Power Series Methods for Differential Equations

We consider the second-order homogeneous linear equation

$$N(x)y'' + P(x)y' + Q(x)y = 0, \tag{1}$$

or, in standard form,

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

For an equation in the standard form (2), we assume that $p(x)$ and $q(x)$ can be expanded in power series about a point x_0 (i.e., series in powers of $x - x_0$), convergent in some interval I centered at x_0. For an equation in form (1), we assume that $N(x)$, $P(x)$, and $Q(x)$ can be expanded in power series about x_0, convergent in I, *and* that $N(x_0) \neq 0$. Our object is to construct the solutions in the form of power series about x_0.
For the purposes of the general theoretical discussion it is more convenient to use the standard form (2). However, in specific problems it is often simpler to use the form (1), as the examples below will show. Also, by making the change of variable $x \to x + x_0$, we may and shall assume that $x_0 = 0$.
We assume, then, that $p(x) = \sum_0^\infty p_k x^k$ and $q(x) = \sum_0^\infty q_k x^k$ on some interval $\{x : |x| < r\}$, and we look for solutions to (2) of the form $y = \sum_0^\infty a_k x^k$. Observe that $a_0 = y(0)$ and $a_1 = y'(0)$, so from the general theory we expect to be able to prescribe a_0 and a_1 arbitrarily.
Substituting these series expansions into (2) yields

$$\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} + \left[\sum_{k=0}^{\infty} p_k x^k\right]\left[\sum_{k=0}^{\infty} k a_k x^{k-1}\right] + \left[\sum_{k=0}^{\infty} q_k x^k\right]\left[\sum_{k=0}^{\infty} a_k x^k\right] = 0.$$

This may be clearer if we write out the first few terms:

$$2\cdot 1a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + \cdots + (p_0 + p_1 x + p_2 x^2 + \cdots)(a_1 + 2a_2 x + 3a_3 x^2 + \cdots)$$
$$+(q_0 + q_1 x + q_2 x^2 + \cdots)(a_0 + a_1 x + a_2 x^2 + \cdots) = 0.$$

Multiplying out the left side gives a new power series. Since its sum is zero, the coefficients of all the powers of x must vanish:

$$\text{(constant term)}: \quad 2a_2 + p_0 a_1 + q_0 a_0 = 0, \tag{3.0}$$
$$\text{(coefficient of } x): \quad 3\cdot 2a_3 + 2p_0 a_2 + p_1 a_1 + q_0 a_1 + q_1 a_0 = 0, \tag{3.1}$$
$$\text{(coefficient of } x^2): \quad 4\cdot 3a_4 + 3p_0 a_3 + 2p_1 a_2 + p_2 a_1 + q_0 a_2 + q_1 a_1 + q_2 a_0 = 0, \tag{3.2}$$

and, in general, for the coefficient of x^k,

$$(k+2)(k+1)a_{k+2} + \sum_{j=0}^{k}(k-j+1)p_j a_{k-j+1} + \sum_{j=0}^{k} q_j a_{k-j} = 0. \tag{3.k}$$

This is called the **recursion formula** for the a_k's. As we observed above, a_0 and a_1 can be given whatever values we like. Once we have chosen them, equation (3.0) determines a_2; then (3.1) determines a_3; then (3.2) determines a_4, and so forth. In this way we determine the whole sequence of coefficients a_k recursively.

The remaining question is whether the resulting series $\sum a_k x^k$ converges, for if it does, equations (3) guarantee that it satisfies the differential equation (2). We won't go through the whole argument, but the idea is simple: use (3.k) to estimate the a_k's in terms of the p_k's and q_k's, and then do a comparison test. Here is the result, stated with a general base point x_0:

Theorem 1 *Let r_p and r_q be the radii of convergence of the series $p(x) = \sum p_k(x - x_0)^k$ and $q(x) = \sum q_k(x - x_0)^k$. If the sequence $\{a_k\}$ is determined by the recursion formulas* (3)*, then the radius of convergence of the series $y = \sum a_k(x - x_0)^k$ is at least* $\min(r_p, r_q)$*, and y satisfies* (2) *on its interval of convergence.*

Many of the important examples of second-order equations are of the form (1) where N, P, and Q are polynomials. In the standard form of such an equation the coefficients $p = P/N$ and $q = Q/N$ are rational functions. It is known from complex function theory that the radius of convergence of the Taylor series of such a function about $x = x_0$ is the distance from x_0 to its nearest singularity—namely, the nearest point where $N(x) = 0$—*in the complex plane.* Combining this with Theorem 1, we have:

Theorem 2 *Suppose $N(x)$, $P(x)$,, and $Q(x)$ are polynomials. The radius of convergence of any power series $y = \sum a_k(x - x_0)^k$ that satisfies* (1) *is at least the distance from x_0 to the nearest zero of N in the complex plane.*

We turn to some examples.

Example 1. Solve the initial value problem

$$(x + 2)y'' + xy' - y = 0, \tag{4}$$

$$y(0) = 1, \qquad y'(0) = -1. \tag{5}$$

Solution: In standard form, (4) becomes $y'' + x(x + 2)^{-1}y' + (x + 2)^{-1}y = 0$. We could expand $p(x) = x(x + 2)^{-1}$ and $q(x) = (x + 2)^{-1}$ into Taylor series without much difficulty, but it's a lot easier to work directly with (4). With $y = \sum_0^\infty a_k x^k$, (4) becomes

$$\sum_0^\infty k(k-1)a_k x^{n-1} + 2\sum_0^\infty k(k-1)a_k x^{k-2} + \sum_0^\infty k a_k x^k - \sum_0^\infty a_k x^k = 0.$$

Shifting the index of summation on the first two sums so that the exponent of x is k, we get

$$\sum_{-1}^\infty (k+1)k a_{k+1} x^k + \sum_{-2}^\infty 2(k+2)(k+1)a_{k+2} x^k + \sum_0^\infty k a_k x^k - \sum_0^\infty a_k x^k = 0.$$

(Note that the terms with $k = -1$ and $k = -2$ in the first two sums aren't really there—their coefficients are zero.) It follows that for all $k \geq 0$,

$$(k+1)k a_{k+1} + 2(k+2)(k+1)a_{k+2} + (k-1)a_k = 0, \text{ or}$$

$$a_{k+2} = -\frac{(k+1)k a_{k+1} + (k-1)a_k}{2(k+2)(k+1)}. \tag{6}$$

From (5) we know that $a_0 = 1$ and $a_1 = -1$. The recursion formula (6) then shows that

$$a_2 = -\frac{1 \cdot 0a_1 + (-1)a_0}{2 \cdot 2 \cdot 1} = -\frac{1}{4}, \qquad a_3 = -\frac{2 \cdot 1a_2 + 0a_1}{2 \cdot 3 \cdot 2} = -\frac{2\left(-\frac{1}{4}\right)}{12} = \frac{1}{24},$$
$$a_4 = -\frac{3 \cdot 2a_3 + 1a_2}{2 \cdot 4 \cdot 3} = -\frac{\frac{6}{24} - \frac{1}{4}}{24} = 0, \quad a_5 = -\frac{4 \cdot 3a_4 + 2a_3}{2 \cdot 5 \cdot 4} = -\frac{0 + \frac{2}{24}}{40} = -\frac{1}{480},$$

and so forth, so that

$$y = 1 - x - \frac{1}{4}x^2 + \frac{1}{24}x^3 - \frac{1}{480}x^5 + \cdots.$$

We could continue to compute the a_k's forever, but there's probably no neat formula for them. Of course, just the first few terms give a good approximation to the solution near $x = 0$. Moreover, *without computing the a_k's at all*, Theorem 2 tells us that the radius of convergence of $\sum a_k x^k$ is at least 2, the distance from 0 to the singular point -2.

Example 2. Suppose we want to look at solutions of the preceding equation (4) near $x = 5$. The series expansions of the solutions about $x = 0$, as in Example 1, are useless here; they probably don't converge beyond $x = 2$. Instead, we look for solutions in the form $y = \sum_0^\infty b_k(x-5)^k$. These will have radius of convergence ≥ 7 (the distance from 5 to the singular point -2). To find them, the easiest way is to make the substitution $t = x - 5$, which turns (4) into

$$(t+7)y'' + (t+5)y' - y = 0,$$

where y is now regarded as a function of t, and turns $\sum b_k(x-5)^k$ into $\sum b_k t^k$. We then proceed as in Example 1.

Example 3. Without calculating any coefficients, what can we say about the radius of convergence of a series solution $\sum a_k x^k$ to the equation

$$(x+2)(x^2 - 2x + 2)y'' + (x-4)y' + 3x^4 y = 0?$$

Solution: The leading coefficient vanishes at $x = -2$ and at $x = 1 \pm i$. The distance from the latter two points to 0 is $\sqrt{2}$, which is less than the distance from -2 to 0. By Theorem 2, the radius of convergence is at least $\sqrt{2}$.

For certain special kinds of equations, including (fortunately) many of the most important equations of applied mathematics, it is possible to find a reasonably simple formula for the general coefficient a_k, so that one can write out the whole power series for the solutions rather than just the first few terms. Here is one example; others will be presented in class.

Example 4. The **Hermite equation** is

$$y'' - 2xy' + 2\lambda y = 0 \qquad (\lambda = \text{ constant}). \tag{7}$$

If we set $y = \sum_0^\infty a_k x^k$, (7) becomes

$$\sum_0^\infty k(k-1)a_k x^{k-2} - 2x\sum_0^\infty k a_k x^{k-1} + 2\lambda \sum_0^\infty a_k x^k = 0.$$

Replacing k by $k+2$ in the first sum and noting that its first two terms vanish, we get

$$\sum_0^\infty (k+2)(k+1)a_{k+2}x^k - \sum_0^\infty 2ka_kx^k + \sum_0^\infty 2\lambda a_kx^k = 0.$$

Hence, for all $k \geq 0$, $(k+2)(k+1)a_{k+2} - 2(\lambda - k)a_k = 0$, or

$$a_{k+2} = \frac{2(k-\lambda)}{(k+1)(k+2)}a_k. \tag{8}$$

Once a_0 is fixed, the recursion formula (8) determines all the a_k with k even:

$$a_2 = \frac{-2\lambda}{1\cdot 2}a_0, \qquad a_4 = \frac{2(2-\lambda)}{3\cdot 4}a_2 = -\frac{2^2\lambda(2-\lambda)}{4!}a_0, \qquad \cdots,$$

$$a_{2n} = -\frac{2^n\lambda(2-\lambda)(4-\lambda)\cdots(2n-2-\lambda)}{(2n)!}a_0.$$

Likewise, (8) determines all the a_k with k odd in terms of a_1:

$$a_3 = \frac{2(1-\lambda)}{2\cdot 3}a_1, \qquad a_5 = \frac{2(3-\lambda)}{4\cdot 5}a_3 = \frac{2^2(1-\lambda)(3-\lambda)}{5!}a_1, \qquad \cdots,$$

$$a_{2n+1} = \frac{2^n(1-\lambda)(3-\lambda)\cdots(2n-1-\lambda)}{(2n+1)!}a_1.$$

Thus the general solution is $y = a_0y_0 + a_1y_1$, where

$$\begin{aligned} y_0 &= 1 - \sum_1^\infty \frac{2^n\lambda(2-\lambda)\cdots(2n-2-\lambda)}{(2n)!}x^{2n}, \\ y_1 &= x + \sum_1^\infty \frac{2^n(1-\lambda)(3-\lambda)\cdots(2n-1-\lambda)}{(2n+1)!}x^{2n+1}. \end{aligned}$$

y_0 and y_1 are particular solutions of (7): y_0 (resp. y_1) is obtained by taking $a_0 = 1$, $a_1 = 0$ (resp. $a_0 = 0$, $a_1 = 1$). Clearly they are a fundamental set of solutions.
By Theorem 2, these series converge everywhere, as the equation (7) has no singular points. You can also verify this easily via the ratio test.
An important feature of the Hermite equation is that when λ is a nonnegative integer, one of the solutions is a polynomial of degree λ. Indeed, when $\lambda = 2m$, the terms with $n > m$ in y_0 vanish because they contain a factor $(2m - \lambda)$, and when $\lambda = 2m+1$, the terms with $n > m$ in y_1 vanish because they contain a factor $(2m+1-\lambda)$. These polynomials, multiplied by constants so that the coefficient of x^λ is 2^λ, are called **Hermite polynomials**. They arise in a number of basic problems in quantum mechanics.