Theorem: (The height bound) - Given $n \geq 2$ & $C_A > 0$ $\exists \varepsilon_0 = \varepsilon_0(n, C_A)$ and $C_0 = C_0(n, C_A) \geq 1$ s.t if $E \in \mathcal{A}(C_A, 8r_0)$ for some $r_0 > 0$ $x_0 \in \partial E$ and

$$e_n(x_0, 4r_0) = e(E, x_0, 4r_0, e_n) \leq \varepsilon_0$$

then $q : \mathbb{R}^{n-1} \times \mathbb{R} \to \{0\} \times \mathbb{R}$, $q(x) = \langle x, e_n \rangle$ satisfies

$$\frac{1}{r_0} \sup \{ |q(x_0) - q(y)| : y \in C(x_0, r_0, e_n) \cap \partial E \} \leq C_0 \, e_n(x_0, 4r_0)^{1/2(n-1)}$$

Pf

Step 1 : Replace E by $F = E_{x_0, r_0} \in \mathcal{A}(C_A, 4)$ $0 \in \partial E$

$$e_n(4) = e(F; 0, 4, e_n) = e(E, x_0, 4r_0, e_n) < \varepsilon_0$$

we want to show that

$$|q(x)| \leq C_0 \, e_n(4)^{1/2(n-1)} \quad \forall x \in C \cap \partial F$$

$C = C(0, 1, e_n)$ Rename $F = E$.

Assume $\varepsilon_0 \leq \omega(n, 1/4)$ ($t_0 = 1/4$ small excess $\Rightarrow$ flatness)

$\Rightarrow \quad |q(x)| \le 1/4 \quad \forall x \in C_2 \cap \partial E \qquad$ if $\quad D_2 = C_2 \cap \mathbb{R}^{n-1} \times \{0\}$

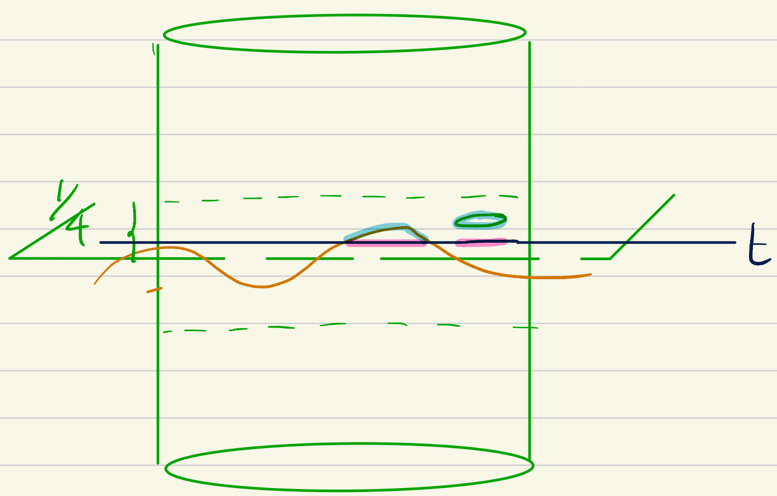
remark following excess measure lemma

$$0 \le \mathcal{H}^{n-1}(M) - \mathcal{H}^{n-1}(D_2) = \xi(D_2) = e_n(2) \le 2^{n-1} e_n(4)$$

where $e_n(s) = e(E, 0, s, e_n)$. For $t \in (-2, 2)$

$$0 \le \mathcal{H}^{n-1}(M \cap \{q(x) > t\}) - \mathcal{H}^{n-1}(E_t \cap D_2) \le e_n(2) \le 2^{n-1} e_n(4)$$

$E_t = \{z : (z, t) \in E\} \qquad$ where $\quad M = \partial^* E \cap C_2$



Step 2 : Consider $f : (-1,1) \to [0, \mathcal{H}^{n-1}(M)]$

$$f(t) = \mathcal{H}^{n-1}(M \cap \{q(x) > t\})$$

Since $|q(x)| < 1/4$ for $x \in M$ because $\varepsilon_0 < \omega(n, 1/4)$

$$f(t) = \mathcal{H}^{n-1}(M) \quad \text{for} \quad t \in (-1, -1/4)$$

$$f(t) = 0 \quad \text{for} \quad t \in (1/4, 1)$$

Moreover f decreasing & right continuous (check). Thus $\exists\ \tau_0$, $|t_0| < 1/4$ s.t

$$f(t) \leq \frac{\mathcal{H}^{n-1}(M)}{2} \quad \text{if} \quad t \geq \tau_0$$

&

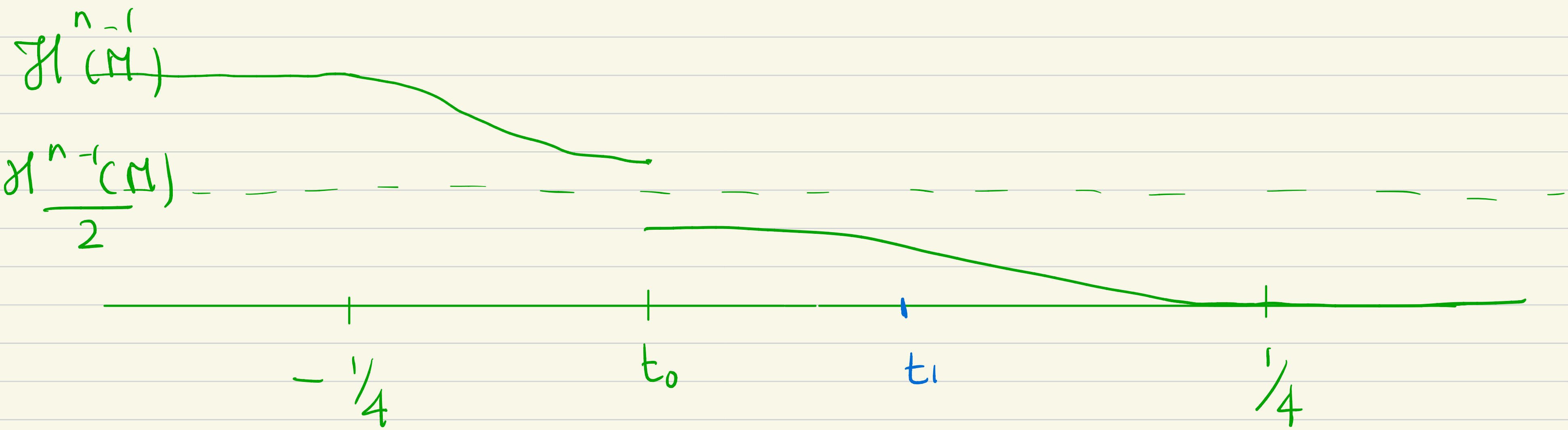
$$f(t) \geq \frac{\mathcal{H}^{n-1}(M)}{2} \quad \text{if} \quad t < t_0$$

Claim : $|q(x) - \tau_0| \leq c(n)\, e_n(4)^{1/2(n-1)} \quad \forall x \in C \cap \partial E$

Note that if this is the case since $0 \in \partial E$ then

$$|t_0| \le c(n)(e_n(4))^{1/2(n-1)} \quad \text{and} \quad \forall x \in C \cap \partial E$$

$$|q(x)| \le |q(x) - t_0| + |t_0| \le 2c(n)\, e_n(4)^{1/2(n-1)}$$



Pf of Claim in 2 steps

Step 3 $\qquad q(x) - t_0 \le c(n)\, e_n(4)^{1/2(n-1)} \quad \forall x \in C \cap \partial E$

a) let $t_1 \in (t_0, 1/4)$ be such

$$f(t) \leq \sqrt{e_n(4)} \qquad \forall t \geq t_1$$

$$f(t) > \sqrt{e_n(4)} \qquad \forall t < t_1$$

Since $\mathcal{H}^{n-1}(M) \geq \mathcal{H}^{n-1}(D_2)$ choosing ε_0 small enough we can guarantee $t_1 \in (t_0, 1/4)$

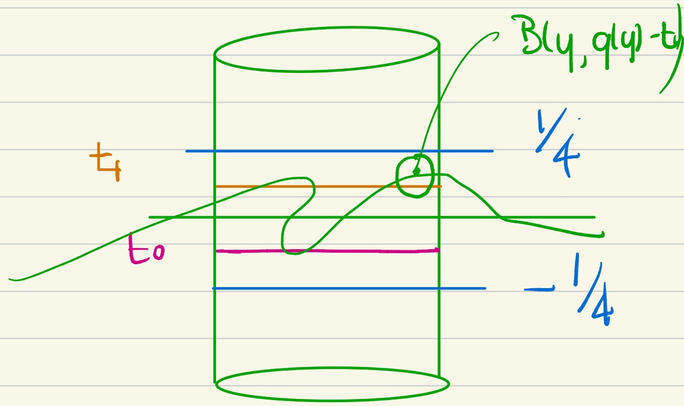
We prove that $q(y) - t_1 \leq C(n)\, e_n(4)^{1/2(n-1)}$ $\forall y \in C \cap \partial E$

if $y \in C \cap \partial E$ & $q(y) > t_1$

since $|q(y)| < 1/4$ & $t_1 \in (t_0, 1/4)$ $\quad 0 < q(y) - t_1 < \frac{1}{2}$

$|t_0| < 1/4 \Rightarrow |t_1| < 1/4$

$$B(y, q(y) - t_1) \subset\subset C_2$$



Since $E \in \mathcal{O}(C_4, 4)$ then

$$c_A^{-1}\,(q(y)-t_1)^{n-1} \le |\mu_E|\,(B(y, q(y)-t_1))$$

if $x \in B(y, q(y)-t_1)$ then $|q(x)-q(y)| < |x-y| < q(y)-t_1$

$\Rightarrow\ -q(y)+t_1 < q(x)-q(y) \Rightarrow q(x) > t_1$

$\Rightarrow\ B(y, q(y)-t_1) \subset \{z \in C_2 : q(z) > t_1\}$

$$c_A^{-1}\,(q(y)-t_1)^{n-1} \le |\mu_E|\,\{C_2 \cap \{q(z)>t_1\}\}$$
$$\le \mathcal{H}^{n-1}\,(\underbrace{C_2 \cap \partial^* E}_{M} \cap \{q(z)>t_1\})$$

if $q(y) > t_1$

$$c_A\,(q(y)-t_1)^{n-1} \le f(t_1) < \sqrt{e_n(4)}$$

$$(\star)\qquad \boxed{q(y)-t_1 \le c(n, C_A)\,e_n(4)^{1/2(n-1)}}$$

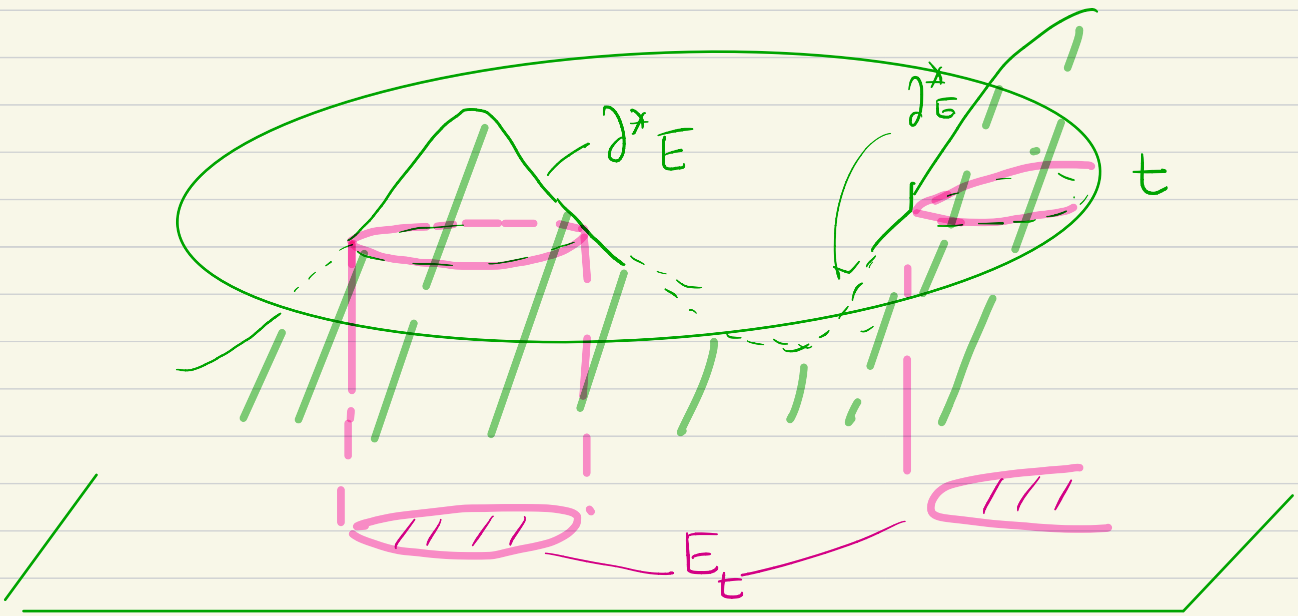
b) We now show that $t_1 - t_0 \le C_n\,e_n(4)^{1/2(n-1)}$ which would complete the proof of Step 3

For a.e t (Δ) $\mathcal{H}^{n-2}(\partial^* E_t \,\Delta\, (\partial^* E)_t) = 0$ by slicing

$E_t = \{ z : (z,t) \in E \}$

$(\partial^* E)_t = \{ z : (z,t) \in \partial^* E \}$

$g : \mathbb{R}^n \to [0,\infty]$ Borel measure



By the co-area formula

$$\int_{\partial^* E} g \underbrace{\sqrt{1-\langle \nu_E, e_n\rangle^2}}_{\substack{\text{Jacobian of} \\ \text{the projection onto } e_n^\perp}} \, d\mathcal{H}^{n-1} = \int_{\mathbb{R}} \left(\int_{(\partial^* E)_t} g \, d\mathcal{H}^{n-2} \right) dt$$

letting $g = \chi_{C_2}$ $\quad = \displaystyle\int_{-2}^{2} \mathcal{H}^{n-2}\left((\partial^* E)_t \cap D_2 \right) dt$

by (Δ)

$$\Rightarrow \int_{-2}^{2} \mathcal{H}^{n-2} (\partial^* E_t \cap D_2) \, dt = \int_M \sqrt{1-\langle \nu_E, e_n\rangle^2} \, d\mathcal{H}^{n-1}$$

$$\leq \sqrt{2} \int_M \sqrt{1-\langle \nu_E, e_n\rangle} \, d\mathcal{H}^{n-1} \leq$$

Cauchy-Schwarz

$$\le \sqrt{2}\ \sqrt{\mathcal{H}^{n-1}(M)}\ \sqrt{\int_M (1-\langle \nu_E, e_n\rangle)\, d\mathcal{H}^{n-1}}$$

$$\le \sqrt{2\mathcal{H}^{n-1}(M)}\ c(n)\sqrt{e_n(2)}$$ since

$$\mathcal{H}^{n-1}(M) \le \mathcal{H}^{n-1}(D_2) + e_n(2) \quad \& \quad e_n(2) \le c(n) e_n(4)$$

$$\le c(n)$$

then $\varepsilon_0 \le 1$

$$\boxed{\int_{-2}^{2} \mathcal{H}^{n-2}(D \cap \partial^* E_t)\, dt \le c(n)\sqrt{e_n(2)}} \qquad (i)$$