $\forall\, x \in \partial E$, $\exists\, x_{n_k} \in \partial E_{n_k}$ s.t $x_{n_k} \longrightarrow x \Rightarrow x \in \operatorname{spt}\mu$

$$\partial E = \operatorname{spt}\mu_E \subset \operatorname{spt}\mu$$

Given $x \in \operatorname{spt}\mu$, $\varepsilon > 0$, for $s < r_0$

$$\mu(B(x,s)) \le \liminf_{n_k\to\infty} |\mu_{E_{n_k}}|(B(x,s)) \le \liminf_{n_k\to\infty} |\mu_{E_{n_k}}|(B(x_{n_k}, (1+\varepsilon)s))$$
$$\le c_A (1+\varepsilon)^{n-1} s^{n-1}$$

$\varepsilon \to 0$ $\quad \mu(B(x,s)) \le C_A s^{n-1} \quad \forall\, r < r_0 \quad \forall\, x \in \operatorname{spt}\mu$

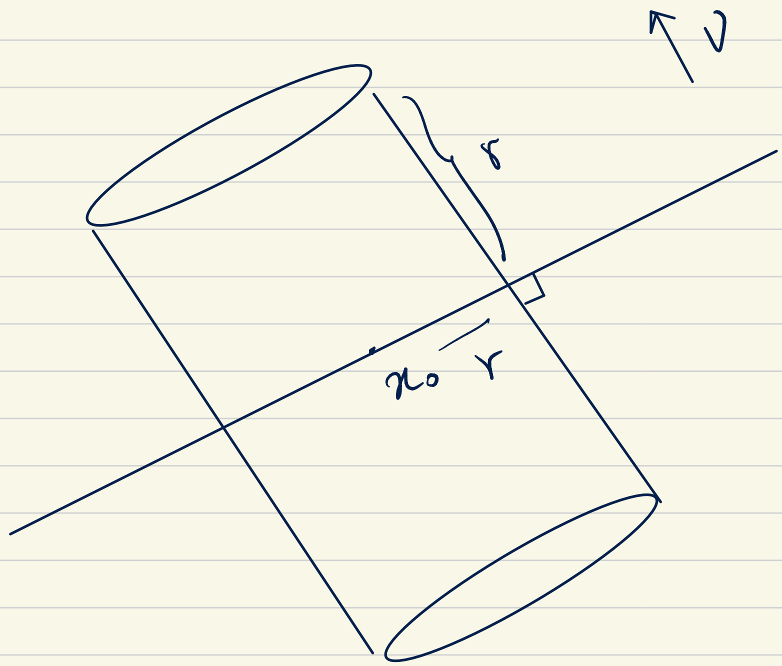
$\forall\, x \in \operatorname{spt}\mu$, by 4 $\exists\, x_{n_k} \in \partial E_{n_k}$ $x_{n_k} \longrightarrow x$ & by $(\star)$

$$c_A^{-1} s^{n-1} \le \mu(B(x,s)) \quad \forall\, r < r_0$$

$\Rightarrow$ 1

If $\nu \in S^{n-1}$, $x_0 \in \mathbb{R}^n$, $r > 0$

$$C(x_0, r, \nu) = \left\{ x = (\bar{x}, t) : \langle x - x_0, \nu \rangle = t \ ; \ |x_0 - \bar{x}| < r \ \& \ |t| < r \right\}$$



Lemma (small excess implies flatness)

Given $C_A \geq 1$, $t_0 \in (0,1)$ $\exists\, \omega(n, t_0, C_A) > 0$ such that if $E \in \mathcal{X}(C_A, 4r_0)$ for some $r_0 > 0$, and $x_0 \in \partial E$ $r \leq 2r_0$ $\nu \in S^{n-1}$

$$e(E, x_0, r, \nu) = \frac{1}{r^{n-1}} \int_{C(x_0, r, \nu) \cap \partial^* E} \frac{|\nu_E - \nu|^2}{2} \, d\mathcal{H}^{n-1}$$

then if $e(E, x_0, 2r_0, \nu) \leq \omega(n, t_0, C_A)$ t.f.h

i) $|\langle x - x_0, \nu \rangle| < t_0 r_0$ $\forall x \in C(x_0, r_0, \nu) \cap \partial E$

ii) $|\{x \in C(x_0, r_0, \nu) \cap E : \langle x - x_0 ; \nu \rangle > t_0 r_0\}| = 0$

(iii) $|\{x \in C(x_0, r_0, \nu) \cap E^c : \langle x - x_0, \nu \rangle < -t_0 r_0\}| = 0$

Small oscillation of the unit normal in $L^2 \Rightarrow \partial E$ is flat (note i) measures the height of ∂E in $C(x_0, r_0, \nu)$)

Pf : Contradiction & compactness argument.

Assume $\exists t_0 \in (0,1)$ $\{E_k\}_{k\in\mathbb{N}} \subset \mathcal{O}\mathcal{C}(C_A, 4r_k)$ $r_k > 0$ $x_k \in \partial E_k$ $\nu_k \in S^{n-1}$

$$e(E_k, x_k, 2r_k, \nu_k) < 2^{-k}$$ and at least one of the following conditions hold for infinitely many k.

(i) $\{x \in C_k \cap \partial E_k : t_0 r_k \leq |q_k(x)| \leq r_k\} \neq \emptyset$

(ii) $|\{x \in C_k \cap E_k : q_k(x) > t_0 r_k\}| > 0$

(iii) $|\{x \in C_k \setminus E_k : q_k(x) < -t_0 r_k\}| > 0$

where

$$C_k = C(x_k, r_k, \nu_k) \quad \&$$

$$q_k(x) = \langle x - x_k, \nu_k \rangle$$

Consider
$$\tilde{F}_k = \frac{1}{r_k}(E_k - x_k) \in \mathfrak{A}(C_A, 4) \qquad (r_k \tilde{F}_k + x_k = E_k)$$

$$e(E_k, x_k, 2r_k, \nu_k) = \frac{1}{(2r_k)^{n-1}} \int_{C(x_k, 2r_k, \nu_k) \cap \partial^* E_k} \frac{1}{2} |\nu_{E_k} - \nu_k|^2 \, d\mathcal{H}^{n-1}(x)$$

$$e(\tilde{F}_k, 0, 2, \nu_k) = \frac{1}{2^{n-1}} \int_{C(0, 2, \nu_k) \cap \partial^* \tilde{F}_k} \frac{1}{2} |\nu_{\tilde{F}_k} - \nu_k|^2 \, d\mathcal{H}^{n-1}$$

modulo rotation $\nu_k \to e_n$ $\quad \tilde{F}_k \longrightarrow F_k \quad 0 \in \partial F_k$

$F_k \in \mathfrak{A}(C_A, 4)$ & $e(F_k, 0, 2, e_n) = e_n(F_k, 0, 2) < 2^{-k}$

and at least one of the following holds for infinitely many k's.

(i) $\{x \in C \cap \partial F_k : \quad t_0 \leq |q(x)| \leq 1\} \neq \emptyset$ where

(ii) $|\{x \in C \cap F_k : q(x) > t_0\}| > 0$ $\qquad \left\{ \begin{array}{l} q(x) = \langle x, e_n \rangle \\ C = C(0, 2, e_n) \end{array} \right.$

(iii) $|\{x \in C \setminus F_k : q(x) < -t_0\}| > 0$