Theorem : (Lipschutz approximation)

Given $n \geq 2$ & $C_A > 0$ $\exists\, \varepsilon_1 = \varepsilon_1(n, C_A)$, $C_1 = C_1(n, C_A)$ and $\delta_0 = \delta_0(n, C_A) > 0$ s.t if $E \in \mathcal{A}(C_A, 16 r_0)$ for some $r_0 > 0$ $x_0 \in \partial E$ and

$$e_n(x_0, 8r_0) = e(E, x_0, 8r_0, e_n) \leq \varepsilon_1$$

then if $M = C(x_0, r_0) \cap \partial E$ &

$$M_0 = \left\{ y \in M : \sup_{0 < s < 4r_0} e_n(y, s) \leq \delta_0 \right\}$$

there exists a Lipschutz function $u : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ with

$$\sup_{\mathbb{R}^{n-1}} \frac{|u|}{r_0} \leq C_1\, e_n(x_0, 8r_0)^{1/2(n-1)} \quad \text{s.t} \qquad \mathrm{Lip}\, u \leq 1$$

$$M_0 \subset M \cap \Gamma \qquad \Gamma = x_0 + \{ (x, u(x)) : x \in D_{r_0} \}$$

i) $\mathcal{H}^{n-1}(M \Delta \Gamma) / r_0^{n-1} \leq C_1\, e_n(x_0, 8r_0)$ Γ covers a large portion of M on C_{r_0}

ii) $\displaystyle \frac{1}{r_0^{n-1}} \int_{D_{r_0}} |\nabla' u|^2 \leq C_1\, e_n(x_0, 8r_0)$ a large piece has small Lipschitz constant and

$$\mathrm{dist}\,(x, (p(x), u(p(x))) = |q(x) - u(p(x))| \leq 2\ \mathrm{dist}\,(p(x), p(M_0))$$

$\forall x \in M$

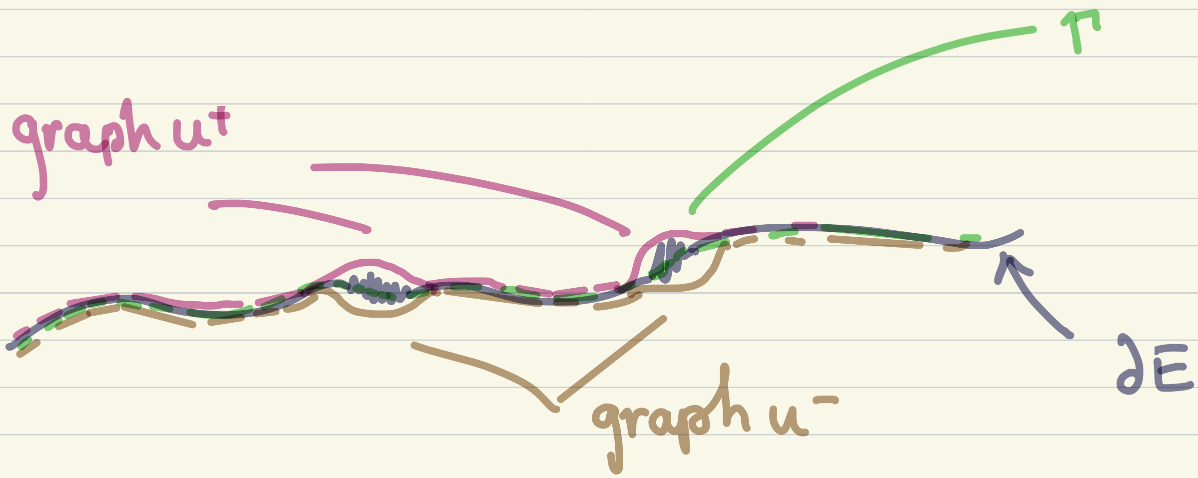
This ensures that there exists Lipschitz functions $u_\pm$ defined by

$$u_+(x) = \begin{cases} u(x) & \text{if } x \in p(M_0) \\ \inf\limits_{y \in p(M_0)} \{u(y) + 2|x-y|\} & x \in D \setminus p(M_0) \end{cases}$$

$$u_-(x) = \begin{cases} u(x) & \text{if } x \in p(M_0) \\ \sup\limits_{y \in p(M_0)} \{u(y) - 2|x-y|\} & x \in D \setminus p(M_0) \end{cases}$$

with the property that

$$u_-(p(x)) \leq q(x) \leq u_+(x) \qquad \forall x \in M$$


graph u^+
Γ
∂E
graph u^-

Pf : as before replace E by E_{x_0,r_0} we can assume that

$E \in \mathcal{A}(c_A, 8)$ $\quad 0 \in \partial E$ $\quad$ en$(8) \le \varepsilon_1$; we need to show

that if $M_0 = \{ y \in M : \sup_{0<s<4} \text{en}(y,s) \le \delta_0(n) \}$; $M = C \cap \partial E$

$\exists\, u : \mathbb{R}^n \to \mathbb{R}$ $\quad \text{Lip}\, u \le 1 \quad \sup_{\mathbb{R}^{n-1}} |u| \le C_1\, \text{en}(8)^{1/2(n-1)}$

$M_0 \subset M \cap \Gamma$, $\Gamma = \text{graph}\, u|_D$ &

i) $\mathcal{H}^{n-1}(M \Delta \Gamma) \le C_1\, \text{en}(8)$

ii) $\displaystyle\int_D |\nabla' u|^2 \le C_1\, \text{en}(8)$

Assume $\varepsilon_1 \le \varepsilon_0(n, c_A) \le \omega(n, 1/4)$ by the height bound

$\sup \{ \langle x, e_n \rangle : x \in C_2 \cap \partial E \} \le C_0\, \text{en}(8)^{1/2(n-1)}$

Moreover for $G \subset D = B_1 \cap \mathbb{R}^{n-1} \times \{0\}$ Borel

$$0 \leq \mathcal{H}^{n-1}(M \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G) \leq e_n(1) \leq 8^{n-1} e_n(8)$$

since $\varepsilon_0 \leq \omega(n, 1/4)$

$$C_2 \cap E \subset \{x \in C_2 \, ; \, q(x) < 1/4\}$$

Step 2: ① M_0 is contained in a Lipschitz graph $\Gamma = \text{graph}\, u$

② $\mathcal{H}^{n-1}(M \setminus M_0) \leq c\, e_n(8)$

③ $\mathcal{H}^{n-1}(M \Delta \Gamma) \leq c\, e_n(8)$

Step 3 $\displaystyle\int_D |D'u|^2 \leq c\, e_n(8)$

Note about ②

For $y \in M \setminus M_0$ $\exists\, s \in (0,4)$ such that $e_n(y,s) > \delta_0$

Thus $\delta_0 \cdot s^{n-1} \leq \int_{C(y,s,e_n)\cap\partial^* E} |\nu_E - e_n|^2/2 \, d\mathcal{H}^{n-1}$

note $\underbrace{C(y,s,e_n)}_{C(y,s)} \subset B(y,\sqrt{2}s) \subset C(0,8,e_n)$

Using Besicovitch there exist N_n disjoint families s.t

$\{B(y_k^l, \sqrt{2}s_k^l)\}_{\substack{y_k^l \in M\setminus M_0 \\ s_k^l \in (0,4)}}$ s.t

$$\delta_0 (s_k^l)^{n-1} \leq \int_{C(y_k,s_k)} \tfrac{1}{2}|\nu_E - e_n|^2 \, d\mathcal{H}^{n-1}$$

& $M\setminus M_0 \subset \bigcup_{l=1}^{N} \bigcup_k B(y_k^l, \sqrt{2}s_k^l)$

$$\mathcal{H}^{n-1}(M\setminus M_0) \leq \sum_{k,l} \mathcal{H}^{n-1}(M\setminus M_0 \cap B(y_k^l, \sqrt{2}s_k))$$

$$\leq \sum_l^N \sum_k \mathcal{H}^{n-1}(M \cap B(y_k^l, \sqrt{2}, s_k^l))$$

$$\leq \sum_{l=1}^N C_A (\sqrt{2})^{n-1} \sum_k (s_k^l)^{n-1}$$

$$\mathcal{H}^{n-1}(M \setminus M_0) \leq \frac{C(n, C_A)}{\delta_0} \sum_{l=1}^{N} \sum_{k} \frac{1}{2} \int_{\underbrace{C(y_k^l, s_k^l)}_{\text{disjoint}} \cap \partial^* E} |\nu_E - e_n|^2 \, d\mathcal{H}^{n-1}$$

$$\leq N\, C(n, C_A) \int_{C(0, 8, e_n) \cap \partial^* E} \frac{1}{2} |\nu_E - e_n|^2 \, d\mathcal{H}^{n-1}$$

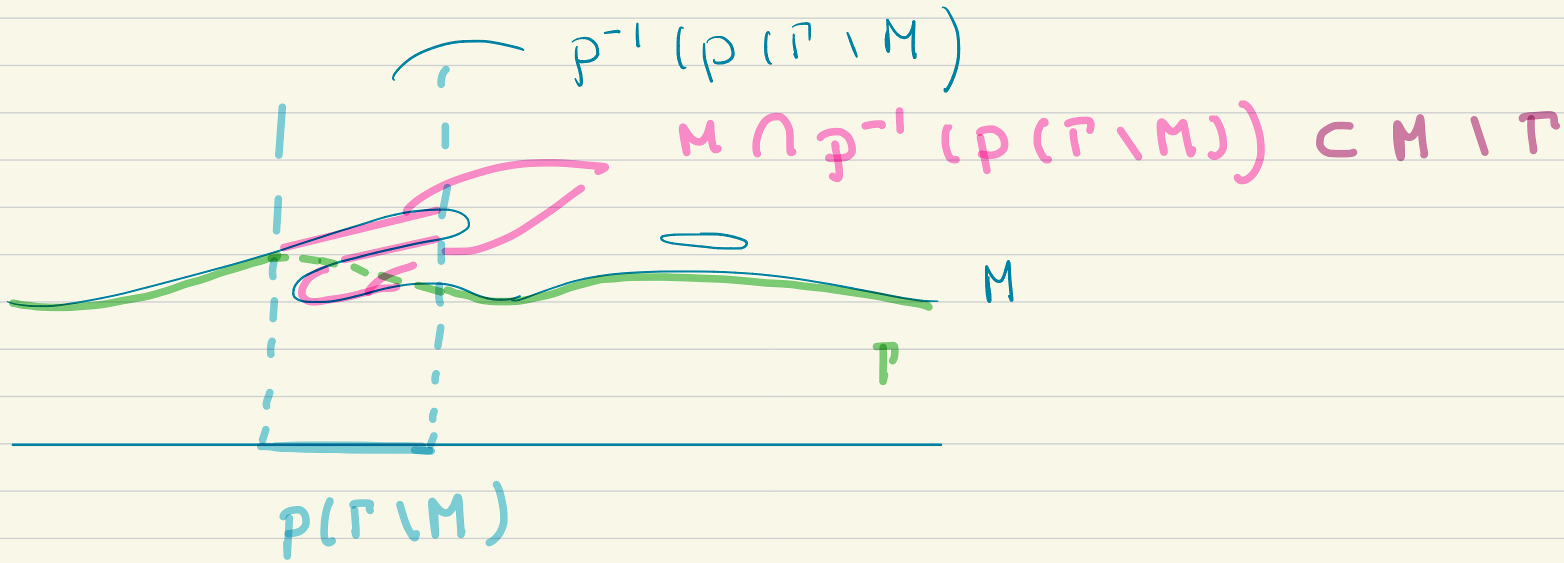
$$\leq C(n, C_A) \, e_n(8)$$

$$\mathcal{H}^{n-1}(M \setminus \Gamma) \leq \mathcal{H}^{n-1}(M \setminus M_0) \leq C(n, C_A) \, e_n(8) \qquad (2)$$

To bound $\mathcal{H}^{n-1}(\Gamma \setminus M)$ recall how to compute the area of a graph

$$\mathcal{H}^{n-1}(\Gamma \setminus M) = \int_{p(\Gamma \setminus M)} \sqrt{1 + |D'u|^2} \, d\bar{x} \leq \sqrt{1 + (\mathrm{Lip}\, u)^2} \, \mathcal{H}^{n-1}(p(\Gamma \setminus M))$$

$$\leq \sqrt{2} \, \mathcal{H}^{n-1}(p(\Gamma \setminus M)) \overset{\text{excess mass lemma}}{\leq} \sqrt{2} \, \mathcal{H}^{n-1}(M \cap p^{-1}(p(\Gamma \setminus M)))$$



$$\mathcal{H}^{n-1}(\Gamma \setminus M) \leq \sqrt{2}\, \mathcal{H}^{n-1}(M \setminus \Gamma) \leq c(n)\, \varepsilon_n(8)$$

$$\Rightarrow \quad \mathcal{H}^{n-1}(\Gamma \,\Delta\, M) \leq c(n)\, \varepsilon_n(8) \qquad (3)$$

Pf of Step 3: For $x \in M \cap \Gamma$ let $\lambda(x) \in \{-1, 1\}$ be such that

$$\nu_E(x) = \lambda(x) \frac{(-\nabla' u(p(x)), 1)}{\sqrt{1 + |\nabla' u(p(x))|^2}}$$

$$\frac{|\nu_E - e_n|^2}{2} = 1 - \langle \nu_E, e_n \rangle \geq \frac{1}{2}\left(1 - \langle \nu_E, e_n\rangle^2\right) \geq \frac{1}{2}|p(\nu_E)|^2$$

$$(1 - \langle \nu_E, e_n\rangle)^2 \geq 0$$

$$p(\nu_E) = \nu_E - \langle \nu_E, e_n \rangle e_n$$

and

$$\langle \nu_E, e_n \rangle = \lambda(x) \frac{1}{\sqrt{1 + |\nabla' u(p(x))|^2}}$$

$$e_n(0,1) \geq \frac{1}{2}\int_{M\cap\Gamma} |p(\nu_E)|^2 \, d\mathcal{H}^{n-1} \geq \frac{1}{2}\int_{M\cap\Gamma} \frac{|\nabla' u(p(x))|^2}{1 + |\nabla' u(p(x))|^2} \, d\mathcal{H}^{n-1}$$

$$\geq \frac{1}{2}\int_{p(M\cap\Gamma)} \frac{|\nabla' u(z)|^2}{\sqrt{1 + |\nabla' u(z)|^2}} \, dz$$

$\mathrm{Lip}\, u \leq 1$

$$\geq \frac{1}{2\sqrt{2}}\int_{p(M\cap\Gamma)} |\nabla' u(z)|^2 \, dz$$

$$\int_D |\nabla' u|^2 = \int_{p(\Gamma \cap \Gamma)} |\nabla' u|^2 + \int_{p(M \Delta \Gamma)} |\nabla' u|^2 \qquad \Gamma = \operatorname{graph} u|_D$$

$$\int_{p(\Gamma \Delta \Gamma)} |\nabla' u|^2 \leq (4 n u)^2 \, \mathcal{H}^{n-1}(p(M \Delta \Gamma)) \qquad J_p \leq 1$$

$$\leq \mathcal{H}^{n-1}(M \Delta \Gamma) \leq C\, e_n(8)$$

thus

$$\int_D |\nabla' u|^2 \, dz \leq C\, e_n(1) + C\, e_n(8) \leq C_n\, e_n(8) \qquad \blacksquare$$