## Math 135 - Winter 2000

## Homework to be done by January 20th

**Section 11.1:** Problems 13, 14, 17, 19, 23, 25, 29, 31, 34, 45, 47, 57, 67, 68.
**Section 11.2:** Problems 1, 7, 10, 13, 16, 29, 32, 49.
**Section 11.3:** Problems 1, 3, 8, 15, 17, 22, 23, 29, 37, 46.

## Problems to be handed in on January 20th

**1.** This problem uses ideas from the note on the other side of the page.

**1.1** Show that

$$\int_2^\infty \frac{dx}{x(\ln x)^p} \quad \text{converges if and only if } p > 1.$$

Find its value in this case.

**1.2** Show that

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$$

converges and that its sum equals 2.1 to one decimal place (i.e. with an error less than .05).

**1.3** Show that

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)}$$

diverges, but that

$$8.10 \leq \sum_{k=2}^{10^{1000}} \frac{1}{k(\ln k)} \leq 8.83.$$

As you can see this series diverges *very* slowly! If you tried to detect the divergence experimentally by adding up a few million terms on the computer you would not succeed.

**2.** Determine whether the series

$$\sum_{k=1}^{\infty} k! \left(\frac{5}{2k}\right)^k$$

converges or diverges.

**A note about the integral test**

The proof of the integral test yields considerably more information that the test itself. In fact let $f : [1, \infty) \to \mathbf{R}$ be a continuous positive nonincreasing function. Then a similar argument to the one presented in class or the one in Salas and Hille (pg 649) shows that if $m \geq 1$ for $n > m$

$$f(m+1) + \cdots + f(n) \leq \int_m^n f(x)\,dx \leq f(m) \cdots f(n-1).$$

These inequalities imply that

$$(1) \qquad \int_m^n f(x)\,dx \leq f(m) + f(m+1) + \cdots + f(n-1) + f(n) \leq f(m) + \int_m^n f(x)\,dx.$$

Note that (1) can be used to estimate the partial sums of the series $\sum f(k)$, whether the series converges or not. If it does converge, one can let $n \to \infty$ in (1) to obtain

$$(2) \qquad \int_m^\infty f(x)\,dx \leq \sum_{k=m}^\infty f(k) \leq f(m) + \int_m^\infty f(x)\,dx.$$

In this case let $\sum_{k=1}^\infty f(k) = L$ and $s_n = \sum_{k=1}^n f(k)$. From (2) we obtain that

$$(3) \qquad \int_{n+1}^\infty f(x)\,dx \leq L - s_n = \sum_{k=n+1}^\infty f(k) \leq f(n+1) + \int_{n+1}^\infty f(x)\,dx,$$

which provides a way to estimate $L$.

**Example:** In order to estimate $L = \sum_{k=1}^\infty \frac{1}{k^2}$, consider

$$\int_m^\infty \frac{dx}{x^2} = \frac{1}{m}.$$

Taking $n = 9$ in (3) we obtain

$$0.1 = \frac{1}{10} \leq L - \sum_{k=1}^9 \frac{1}{k^2} \leq f(10) + \frac{1}{10} = 0.11.$$

Since

$$\sum_{k=1}^9 \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{81} = 1.5397...,$$

we conclude that $L$ is about 1.64. (The exact value is $\frac{1}{6}\pi^2 \sim 1.6449341$.)