Theorem : (The height bound) - Given $n \geq 2$ & $C_A > 0$ $\exists \varepsilon_0 = \varepsilon_0(n, C_A)$ and $C_0 = C_0(n, C_A) \geq 1$ s.t if $E \in \mathcal{A}(C_A, 8r_0)$ for some $r_0 > 0$ $x_0 \in \partial E$ and

$$e_n(x_0, 4r_0) = e(E, x_0, 4r_0, e_n) \leq \varepsilon_0$$

then $q : \mathbb{R}^{n-1} \times \mathbb{R} \to \{0\} \times \mathbb{R}$, $q(x) = \langle x, e_n \rangle$ satisfies

$$\frac{1}{r_0} \sup \{ |q(x_0) - q(y)| : y \in C(x_0, r_0, e_n) \cap \partial E \} \leq C_0 \, e_n(x_0, 4r_0)^{1/2(n-1)}$$

Pf

Step 1 : Replace E by $F = E_{x_0, r_0} \in \mathcal{A}(C_A, 4)$ $0 \in \partial E$

$$e_n(4) = e(F; 0, 4, e_n) = e(E, x_0, 4r_0, e_n) < \varepsilon_0$$

we want to show that

$$|q(x)| \leq C_0 \, e_n(4)^{1/2(n-1)} \quad \forall x \in C \cap \partial F$$

$C = C(0, 1, e_n)$ Rename $F = E$.

Assume $\varepsilon_0 \leq \omega(n, 1/4)$ ($t_0 = 1/4$ small excess $\Rightarrow$ flatness)

$\Rightarrow \quad |q(x)| \leq 1/4 \quad \forall x \in C_2 \cap \partial E \qquad$ if $\quad D_2 = C_2 \cap \mathbb{R}^{n-1} \times \{0\}$

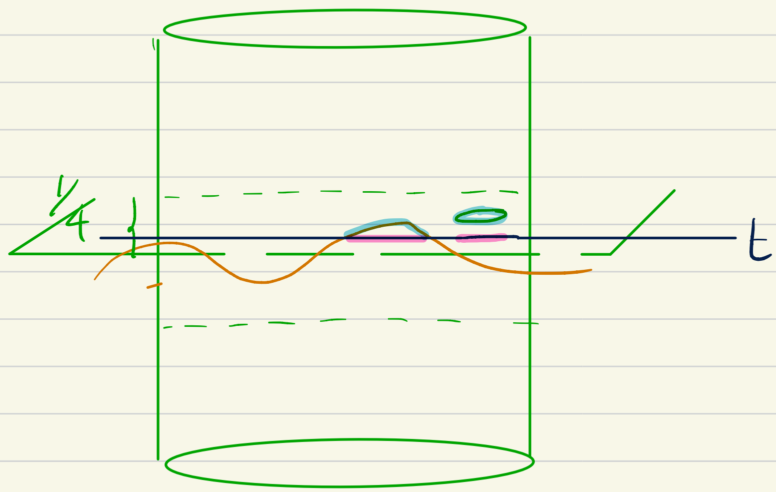
remark following excess measure lemma

$$0 \leq \mathcal{H}^{n-1}(M) - \mathcal{H}^{n-1}(D_2) = \xi(D_2) = e_n(2) \leq 2^{n-1} e_n(4)$$

where $e_n(s) = e(E, 0, s, e_n)$. For $t \in (-2, 2)$

$$0 \leq \mathcal{H}^{n-1}(M \cap \{q(x) > t\}) - \mathcal{H}^{n-1}(E_t \cap D_2) \leq e_n(2) \leq 2^{n-1} e_n(4)$$

$E_t = \{z : (z, t) \in E\} \qquad$ where $\quad M = \partial^* E \cap C_2$



Step 2 : Consider $f : (-1,1) \to [0, \mathcal{H}^{n-1}(M)]$

$$f(t) = \mathcal{H}^{n-1}(M \cap \{q(x) > t\})$$

Since $|q(x)| < 1/4$ for $x \in M$ because $\varepsilon_0 < \omega(n, 1/4)$

$$f(t) = \mathcal{H}^{n-1}(M) \quad \text{for} \quad t \in (-1, -1/4)$$

$$f(t) = 0 \quad \text{for} \quad t \in (1/4, 1)$$

Moreover f decreasing & right continuous (check). Thus $\exists\, \tau_0$, $|t_0| < 1/4$ s.t

$$f(t) \leq \frac{\mathcal{H}^{n-1}(M)}{2} \quad \text{if} \quad t \geq \tau_0$$

&

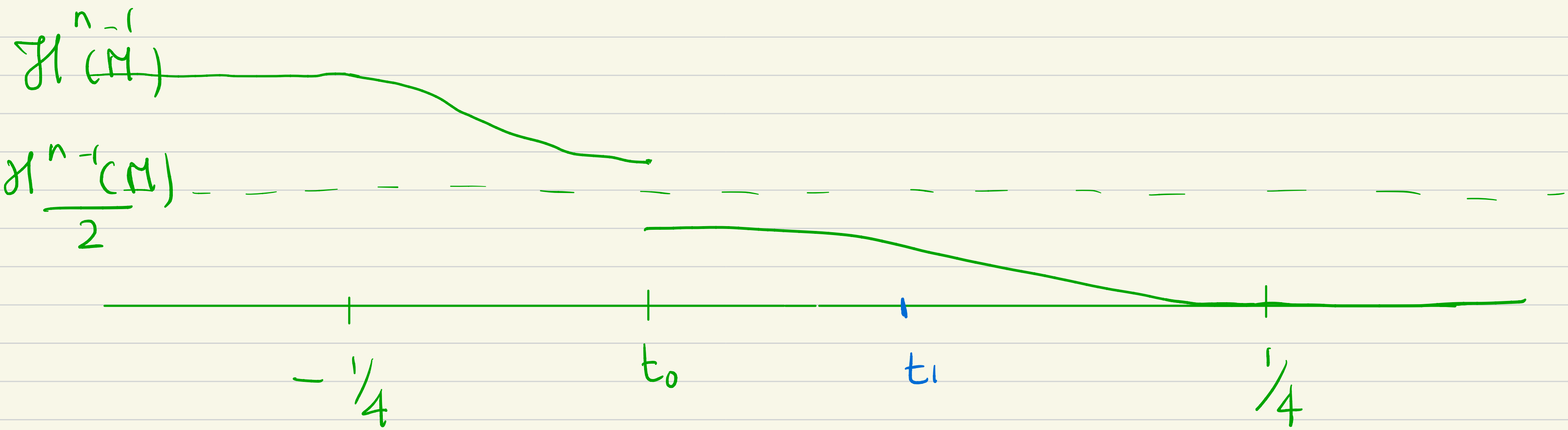
$$f(t) \geq \frac{\mathcal{H}^{n-1}(M)}{2} \quad \text{if} \quad t < t_0$$

Claim : $|q(x) - \tau_0| \leq c(n)\, e_n(4)^{1/2(n-1)} \quad \forall x \in C \cap \partial E$

Note that if this is the case since $0 \in \partial E$ then

$$|t_0| \le c(n)(e_n(4))^{1/2(n-1)} \quad \text{and} \quad \forall x \in C \cap \partial E$$

$$|q(x)| \le |q(x) - t_0| + |t_0| \le 2c(n)\, e_n(4)^{1/2(n-1)}$$



Pf of Claim in 2 steps

Step 3 $\quad q(x) - t_0 \le c(n)\, e_n(4)^{1/2(n-1)}$ $\quad \forall x \in C \cap \partial E$

a) let $t_1 \in (t_0, 1/4)$ be such

$$f(t) \leq \sqrt{e_n(4)} \qquad \forall t \geq t_1$$

$$f(t) > \sqrt{e_n(4)} \qquad \forall t < t_1$$

$$(\ast) \qquad \boxed{|g(y) - t_1| \leq c(n, C_A)\, e_n(4)^{1/2(n-1)}}$$

b) We now show that $t_1 - t_0 \leq C_n\, e_n(4)^{1/2(n-1)}$ which would complete the proof of step 3

For a.e t $\quad (\Delta)\ \mathcal{H}^{n-2}\left(\partial^* E_t \,\Delta\, (\partial^* E)_t\right) = 0$

$$E_t = \{ z : (z,t) \in E \}$$

$$(\partial^* E)_t = \{ z : (z,t) \in \partial^* E \}$$

thus for a.e t

$$\mathcal{H}^{n-1}(\partial^* E_t \cap D_2) = \mathcal{H}^{n-1}((\partial^* E)_t \cap D_2)$$

By the co_area formula & Cauchy-Schwarz we have

$$\int_{-2}^{2} \mathcal{H}^{n-2}(D \cap \partial^* E_t)\, dt \le c(n) \sqrt{\mathcal{H}^{n-1}(M)} \sqrt{en(2)}$$

&

$$\mathcal{H}^{n-1}(M) \le \mathcal{H}^{n-1}(D_2) + en(2) \quad \& \quad en(2) \le c(n) en(4)$$

$$\le c(n) \qquad \qquad \mathcal{E}_0 \le 1$$

then

$$\boxed{\int_{-2}^{2} \mathcal{H}^{n-2}(D_2 \cap \partial^* E_t)\, dt \le c(n) \sqrt{en(2)}} \qquad (i)$$

Since for $t \in (t_0, 2)$

$$\mathcal{H}^{n-1}(E_t \cap D_2) \le \mathcal{H}^{n-1}(M \cap \{q|x| > t\}) = f(t)$$

$$\le \frac{\mathcal{H}^{n-1}(M)}{2} \le \frac{\mathcal{H}^{n-1}(D_2) + 2^{n-1} en(4)}{2}$$

& $\mathcal{E}_0$ small enough dep n. $\le \frac{3}{4} \mathcal{H}^{n-1}(D_2)$

Since

$$\mathcal{H}^{n-1}(E_t \cap D_2) \leq \tfrac{3}{4}\, \mathcal{H}^{n-1}(D_2)$$

applying the local perimeter bound on volume (Maggi Prop. 12.37)

$$|\mu_{E_t}|(D_2) \geq c(n)\, |E_t \cap D_2|^{n-2/n-1}$$

$$(2) \quad \boxed{\mathcal{H}^{n-2}(\partial^* E_t \cap D_2) \geq c(n)\, \mathcal{H}^{n-1}(E_t \cap D_2)^{n-2/n-1}}$$

Combining (1) & (2)

$$c(n) \int_{t_0}^{2} \mathcal{H}^{n-1}(E_t \cap D_2)^{n-2/n-1}\, dt \leq \int_{-2}^{2} \mathcal{H}^{n-2}(\partial^* E_t \cap D_2)\, dt$$

(3)

$$\leq c(n) \sqrt{e_n(2)} \leq c'(n) \sqrt{e_n(4)}$$

for $t \in (t_0, t_1)$

$$\mathcal{H}^{n-1}(E_t \cap D_2) \geq \mathcal{H}^{n-1}(M \cap \{q(x) > t\}) - e_n(2)$$

$$\mathcal{H}^{n-1}(E_t \cap D_2) \geq \mathcal{H}^{n-1}(M \cap \{q(x) > t\}) - \varepsilon_n(2)$$

$$\geq \sqrt{\varepsilon_n(4)} - 2^{n-1}\varepsilon_n(4)$$

(4) $\boxed{\mathcal{H}^{n-1}(E_t \cap D_2) \geq c(n)\sqrt{\varepsilon_n(4)}}$ for ε_0 small enough

Combining (3) & (4) we have.

$$c_n (t_0 - t_1)(\varepsilon_n(4))^{\frac{n-2}{2(n-1)}} \leq C(n)\, \varepsilon_n(4)^{1/2}$$

$$\Rightarrow \boxed{\begin{aligned} t_0 - t_1 &\leq c(n)\, \varepsilon_n(4)^{\frac{1}{2} - \frac{n-2}{2(n-1)}} \\ &\leq c(n)\, \varepsilon_n(4)^{1/2(n-1)} \end{aligned}} \quad (\ast\ast)$$

Thus

by $(\ast)$ & $(\ast\ast)$ we prove step 3 namely

$\forall x \in C \cap \partial E$ $\quad q(x) - t_0 \leq c(n)\, \varepsilon_n(4)^{1/2(n-1)}$

applying the same argument for $\mathbb{R}^n \setminus E \in \mathcal{O}(C C_A, 4)$ we conclude that

$$t_0 - g(x) \leq c(n)\, e(4)^{1/2(n-1)} \qquad \forall x \in C \cap \partial E \qquad \blacksquare$$

Theorem : (Lipschutz approximation)

Given $n \geq 2$ & $C_A > 0$ $\exists\, \varepsilon_1 = \varepsilon_1(n, C_A)$, $C_1 = C_1(n, C_A)$ and $\delta_0 = \delta_0(n, C_A) > 0$ s.t if $E \in \mathcal{A}(C_A, 16 r_0)$ for some $r_0 > 0$ $x_0 \in \partial E$ and

$$e_n(x_0, 8r_0) = e(E, x_0, 8r_0, e_n) \leq \varepsilon_1$$

then if $M = C(x_0, r_0) \cap \partial E$ &

$$M_0 = \left\{ y \in M : \sup_{0 < s < 8r_0} e_n(y, s) \leq \delta_0 \right\}$$

there exists a Lipschutz function $u : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ with

$$\sup_{\mathbb{R}^{n-1}} \frac{|u|}{r_0} \leq C_1\, e_n(x_0, 8r_0)^{1/2(n-1)} \quad \text{s.t}$$

$\operatorname{Lip} u \leq 1$

$$M_0 \subset M \cap \Gamma \qquad \Gamma = x_0 + \{ (x, u(x)) : x \in D_{r_0} \}$$

i) $$\mathcal{H}^{n-1}(M \Delta \Gamma) / r_0^{\,n-1} \leq C_1\, e_n(x_0, 8r_0)$$

Γ covers a large portion of M on C_{r_0}

ii) $$\frac{1}{r_0^{\,n-1}} \int_{D_{r_0}} |\nabla' u|^2 \leq C_1\, e_n(x_0, 8r_0)$$

a large piece has small Lipschitz constant