Theorem (FIRST VARIATION OF PERIMETER)

If $A \subset \mathbb{R}^n$ is an open set, E a set of locally finite perimeter, and $\{f_t\}_{|t|<\varepsilon}$ is a local variation in A with initial velocity T for $\{f_t\}_{|t|<\varepsilon}$ then

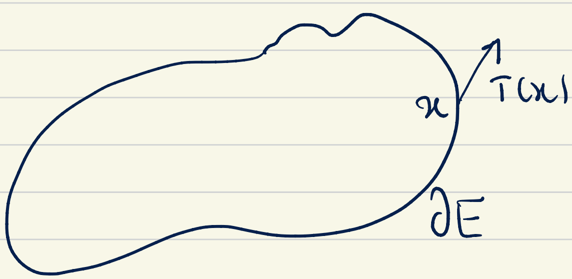
$$|\mu_{f_t(E)}|(A) = |\mu_E|(A) + t \int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} + O(t^2)$$

where $\mathrm{div}_E T : \partial^* E \longrightarrow \mathbb{R}$

$$\mathrm{div}_E T(x) = \mathrm{div}\, T(x) - \nu_E(x) \cdot \nabla T(x) \nu_E(x)$$

Boundary divergence of T on E.

Remark: If E is a C^2 domain in $\mathbb{R}^n$



$$\mathrm{div}_E T = \mathrm{div}^{\partial E} T$$

$$\int_{\partial E} \mathrm{div}^{\partial E} T \, d\mathcal{H}^{n-1} = \int_{\partial E} \langle T, \vec{H}_{\partial E} \rangle \, d\mathcal{H}^{n-1} \quad (\partial(\partial E) = \emptyset)$$

Thus

$$\frac{d}{dt} |\mu_{f_t(E)}|(A) \Big|_{t=0} = \int_{\partial E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial E} \langle T, \vec{H}_{\partial E} \rangle \, d\mathcal{H}^{n-1}$$

Pf of theorem: By the proposition $g_t = f_t^{-1}$

$$|\mu_{f_t(E)}|(A) = \int_{A \cap \partial^* E} Jf_t \, |(\nabla g_t \circ f_t)^* \nu_E| \, d\mathcal{H}^{n-1}$$

$$\nabla f_t = id + t \nabla T + o(t^2)$$

$$\nabla g_t \circ f_t = (\nabla f_t)^{-1} = (id + t\nabla T + o(t^2))^{-1}$$

$$= id - t\nabla T + o(t^2)$$

$$Jf_t = \det(id + t\nabla T + o(t^2)) = 1 + t \,\mathrm{div}\, T + o(t^2)$$

$$|(\nabla g_t \circ f_t)^* \nu_E|^2 = |\nu_E - t\nabla T(\nu_E)|^2 + o(t^2)$$

$$= 1 - 2t\langle \nabla T(\nu_E), \nu_E \rangle + o(t^2)$$

thus since $\sqrt{1+\eta} = 1 + \frac{\eta}{2} + o(\eta^2)$ for $|\eta|$ small

$$|(\nabla g_t \circ f_t)^* \nu_E| = 1 - t \langle \nabla T(\nu_E), \nu_E \rangle + o(t^2)$$

$$Jf_t \, |(\nabla g_t \circ f_t)^* \nu_E| = (1 + t \operatorname{div} T)(1 - t\langle \nabla T(\nu_E), \nu_E \rangle + o(t^2))$$

$$= 1 + t \operatorname{div} T - t \langle \nabla T(\nu_E), \nu_E \rangle + o(t^2)$$

$$|\mu_{f_t(E)}|(A) = |\mu_{f(E)}|(A) + t \int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} + o(t^2).$$

$\blacksquare$

<u>Question</u>: If $E = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t > u(x)\}$

$u \in C^2$, compute H_E. If $\bar{x} = (x,t)$ $\tau_1(\bar{x}) \ldots \tau_{n-1}(\bar{x})$ o.n basis for $T_{\bar{x}} \partial E$ then

$$v_i = \left(e_i + \frac{\partial u}{\partial x_i} e_n\right) \qquad \nu_{\partial E}(\bar{x}) = \frac{(-Du, 1)}{\sqrt{1+|Du|^2}}$$

Find H_E

Stationary sets & Monotonicity of density ratios

Definition: We say that a set of locally finite perimeter E is stationary for perimeter in an open set A if

(1) $\operatorname{spt} \mu_E = \partial E$

(2) $\frac{d}{dt} |\mu_{f_t(E)}|(A) \Big|_{t=0} = 0$ whenever $\{f_t\}_{|t|<\varepsilon}$ is a local variation in A.

Remark: A perimeter minimizer in A is stationary for perimeter in A.

Note that if E is a set of locally finite perimeter with bounded 1^{st} variation in the sense that

$$\sup \left\{ \int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} : T \in C_c^1(A, \mathbb{R}^n) \ \|T\|_\infty \leq 1 \right\} < \infty$$

then by the Riesz Representation Theorem there exists a vector valued measure $\vec{H}_E$ such that

$$\int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial^* E} T \cdot d\vec{H}_E$$

$\vec{H}_E$ generalized mean curvature

note that $|\vec{H}_E| \ll \mathcal{H}^{n-1} \llcorner \partial^* E$

<u>Definition</u>: We say that a set E of l.f.p has distributional mean curvature $\vec{H}_E$ in A if $\vec{H}_E = H_E \nu_E$ for some $H_E \in L^1_{loc}(A \cap \partial^* E; \mathcal{H}^{n-1})$ s.t

$$\boxed{\int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial^* E} \langle T, \nu_E \rangle H_E \, d\mathcal{H}^{n-1}} \quad \forall T \in C^1_c(A, \mathbb{R}^n)$$

<u>Proposition</u>: A set of l.f.p E is stationary for perimeter in an open set A iff

$$\int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} = 0 \qquad \forall T \in C^1_c(A, \mathbb{R}^n)$$

$$(\iff H_E \equiv 0)$$

Monotonicity formula

Assume $\underline{E}$ set l.f.p with generalized mean curvature in A

let $\overline{B_\rho(\xi)} \subset A$, consider $T(x) = \gamma(r)(x-\xi)$ with

$r = |x-\xi|$ $\quad \gamma \in C^1(\mathbb{R})$ s.t

$$\chi_{B_{\rho/2}(\xi)} \le \gamma \le \chi_{B_\rho(\xi)} \qquad \gamma'(t) \le 0, \quad \text{apply}$$

$$\int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial^* E} \langle T, \nu_E \rangle H_E \, d\mathcal{H}^{n-1}$$

$$\nabla T(x) = \gamma(r)\, \mathrm{id} + \gamma'(r) \frac{(x-\xi)}{|x-\xi|} \otimes (x-\xi)$$

$$\operatorname{div}_E T = \operatorname{div} T - \langle \nu_E, \nabla T(\nu_E) \rangle$$

$$= n\gamma(r) + \gamma'(r) r - \gamma(r) - \gamma'(r) r \left\langle \frac{x-\xi}{r}, \nu_E \right\rangle^2$$

$$= (n-1)\gamma(r) + r\gamma'(r) - r\gamma'(r) \left\langle \frac{x-\xi}{r}, \nu_E \right\rangle^2$$

Since
$$\int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial^* E} \langle T, \vec{H}_E \rangle \, d\mathcal{H}^{n-1}$$

$$(n-1) \int_{\partial^* E} \gamma(r) \, d\mathcal{H}^{n-1} + \int_{\partial^* E} r\gamma'(r) = \int_{\partial^* E} H_E \, \gamma(r) \langle x-\xi, \nu_E \rangle \, d\mathcal{H}^{n-1}$$
$$+ \int_{\partial^* E} r\gamma'(r) \left\langle \frac{x-\xi}{r}, \nu_E \right\rangle^2$$

Let $\phi(t) = 1 \quad t \leq 1/2 \qquad \phi(t) = 0 \quad$ if $\quad t \geq 1$

$\phi'(t) \leq 0 \qquad \gamma(r) = \phi\left(\frac{r}{\rho}\right) \qquad \overline{B_\rho(\xi)} \subset A$

$$r\gamma'(r) = \frac{r}{\rho} \phi'\left(\frac{r}{\rho}\right) = -\rho \frac{d}{d\rho} \phi\left(\frac{r}{\rho}\right)$$

Thus if

$$I(\rho) = \int_{\partial^* E} \phi\left(\frac{r}{\rho}\right) d\mathcal{H}^{n-1} \qquad J(\rho) = \int_{\partial^* E} \phi\left(\frac{r}{\rho}\right) \left\langle \frac{x-\xi}{r}, \nu_E \right\rangle^2 d\mathcal{H}^{n-1}$$

$$L(\rho) = \int_{\partial^* E} \phi\left(\frac{r}{\rho}\right) \langle x-\xi, \vec{H}_E \rangle \, d\mathcal{H}^{n-1}$$ we have

$$\boxed{(n-1)\, I(\rho) - \rho \frac{d}{d\rho} I(\rho) = -\rho \frac{d}{d\rho} J(\rho) + L(\rho)}$$