Math 135 - Winter 2000

Homework to be done by February 24th

The following problems pertain to the material in the notes on power series methods for differential equations; answers are on the back. Below them are some problems from Salas–Hille to get us started on the final group of topics for this quarter.

In Problems 1–3, find the general power series solution of the equation about $x = 0$. This includes finding a formula for the kth term of the series.

1. $y'' - xy' - y = 0$.
2. $y'' + x^2 y = 0$.
3. $(2 + x^2)y'' + 3xy' + y = 0$.

In Problems 4–5, identify the singular points of the equation and tell whether they are regular or irregular.

4. $2x^3(4 - x^2)y'' + 2y' + 3xy = 0$.
5. $(x^2 - 1)^2 y'' + x(1 - x)y' + (1 + x)y = 0$.
6. The equation $x(1 - x)y'' + [c - (a + b + 1)x]y' - aby = 0$ (a, b, c constants) is called the *hypergeometric equation*. Show that its singular points 0 and 1 are both regular, and find the characteristic exponents at each point. (To handle the point 1, make the change of variable $t = x - 1$.)

The equations in Problems 7–10 have a regular singular point at $x = 0$. (a) Find the characteristic exponents r_1 and r_2 (taking r_1 to be the larger one). (b) Find a solution of the form $y = \sum_0^\infty a_k x^{k+r}$ with $r = r_1$. (c) Find another such solution with $r = r_2$ if possible, or show that no such solution exists.

7. $xy'' + y = 0$.
8. $x^2 y' + xy' - (x + 2)y = 0$.
9. $(x - x^2)y'' + (3 - 6x)y' - 6y = 0$.
10. $xy'' + (1 - x)y' + \lambda y = 0$ ($\lambda =$ constant).

Notes:
(1) Observe that the equation of Problem 9 is a special case of the hypergeometric equation in Problem 6.
(2) The equation of Problem 10 is called the *Laguerre equation*. You should find that when λ is a nonnegative integer, the solution you get is a polynomial of degree λ. These polynomials, called *Laguerre polynomials*, are used in describing the quantum-mechanical wave functions of electrons moving in atomic orbits.

Section 12.1: Problems 5, 7, 9, 13, 21, 39.
Setion 12.3: 1, 3, 9, 11, 15, 19, 20, 21, 23, 27, 35, 37.

The **second midterm exam** will be on **Friday, February 25**. It will cover Sections 11.5–11.8 and 18.3–18.4 plus the material in the four sets of notes handed out in the past month.

ANSWERS TO PROBLEMS ON THE OTHER SIDE

1. $a_0 \sum_0^\infty x^{2n}/2^n n! + a_1 \sum_0^\infty x^{2n+1}/[1 \cdot 3 \cdot 5 \cdots (2n+1)]$.

2. $a_0 \left[1 + \sum_1^\infty \frac{(-1)^n x^{4n}}{[3 \cdot 7 \cdots (4n-1)][4 \cdot 8 \cdots (4n)]} \right] + a_1 \left[x + \sum_1^\infty \frac{(-1)^n x^{4n+1}}{[4 \cdot 8 \cdots (4n)][5 \cdot 9 \cdots (4n+1)]} \right]$.

3. $a_0 \left[1 + \sum_1^\infty \frac{(-1)^n 1 \cdot 3 \cdots (2n-1) x^{2n}}{2^{2n} n!} \right] + a_1 \left[x + \sum_1^\infty \frac{(-1)^n n! x^{2n+1}}{3 \cdot 5 \cdots (2n+1)} \right]$.

4. -2 regular, 0 irregular, 2 regular.

5. 1 regular, -1 irregular.

6. Exponents are 0 and $1 - c$ at 0, 0 and $c - a - b$ at 1.

7. $r_1 = 1$, $r_2 = 0$, $y_1 = \sum_0^\infty (-1)^k x^{k+1}/k!(k+1)!$, no y_2.

8. $r_1 = \sqrt{2}, r_2 = -\sqrt{2}$. $y_1 = x^{\sqrt{2}} + \sum_1^\infty x^{k+\sqrt{2}}/[k!(1+2\sqrt{2})(2+2\sqrt{2}) \cdots (k+2\sqrt{2})]$,
$y_2 = x^{-\sqrt{2}} + \sum_1^\infty x^{k-\sqrt{2}}/[k!(1-2\sqrt{2})(2-2\sqrt{2}) \cdots (k-2\sqrt{2})]$

9. $r_1 = 0$, $r_2 = -2$, $y_1 = \sum_0^\infty (k+1) x^k = (1-x)^{-2}$, $y_2 = x^{-2}$.

10. $r_1 = r_2 = 0$. $y_1 = 1 + \sum_1^\infty \frac{(-\lambda)(1-\lambda) \cdots (k-1-\lambda)}{(k!)^2} x^k$, no y_2.

Problems to be handed in on February 24th

Problem 1. Consider the *Chebyshev equation*

$$(1-x^2)y'' - xy' + a^2 y = 0 \qquad (a = \text{ constant}).$$

(Note: Chebyshev = Tchebycheff = Tschebischev = Čebyšev = ...)

1.1 Find the general power series solution of this equation about $x = 0$. (Find the whole series—i.e., give the formula for the kth term.)

1.2 Show that when a is a nonnegative integer, one of these solutions is a polynomial of degree a. (These polynomials, multiplied by constants so that their value at $x = 1$ is 1, are called *Chebyshev polynomials*, and whole books have been written about them and their applications. Here's the simplest one: The n-tuple angle formula for cosines can be written as $\cos n\theta = P_n(\cos\theta)$ where P_n is the Chebyshev polynomial of degree n.)

Problem 2. Consider the equation

$$xy'' + (2x^2 - 3)y' + 4xy = 0.$$

2.1 Show that $x = 0$ is a regular singular point, and that the characteristic exponents there are 4 and 0.

2.2 Find a series solution y_1 corresponding to the exponent 4.

2.3 Express the solution y_1 in closed form. (I.e., sum the series. It's a simple elementary function.)

2.4 Find a second solution y_2 corresponding to the exponent 0, or show that no such solution exists.