Theorem : (Lipschitz approximation)

Given $n \geq 2$ & $C_A > 0$ $\exists\, \varepsilon_1 = \varepsilon_1(n, C_A)$, $C_1 = C_1(n, C_A)$ and $\delta_0 = \delta_0(n, C_A) > 0$ s.t if $E \in \mathcal{A}(C_A, 16 r_0)$ for some $r_0 > 0$ $x_0 \in \partial E$ and

$$e_n(x_0, 8r_0) = e(E, x_0, 8r_0, e_n) \leq \varepsilon_1$$

then if $M = C(x_0, r_0) \cap \partial^* E$ &

$$M_0 = \{ y \in M : \sup_{0 < s < 4r_0} e_n(y, s) \leq \delta_0 \}$$

there exists a Lipschitz function $u : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ with

$$\sup_{\mathbb{R}^{n-1}} \frac{|u|}{r_0} \leq C_1\, e_n(x_0, 8r_0)^{1/2(n-1)} \quad \text{s.t} \qquad \mathrm{Lip}\, u \leq 1$$

$$M_0 \subset M \cap \Gamma \qquad \Gamma = x_0 + \{ (x, u(x)) : x \in D_{r_0} \}$$

i) $\mathcal{H}^{n-1}(M \Delta \Gamma) / r_0^{n-1} \leq C_1\, e_n(x_0, 8r_0)$ $\quad$ Γ covers a large portion of M on C_{r_0}

ii) $\dfrac{1}{r_0^{n-1}} \displaystyle\int_{D_{r_0}} |\nabla' u|^2 \leq C_1\, e_n(x_0, 8r_0)$ $\quad$ a large piece has small Lipschitz constant

Pf : as before replace E by E_{x_0,r_0} we can assume that

$E \in \mathcal{A}(C_A, 8)$ $\quad 0 \in \partial E$ $\quad e_n(8) \leq \varepsilon_1$; we need to show

that if $M_0 = \{ y \in M : \sup\limits_{0<s<8} e_n(y,s) \leq \delta_0(n) \}$; $M = C \cap \partial E$

$\exists\, u : \mathbb{R}^n \to \mathbb{R}$ $\quad \mathrm{Lip}\, u \leq 1$ $\quad \sup\limits_{\mathbb{R}^{n-1}} |u| \leq C_1\, e_n(8)^{1/2(n-1)}$

$M_0 \subset M \cap \Gamma$, $\Gamma = \mathrm{graph}\, u|_D$ &

i) $\mathcal{H}^{n-1}(M \Delta \Gamma) \leq C_1\, e_n(8)$

ii) $\displaystyle\int_D |\nabla' u|^2 \leq C_1\, e_n(8)$

Assume $\varepsilon_1 \leq \varepsilon_0(n, C_A) \leq \omega(n, 1/4)$ by the height bound

$\sup \{ \langle x, e_n \rangle : x \in C_2 \cap \partial E \} \leq C_0\, e_n(8)^{1/2(n-1)}$

Moreover for $G \subset D = B_1 \cap \mathbb{R}^{n-1} \times \{0\}$ Borel

$$0 \leq \mathcal{H}^{n-1}(M \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G) \leq e_n(1) \leq 8^{n-1} e_n(8)$$

since $\varepsilon_0 \leq \omega(n, 1/4)$

$$C_2 \cap E \subset \{x \in C_2 \,;\, q(x) < 1/4\}$$

Step 2: ① M_0 is contained in a Lipschutz graph $\Gamma = \operatorname{graph} u$

② $\mathcal{H}^{n-1}(M \setminus M_0) \leq c\, e_n(8)$

③ $\mathcal{H}^{n-1}(M \Delta \Gamma) \leq c\, e_n(8)$

Step 3

$$\int_D |D' u|^2 \leq c\, e_n(8)$$

Pf of step 2, let $y \in M_0$, $e_n(y, s) \leq \delta_0$ for all $s \in (0, 4)$. Let $x \in M$, $x \neq y$ with $\|x - y\| < 1$

thus $e_n(y, 4\|x-y\|) < \delta_0$ since $y \in M_0$ then provided that $\delta_0 < \varepsilon_0(n, C_A)$ by the height bound since

$x \in C(y, \|x-y\|, e_n) \cap \partial E$ then

$$|q(x) - q(y)| \leq C_0 \, e_n(y, 4\|x-y\|)^{1/2(n-1)} \|x-y\|$$

$$|q(x) - q(y)| \leq C_0 \, \delta_0^{1/2(n-1)} \|x-y\|$$

$L = 2C_0 \delta_0^{1/2(n-1)}$ choose δ_0 small enough so $L < 1/2$

then

$$|q(x) - q(y)| \leq C_0 \delta_0^{1/2(n-1)} \left(|p(x) - p(y)| + |q(x) - q(y)|\right)$$

p projection onto $\langle e_n \rangle^{\perp}$ thus

check that for $|x-y| \geq 1$ also holds if

(1) $\boxed{|q(x) - q(y)| \leq 2C_0 \delta_0^{1/2(n-1)} |p(x) - p(y)| < \frac{1}{2} |p(x) - p(y)|}$ ε_1 small enough

Define $u : p(M_0) \longrightarrow \mathbb{R}$ $u(\bar{x}) = q(x)$ for $x \in M_0$

s.t $p(x) = \bar{x}$

Check: i) is u well defined

ii) u Lipschitz

i) assume $\exists\, x, z \in M_0$ s.t $p(x) = p(z) = \bar{x}$ then

then by (1)

$$|q(x) - q(z)| < \frac{1}{2} |p(x) - p(z)| = 0$$

thus $u(\bar{x}) = q(x) = q(z)$ well defined

take $\bar{x}, \bar{y} \in p(M_0)$ s.t $p(x) = \bar{x}$ & $p(y) = \bar{y}$

$$|u(\bar{x}) - u(\bar{y})| = |q(x) - q(y)| \le 2C_0\, \delta_0^{1/2(n-1)} |p(x) - p(y)|$$

$$\le \underbrace{2C_0\, \delta_0^{1/2(n-1)}}_{L} |\bar{x} - \bar{y}|$$

Since $M_0 \subset M$ by the height bound for $x \in M_0$

$$|q(x)| = |u(p(x))| \le C_0\, \text{en}(8)^{1/2(n-1)}$$

Thus u satisfies the properties we want on $p(M_0)$. We now extend u to a function on all of $\mathbb{R}^{n-1}$ as follows for $z \in \mathbb{R}^{n-1}$ let $\bar{u}(z) = \min_{y \in p(\Gamma_0)} \{ u(y) + L|z-y| \}$ $\quad L = 2C_0 \delta_0^{1/2(n-1)}$

$\bar{u}$ is Lipschitz with constant $L < 1/2$

Extend u by modifying $\bar{u}$ has follows

$$\hat{u}(z) = \begin{cases} \min\{ \bar{u}(z), C_0\, en(8)^{1/2(n-1)} \} & \text{if } \bar{u}(z) \geq 0 \\ -\min\{ -\bar{u}(z), C_0\, en(8)^{1/2(n-1)} \} & \text{if } \bar{u}(z) < 0 \end{cases}$$

Claim $\hat{u}(z)$ is a Lipschitz extension of u and satisfies

$$\sup |\hat{u}| \leq C_0\, en(8)^{1/2(n-1)} \qquad \text{(check)}$$

Moreover $M_0 \subset \Gamma = \text{graph } u$ ①

For $y \in M \setminus M_0$ $\exists\, s \in (0,4)$ such that $en(y,s) > \delta_0$

Thus $\quad \delta_0 \, s^{n-1} \leq \displaystyle\int_{C(y,s,e_n)\cap\partial^* E} |\nu_E - e_n|^2/2 \, d\mathcal{H}^{n-1}$

note $\underbrace{C(y,s,e_n)}_{C(y,s)} \subset B(y,\sqrt{2}s) \subset C(0,8,e_n)$

Using Besicovitch there exist N_n disjoint families s.t

$\{B(y_k^l, \sqrt{2}\, s_k^l)\}_{\substack{y_k^l \in M\setminus M_0 \\ s_k^l \in (0,4)}}$ s.t

$$\delta_0 (s_k^l)^{n-1} \leq \int_{C(y_k, s_k)} \tfrac{1}{2} |\nu_E - e_n|^2 \, d\mathcal{H}^{n-1}$$

&

$$M \setminus M_0 \subset \bigcup_{l=1}^{N} \bigcup_k B(y_k^l, \sqrt{2}\, s_k^l)$$

$$\mathcal{H}^{n-1}(M\setminus M_0) \leq \sum_{k,l} \mathcal{H}^{n-1}(M\setminus M_0 \cap B(y_k^l, \sqrt{2}\, s_k))$$

$$\leq \sum_{l}^{N} \sum_{k} \mathcal{H}^{n-1}(M \cap B(y_k^l, \sqrt{2}, s_k^l))$$

$$\leq \sum_{l=1}^{N} C_A (\sqrt{2})^{n-1} \sum_k (s_k^l)^{n-1}$$

$$\mathcal{H}^{n-1}(M \setminus M_0) \leq \frac{C(n, C_A)}{\delta_0} \sum_{l=1}^{N} \sum_{k} \frac{1}{2} \int_{\underbrace{C(y_k^l, s_k^l)}_{\text{disjoint}} \cap \partial^* E} |\nu_E - e_n|^2 \, d\mathcal{H}^{n-1}$$

$$\leq N\, C(n, C_A) \int_{C(0, 8, e_n) \cap \partial^* E} \frac{1}{2} |\nu_E - e_n|^2 \, d\mathcal{H}^{n-1}$$

$$\leq C(n, C_A) \, e_n(8)$$