Monotonicity formula

Assume E set l.f.p with generalized mean curvature in A
let $\overline{B_\rho(\xi)} \subset A$, consider $T(x) = \gamma(r)(x-\xi)$ with
$r = |x-\xi|$ $\gamma \in C^1(\mathbb{R})$ s.t

$$\chi_{B_{\rho/2}(\xi)} \leq \gamma \leq \chi_{B_\rho(\xi)} \qquad \gamma'(t) \leq 0 \,, \quad \text{apply}$$

$$\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial^* E} \langle T, \nu_E \rangle H_E \, d\mathcal{H}^{n-1}$$

$$\nabla T(x) = \gamma(r)\, \mathrm{id} + \gamma'(r) \frac{(x-\xi)}{|x-\xi|} \otimes (x-\xi)$$

$$\mathrm{div}_E T = \mathrm{div}\, T - \langle \nu_E , \nabla T(\nu_E) \rangle$$

$$= n\gamma(r) + \gamma'(r) r - \gamma(r) - \gamma'(r) r \left\langle \frac{x-\xi}{r}, \nu_E \right\rangle^2$$

$$= (n-1)\gamma(r) + r\gamma'(r) - r\gamma'(r) \left\langle \frac{x-\xi}{r}, \nu_E \right\rangle^2$$

Since
$$\int_{\partial^* E} \operatorname{div}_E T \, d\mathcal{H}^{n-1} = \int_{\partial^* E} \langle T, \vec{H}_E \rangle \, d\mathcal{H}^{n-1}$$

$$(n-1)\int_{\partial^* E} \gamma(r)\, d\mathcal{H}^{n-1} + \int_{\partial^* E} r\gamma'(r) = \int_{\partial^* E} H_E\, \gamma(r) \langle x-\xi, \nu_E \rangle \, d\mathcal{H}^{n-1}$$

$$+ \int_{\partial^* E} r\gamma'(r) \left\langle \frac{x-\xi}{r}, \nu_E \right\rangle^2$$

Let $\phi(t) = 1 \quad t \leq 1/2 \qquad \phi(t) = 0 \quad \text{if} \quad t \geq 1$

$\phi'(t) \leq 0 \qquad \gamma(r) = \phi\left(\frac{r}{\rho}\right) \qquad \overline{B_\rho(\xi)} \subset A$

$$r\gamma'(r) = \frac{r}{\rho}\, \phi'\left(\frac{r}{\rho}\right) = -\rho \frac{d}{d\rho} \phi\left(\frac{r}{\rho}\right)$$

Thus if

$$I(\rho) = \int_{\partial^* E} \phi\left(\frac{r}{\rho}\right) d\mathcal{H}^{n-1} \qquad J(\rho) = \int_{\partial^* E} \phi\left(\frac{r}{\rho}\right) \left\langle \frac{x-\xi}{r}, \nu_E \right\rangle^2 d\mathcal{H}^{n-1}$$

$$L(\rho) = \int_{\partial^* E} \phi\left(\frac{r}{\rho}\right) \langle x-\xi, \vec{H}_E \rangle \, d\mathcal{H}^{n-1}$$ we have

$$\boxed{(n-1)\, I(\rho) - \rho \frac{d}{d\rho} I(\rho) = -\rho \frac{d}{d\rho} J(\rho) + L(\rho)}$$

multiply by ρ^{-n}

$$(n-1)\rho^{-n} I(\rho) - \rho^{-(n-1)} \frac{d}{d\rho} I(\rho) = -\rho^{-(n-1)} \frac{d}{d\rho} J(\rho) + \rho^{-n} L(\rho)$$

$$-\frac{d}{d\rho}\left(\rho^{-(n-1)} I(\rho)\right) = -\rho^{-(n-1)} \frac{d}{d\rho} J(\rho) + \rho^{-n} L(\rho)$$

(A) $$\boxed{\frac{d}{d\rho}\left[\rho^{-(n-1)} I(\rho)\right] = \rho^{-(n-1)} \frac{d}{d\rho} J(\rho) - \rho^{-n} L(\rho)}$$

let $\phi \longrightarrow \chi_{(-\infty,1)}$ in the distribution sense we obtain at

$$\mu = \mathcal{H}^{n-1} \llcorner \partial^* E$$

$$(**) \quad \boxed{\frac{d}{d\rho}\left[\frac{\mu(B_\rho(\xi))}{\rho^{n-1}}\right] = \frac{d}{d\rho}\int_{B_\rho(\xi)} \left\langle \frac{x-\xi}{|x-\xi|}, \nu_E\right\rangle^2 \frac{1}{r^{n-1}}\, d\mu \\ - \rho^{-n}\int_{B_\rho(\xi)} \langle x-\xi, \vec{H}_E\rangle\, d\mu} \quad \text{a.e } \rho$$

$\frac{d}{d\rho}\left[\rho^{-(n-1)} I(\rho)\right]$ integration by parts against $\Psi \in C_c^1(\mathbb{R})$

$$\rho^{-(n-1)} \frac{d}{d\rho}\int_{\partial^* E} \phi\left(\frac{r}{\rho}\right) \left\langle \frac{x-\xi}{|x-\xi|}, \nu_E\right\rangle^2 d\mathcal{H}^{n-1}$$

a.e ρ

$$\longrightarrow \rho^{-(n-1)} \frac{d}{d\rho}\int_{\partial^* E \cap B_\rho(\xi)} \left\langle \frac{x-\xi}{|x-\xi|}, \nu_E\right\rangle^2 d\mathcal{H}^{n-1} = \rho^{-(n-1)}\int_{\partial B_\rho(\xi)} \left\langle \frac{x-\xi}{|x-\xi|}, \nu_E\right\rangle^2 d\mu$$

$$= \int_{\partial B_\rho(\xi)} \left\langle \frac{x-\xi}{|x-\xi|}, \nu_E \right\rangle^2 \frac{d\mu}{\rho^{n-1}} = \frac{d}{d\rho} \int_{B_\rho(\xi)} \left\langle \frac{x-\xi}{|x-\xi|}, \nu_E \right\rangle^2 \frac{d\mu}{r^{n-1}}$$

Theorem: If E is stationary for perimeter in an open set A, $\xi \in A$ then the density ratio for $\mu = \mathcal{H}^{n-1} \llcorner \partial^* E$

$$\frac{\mu(B_\rho(\xi))}{\omega_{n-1}\rho^{n-1}}$$ is increasing as a function of $\rho \in (0, \operatorname{dist}(\xi, \partial A))$

Pf: If $H_E = 0$ then $(\star\star)$ yields

$$\frac{d}{d\rho}\left(\frac{\mu(B_\rho(\xi))}{\rho^{n-1}} \right) = \frac{d}{d\rho} \underbrace{\int_{B_\rho(\xi)} \frac{1}{r^{n-1}} \left\langle \frac{x-\xi}{|x-\xi|}, \nu_E \right\rangle^2 d\mathcal{H}^{n-1}}_{\text{increasing function of } \rho} \geq 0$$

<u>Corollary</u> : If E is stationary for perimeter in an open set A then

$$\frac{\mu(B_\rho(\xi))}{\omega_{n-1}\rho^{n-1}} \geq 1 \qquad \forall \xi \in A \cap \partial E \qquad \forall r \in (0, \operatorname{dist}(\xi, \partial A))$$

in particular $\mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0$

Moreover

$$\Theta^{n-1}(\partial E, x) \geq 1 \qquad \forall x \in A \cap \partial E \qquad (*)$$

$$\Theta^{n-1}(\partial E, x) = 1 \qquad \forall x \in A \cap \partial^* E$$

<u>Pf</u>: By monotonicity $\forall \xi \in A \cap \partial^* E$ $\forall r \in (0, \operatorname{dist}(\xi, \partial A))$

$$\frac{\mu(B_r(\xi))}{\omega_{n-1} r^{n-1}} \geq \lim_{\rho \to 0^+} \frac{\mu(B_\rho(\xi))}{\omega_{n-1}\rho^{n-1}} = \Theta^{n-1}(\partial^* E, \xi) = 1$$

if $\xi \in (\partial E \setminus \partial^* E) \cap A$ $\exists \xi_k \in A \cap \partial^* E$ $\xi_k \to \xi$

given $\varepsilon > 0$ small for k large enough (by monotonicity density exists)

$$\mu(B_r(\xi)) \geq \mu(B_{r(1-\varepsilon)}(\xi_k)) \geq \omega_{n-1} r^{n-1}(1-\varepsilon)^{n-1} \qquad \forall \varepsilon > 0$$

$$\mu(B_r(\xi)) \geq \omega_{n-1} r^{n-1} \implies \Theta^{n-1}(\mu, \xi) \geq 1$$

$(*)$

$\Rightarrow \quad \mathcal{H}^{n-1}(A \cap \partial^* E) \geq \mathcal{H}^{n-1}(A \cap \partial E)$ bec $\mu = \mathcal{H}^{n-1} \llcorner \partial^* E$

$\Rightarrow \quad \mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0$

thus $\quad \Theta^{n-1}(\partial^* E, \xi) = \Theta^{n-1}(\mu, \xi) = \Theta^{n-1}(\partial E, \xi)$

$\forall \xi \in A \cap \partial E$ $\blacksquare$

Theorem: If $n \geq 2$, $A \subset \mathbb{R}^n$ open, E local perimeter minimizer in A then

(1) $A \cap \partial^* E$ is an analytic hypersurface with vanishing mean curvature

(2) $A \cap \partial^* E$ is relatively open in $A \cap \partial E$

The singular set $\Sigma(E, A) = A \cap (\partial E \setminus \partial^* E)$ satisfies

i) $2 \leq n \leq 7$ $\quad \Sigma(E, A) = \emptyset$

ii) $n = 8$ $\quad \Sigma(E, A)$ has no accummulation points

iii) $n \geq 9$ then $\dim_{\mathcal{H}} \Sigma(E, A) \leq n - 8$

These assertions are sharp, i.e there exists a perimeter minimizer in $\mathbb{R}^8$ (Simons cone) s.t. $\mathcal{H}^0(\Sigma(E, A)) = 1$. Moreover there exists perimeter minimizers in $\mathbb{R}^n$ s.t $\mathcal{H}^{n-8}(\Sigma(E, A)) = +\infty$

Step 1 : $A \cap \partial^* E$ is locally a $C^{1,\gamma}$ hypersurface $\forall \gamma \in (0,1)$

Step 2 : $A \cap \partial^* E$ is real analytic (standard PDE theory)

Step 3 : Structure of the singular set $\Sigma(E,A)$, we know $\mathcal{H}^{n-1}(\Sigma(E,A)) = 0$ to improve we need to study blow-ups $E_{x,r}$ at points $x \in \Sigma(E,A)$. They converge to cones (by the monotonicity formula). This cônes are global perimeter minimizers. (Federer reduction argument)

Theorem : Given $n \geq 2$, there exist positive constants $\varepsilon_1(n)$, $C_1(n)$ and $\delta_0(n)$ s.t if E is a local perimeter minimizer $r_0 > 0$ $x_0 \in \partial E$ and

$$e_n(x_0, 8r_0) = e(E, x_0, 8r_0, e_n) \leq \varepsilon_1 \quad \text{then if}$$

$$M = C(x_0, r_0) \cap \partial^* E \quad \& \quad M_0 = \left\{ y \in M : \sup_{0<s<4r_0} e_n(y,s) \leq \delta_0 \right\}$$

there exists a Lipschitz function $u : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ with $\operatorname{Lip} u \leq 1$

$$\sup_{\mathbb{R}^{n-1}} \frac{|u|}{r_0} \leq C_1 \, e_n(x_0, 8r_0)^{1/2(n-1)} \quad \text{s.t}$$

$$M_0 \subset M \cap \Gamma \qquad \Gamma = x_0 + \{ (x, u(x)) : x \in D_{r_0} \}$$

i) $$\mathcal{H}^{n-1}(M \Delta \Gamma) / r_0^{n-1} \leq C_1 \, e_n(x_0, 8r_0)$$

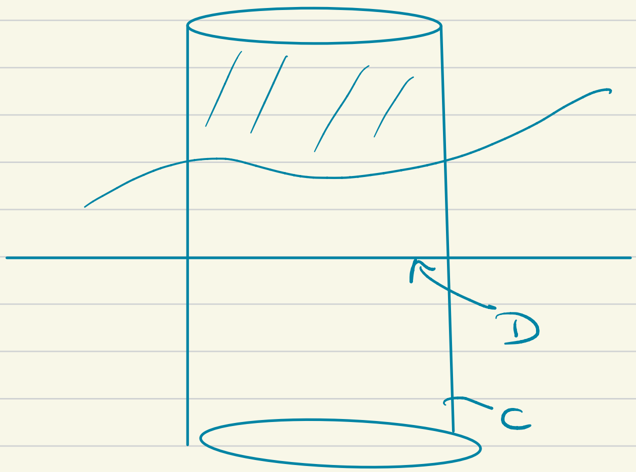
ii) $$\frac{1}{r_0^{n-1}} \int_{D_{r_0}} |\nabla' u|^2 \leq C_1 \, e_n(x_0, 8r_0)$$

u almost harmonic

$\forall \varphi \in C_c^1(D_{r_0})$

iii) $$\frac{1}{r_0^{n-1}} \left| \int_{D_{r_0}} \nabla' u \cdot \nabla' \varphi \right| \leq C_1 \sup_{D_{r_0}} |\nabla' \varphi| \, e_n(x_0, 8r_0)$$

Motivation for iii) : Consider $E \cap C$ = area above the graph of u over $D \subset \mathbb{R}^{n-1}$



area of $E \cap C$

$$\mathcal{F}(u) = \int_D \sqrt{1+|Du|^2}\,dx$$

Assume that E minimizes area among all domains above graphs over D such that $u = u_0$ on ∂D

$$\forall \xi \in C_c^\infty(D) \qquad s \in (-\varepsilon, \varepsilon) \qquad u + s\xi = u_0 \quad \text{on } \partial D$$

and $\mathcal{F}(u+s\xi) \leq \mathcal{F}(u)$

$$\mathcal{F}(u+s\xi) = \int_D \sqrt{1+|Du+sD\xi|^2}\,dx = \int_D \sqrt{1+|Du|^2+2sDu.D\xi+s^2|D\xi|^2}$$

$$\frac{d}{ds}\mathcal{F}(u+s\xi)\Big|_{s=0} = \int_D \frac{Du.D\xi}{\sqrt{1+|Du|^2}}\,dx = 0 \qquad \forall \xi \in C_c^\infty(D)$$

$$\Rightarrow \quad -\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \qquad \text{vanishing mean curvature}$$

if $|Du|$ is very small then

$$\mathcal{F}(u) \sim \underbrace{\int_D 1 + \frac{1}{2}|Du|^2}_{\mathcal{G}(u)} \qquad \text{f first variation}$$

similar to 1st of $\mathcal{G}(u)$ $\qquad \xi \in C_c^\infty(D)$

$$\mathcal{G}(u+s\xi) = \int_D \left(1 + \frac{1}{2}|Du|^2 + s\,Du\cdot D\xi + \frac{1}{2}s^2|D\xi|^2\right)dx$$

$$\frac{d}{ds}\mathcal{G}(u+s\xi)\Big|_{s=0} = \int_D \underbrace{Du\cdot D\xi}_{\text{iii)}} = 0 \quad \Rightarrow \quad -\operatorname{div} Du = 0$$

u harmonic $<$ $\overset{\shortparallel}{\Delta u}$

iii) means u there almost harmonic

Pf of iii) : We reduce the proof of "almost harmonicity" to "almost vanishing mean curvature" ; i.e we show that $\forall \xi \in C_c^\infty(D_{r_0})$

$$\frac{1}{r_0^{n-1}} \left| \int_{D_{r_0}} \frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}} \, dx \right| \leq C(n) \sup_{D_{r_0}} |\nabla' \varphi| \; e_n(x_0, 8r_0)$$

note that since $\mathrm{Lip}\, u \leq 1$ & $\displaystyle\int_{D_{r_0}} |\nabla' u|^2 \leq r_0^{n-1} c(n)\, e_n(x_0, 8r_0)$

Recall wlog $r_0 = 1$ $x_0 = 0$ ✓

$$\int_D \left| \frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}} - \nabla' u \cdot \nabla' \varphi \right| \leq \int_D |\nabla' u \cdot \nabla' \varphi| \left| \frac{\sqrt{1+|D' u|^2} - 1}{\sqrt{1+|D' u|^2}} \right| dx$$

$$\leq \int_D |\nabla' \varphi| \underbrace{|\nabla' u|}_{\leq 1} \frac{|D' u|^2}{\sqrt{1+|D'u|^2}\left(1+\sqrt{1+|D'u|^2}\right)}$$

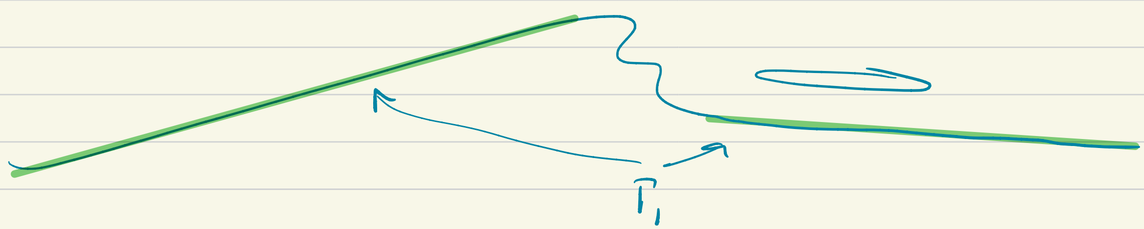
$$\leq \sup_D |\nabla' \varphi| \int_D |\nabla' u|^2 \leq C \, e_n(0, 8) \sup_{D_8} |\nabla' \varphi|$$

Thus enough to estimate $\displaystyle\int_{D_8} \frac{D'u \cdot D'\varphi}{\sqrt{1+|D'u|^2}}$

Step 1 : $\Gamma_1 = M \cap \text{graph } u \cap \left\{ x \in \partial E : \nu_E(x) = \frac{(-\nabla' u(p(x)), 1)}{\sqrt{1+|\nabla' u(p(x))|^2}} \right\}$

$\uparrow$ unit normal pointing in the right direction

$M = \partial^* E \cap C$



$\mathcal{H}^{n-1}(M \Delta \Gamma_1) \leq c_n \, e_n(0,8)$

Step 2 : Back to the 1st variation $\displaystyle\int_{\partial^* E} \operatorname{div} T \, d\mathcal{H}^{n-1} = 0$

$T \in C^1_c(C_2, \mathbb{R}^n)$

given $\varphi \in C_c^\infty(D)$ $\quad \alpha \in C_c^\infty([-1,1],[0,1])$ $\quad \alpha(s)=1$ $\quad |s|<1/4$

$$T(x) = \alpha(q(x))\,\varphi(p(x)) \qquad q(x) = \langle x, e_n\rangle$$

$$q(x)e_n + p(x) = x.$$

$$\int_{\partial^* E} \operatorname{div}_E T = 0 \implies \int_M \langle \nu_E, e_n\rangle \langle \nu_E, \bar\nabla'\varphi(p(x))\rangle \, d\mathcal{H}^{n-1} = 0$$

$$M = \underbrace{M \setminus T_1}_{\text{small}} \cup M \cap T_1$$

$$\int_{M\cap T_1} \langle \nu_E, e_n\rangle \langle \nu_E, \bar\nabla'\varphi(p(x))\rangle \, d\mathcal{H}^{n-1} = \int_{M\cap T_1} \frac{\bar\nabla'\varphi(p(x))\,\bar\nabla' u(p(x))}{1+|\bar\nabla' u(p(x))|^2}\, d\mathcal{H}^{n-1}$$

$$= \int_{p(M\cap T_1)} \frac{\nabla'\varphi(z)\,\bar\nabla' u(z)}{\sqrt{1+|\bar\nabla' u(z)|^2}}\, dz = 0$$

$$\int_D \frac{\nabla' u \; \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}} = \underbrace{\int_{p(M \cap \Gamma_1)}}_{0} + \underbrace{\int_{p(M \setminus \Gamma_1)}}_{\text{small}}$$

$$\left| \int_{p(M \setminus \Gamma_1)} \frac{\nabla' u \cdot \nabla' \varphi}{\sqrt{1+|\nabla' u|^2}} \right| \underset{\substack{\uparrow \\ \operatorname{Lip} u \leq 1}}{\leq} 2 \sup_D |\nabla' \varphi| \, \mathcal{H}^{n-1}(p(M \setminus \Gamma_1))$$

$$\leq 2 \sup_D |\nabla' \varphi| \, \mathcal{H}^{n-1}(M \triangle \Gamma_1)$$

$$\leq \varepsilon \sup_D |\nabla' \varphi| \quad \text{en } (0,8) \qquad \blacksquare$$

Theorem (Reverse Poincaré inequality)

There exist $c_2(n) > 0$, $\varepsilon_2(n) > 0$ s.t. if E is a perimeter minimizer, $x_0 \in \partial E$, $r_0 > 0$ s.t.

measures L^2 distance to $(n-1)$ plane

$$e(E, x_0, 4r_0, \nu) \leq \varepsilon_2(n) \quad \text{then}$$

$$e(E, x_0, r, \nu) \leq c_2 \inf_{c \in \mathbb{R}} \frac{1}{r^{n-1}} \int_{C(x,r,\nu) \cap \partial^* E} \left| \frac{\langle y - x, \nu \rangle - c}{r} \right|^2 d\mathcal{H}^{n-1}(y)$$

Theorem (Excess improvement by tilting)

Given $\alpha \in (0, 1/72)$ there exist $\varepsilon_3(n, \alpha) > 0$, $c_3(n)$ s.t. if E is a local perimeter minimizer $x_0 \in \partial E$ $r_0 > 0$ $\nu_0 \in S^{n-1}$ and

$$e(x_0, r_0, \nu_0) \leq \varepsilon_3 \quad \text{then there exists } \nu_1 \in S^{n-1}$$

s.t.

$$e(x_0, \alpha r_0, \nu_1) \leq c_3 \alpha^2 e(x_0, r_0, \nu_0)$$

&

$$|\nu_1 - \nu_0| \leq c_3 e(x_0, r_0, \nu_0)$$