Second-Order Linear Differential Equations

We consider the general second-order linear differential equation

$$N(x)y'' + P(x)y' + Q(x)y = G(x). \tag{1}$$

We shall say that an equation of this sort is in **standard form** if the coefficient of y'' is 1:

$$y'' + p(x)y' + q(x)y = g(x). \tag{2}$$

(1) can be put into the form (2) by dividing through by $N(x)$, and for the purposes of this discussion we shall assume that this has been done. The equation (1) (or (2)) is said to be **homogeneous** or **reduced** if $G \equiv 0$ (or $g \equiv 0$), **inhomogeneous** if not.

We assume that the functions p, q, and g in (2) *are continuous on an interval $I = (\alpha, \beta)$* (*which might be the whole line*), *and we seek solutions y on this interval.* (If the equation is given in the form (1), this means that we need the functions N, P, Q, and G to be continuous on I *and* N to be non-vanishing on I. Points where N vanishes, or where p and/or q have singularities, are called **singular points** of the equation (1) or (2). The general theory does not apply on an interval that contains singular points—but there is more to be said about them, as we shall see later.)
It is convenient to denote the left hand side of (2) by $L[y]$; thus L is the *differential operator*

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x).$$

L is a *linear* operator; that is,

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2] \qquad (c_1, c_2 \text{ constants}). \tag{3}$$

In particular, if $L[y_1] = L[y_2] = 0$ then $L[c_1y_1 + c_2y_2] = 0$ for all constants c_1 and c_2. (This fact is often called the **superposition principle** in the physics literature.)

The fundamental existence and uniqueness theorem is as follows:

Theorem 1 *Suppose p, q, and g are continuous on the interval I, and $x_0 \in I$. For any numbers a and b there is a unique solution y of the equation (2) that satisfies $y(x_0) = a$ and $y'(x_0) = b$.*

The proof of this theorem is beyond the scope of this course. It can be found, for example, in Appendix 5 of *Fourier Analysis and its Applications* by G. B. Folland, or (usually in a more general form) in many advanced books on ordinary differential equations.
Theorem 1 tells us that a solution y of (2) is completely determined by the two constants $y(x_0)$ and $y'(x_0)$, which may be freely chosen. We say that *the solution space for* (2) *is two-dimensional.*

We begin by analyzing the homogeneous equation $L[y] = 0$; we shall return to the general case later. As we observed above, if y_1 and y_2 are solutions of $L[y] = 0$ then so is $c_1y_1 + c_2y_2$ for any constants c_1 and c_2. Thus, if we can find two solutions that are genuinely different (one is not a constant multiple of the other), we obtain a two-parameter family of solutions this way, and according to the preceding paragraph, there is hope that *all* solutions will belong to this family. More precisely, the question is whether we can find c_1 and c_2 so that

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= a, \\ c_1y_1'(x_0) + c_2y_2'(x_0) &= b, \end{aligned} \tag{4}$$

for any given numbers a and b. This is a system of two linear equations for the two unknowns c_1 and c_2, and it has a unique solution for any a and b if and only if the determinant of the coefficient matrix, $y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0)$, is nonzero. This determinant plays a central role in the theory; it is called the **Wronskian** of y_1 and y_2 at x_0:

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x). \tag{5}$$

We are now ready for the main theorem about solutions of $L[y] = 0$.

Theorem 2 *Suppose p and q are continuous on the interval I, and y_1 and y_2 are solutions of $L[y] = 0$ on I. The following conditions are equivalent:*

(a) $W(y_1, y_2)(x_0) \neq 0$ for every point $x_0 \in I$.

(b) $W(y_1, y_2)(x_0) \neq 0$ for some point $x_0 \in I$.

(c) Every solution of $L[y] = 0$ on I is of the form $c_1y_1 + c_2y_2$.

(d) y_1 and y_2 are not constant multiples of each other.

Proof.

(a) $\Rightarrow$ (b): This is trivial.

(b) $\Rightarrow$ (c): Suppose $L[y] = 0$. If (b) holds, we can solve the equations (4) with $a = y(x_0)$ and $b = y'(x_0)$. But then y and $c_1y_1 + c_2y_2$ both solve the differential equation and both have the same value and the same slope at x_0. By the uniqueness in Theorem 1, they are equal.

(c) $\Rightarrow$ (a): By Theorem 1, if $x_0 \in I$ there is a solution of $L[y] = 0$ with any specified values of $y(x_0)$ and $y'(x_0)$. If (c) holds, so that $y = c_1y_1 + c_2y_2$, this means that we can always solve (4) for arbitrary a and b. Hence the determinant $W(y_1, y_2)(x_0)$ is nonzero.

(d) $\iff$ (a): We shall show that (d) is false (i.e., $y_2 = cy_1$) precisely when (a) is false (i.e., $W(y_1, y_2)(x_0) = 0$ for some x_0). Clearly, if $y_2 = cy_1$ then $W(y_1, y_2) = y_1cy_1' - cy_1y_1' \equiv 0$. On the other hand, if $W(y_1, y_2)(x_0) = 0$ then $y_2(x_0)/y_1(x_0) = y_2'(x_0)/y_1'(x_0)$. Denoting this common value by c, we have $y-2(x_0)=cy_1(x_0)$ and $y_2'(x_0)=cy_1'(x_0)$. By the uniqueness in Theorem 1, $y_2=cy_1$. (The little extra argument to dispose of the case where $y_1(x_0)=0$ or $y_1'(x_0)=0$ is left to the reader.) ■

A pair y_1, y_2 of solutions of $L[y]=0$ that satisfies the conditions in Theorem 2 is called a **fundamental set of solutions**, and condition (c) says that if we know a fundamental set of solutions then we know all solutions.

Reduction of Order.

If we can just find *one* nontrivial solution y_1, there is a simple device that reduces the search for a second one to the calculation of a couple of integrals. The idea is to look for a second solution in the form $y_2=vy_1$, where v is a nonconstant function. Indeed, we have

$$\begin{aligned} L[vy_1]=(vy_1)''+p(vy_1)'+qvy_1 &= v''y_1+2v'y_1+vy_1''+pv'y_1+pvy_1'+qvy_1 \\ &= vL[y_1]+v''y_1+v'(2y_1'+py_1). \end{aligned}$$

Since $L[y_1]=0$, we will have $L[vy_1]=0$ provided that

$$y_1v''+(2y_1'+py_1)v'=0. \tag{6}$$

This is a separable (and linear) first-order equation for v', so we can solve it and integrate to find v. v will be non-constant provided we don't take $v'=0$. This device is called *reduction of order*, since it reduces finding a second solution of $L[y]=0$ to solving a first-order equation.

Example. It is easy to check that $y_1(x)=x^2$ satisfies

$$x^2y''-3xy'+4y=0. \tag{7}$$

To find a second solution for a fundamental set, try $y_2(x)=x^2v(x)$. Plugging this into (7) yields

$$x^2(x^2v''+4xv'+2v)-3x(2xv+x^2v')+4(x^2v),$$

which simplifies to

$$x^4v''+x^3v'=0, \quad \text{or } \frac{v''}{v'}=-\frac{1}{x}.$$

Integrating both sides gives $\log|v'|=-\log|x|$, or $v'=x^{-1}$; then $v=\log|x|$ and the second solution is $y_2=x^2\log|x|$. We omitted the constants of integration because we only need one v. Alternatively, if we keep both constants of integration in this calculation, we get $v=c_1+c_2\log|x|$. The corresponding solution on (7) is $x^2v=c_1y_1+c_2y_2$.

The Inhomogeneous Equation.

Once we have found a fundamental set of solutions (and hence all solutions) to the homogeneous equation $L[y] = 0$, there is a straightforward procedure, called **variation of parameters**, for finding a solution of the inhomogeneous equation $L[y] = g$. It is described in Section 18.4 of Salas-Hille. The book considers only equations with constant coefficients, but the technique and the results work perfectly well in the general case. There is just one possible source of error: to apply variation of parameters, the equation needs to be in the standard form (2). (If it is in the form (1), the input for the variation-of-parameters machine is not $G(x)$ but $G(x)/N(x)$.)

More precisely assume that the general solution of the homogeneous equation associated with equation (2) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x). \tag{8}$$

To find a particular solution of the inhomogeneous equation the crucial idea is to replace the constants c_1 and c_2 by functions $u_1(x)$ and $u_2(x)$, respectively, this gives

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x). \tag{9}$$

Then we try to determine $u_1(x)$ and $u_2(x)$ so that (9) is a solution of the inhomogeneous equation (2). Proceeding exactly as in Section 18.4 of Salas-Hille we impose that

$$u_1'(x) y_1(x) + u_2'(x) y_2(x) = 0. \tag{10}$$

Differentiating y_p and using the fact that it satisfies (2) we obtain that $u_(x)$ and $u_2(x)$ satisfy

$$u_1'(x) y_1'(x) + u_2'(x) y_2'(x) = g(x). \tag{11}$$

Equations (10) and (11) form a system of two linear equations in the derivatives $u_1'(x)$ and $u_2'(x)$ of the unknown functions. Solving the system we obtain

$$u_1'(x) = -\frac{y_2(x) g(x)}{W(y_1, y_2)(x)}, \qquad u_2'(x) = \frac{y_1(x) g(x)}{W(y_1, y_2)(x)}, \tag{12}$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. Note that the division by $W(y_1, y_2)$ is permissible since y_1 and y_2 are a fundamental set of solutions, and therefore the Wronskian is nonzero. By integrating (12), we obtain that

$$u_1(x) = -\int \frac{y_2(x) g(x)}{W(y_1, y_2)(x)}\, dx + C, \qquad u_2(x) = \int \frac{y_1(x) g(x)}{W(y_1, y_2)(x)}\, dx + C. \tag{13}$$

Finally, substituting from (13) in (9) gives the general solution of (2).

Theorem 3 *If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 form a fundamental set of solutions for the homogeneous equation corresponding to the inhomogeneous equation*

$$y'' + p(x)y' + q(x)y = g(x),$$

then a particular solution of (2) is

$$y_p(x) = -y_1(x)\int \frac{y_2(x)g(x)}{W(y_1,y_2)(x)}\,dx + y_2(x)\int \frac{y_1(x)g(x)}{W(y_1,y_2)(x)}\,dx, \tag{14}$$

and the general solution of (2) is

$$y(x) = c_1 y_(x) + c_2 y_2(x) + y_p(x) \tag{15}$$

How to find solutions? By using reduction of order and variation of parameters, one can reduce the problem by finding the general solution $L[y] = g$ to finding one nonzero solution of $L[y] = 0$. But in contrast to the situation for first-order linear equations, there is *no* general procedure for solving $L[y] = 0$ in elementary terms. We shall concentrate only on some types of equations—fortunately, the most important ones—where solution methods are available. First, if the coefficients p and q are constant, the solution is easy; it is described in Section 18.3 of Salas-Hille. Second, if the coefficients p and q can be expanded in power series, one can obtain solutions in the form of power series; we shall say more about this later.

Euler Equations. There is one other case of (1) that can be solved easily, the *Euler equation*

$$x^2y'' + axy' + by = 0 \qquad (a, b \text{ constants}). \tag{16}$$

The idea is similar to the constant-coefficient case. Namely, try $y = x^r$. Plugging this into (16) yields $[r(r-1) + ar + b]x^r = 0$, so x^r will be a solution of (16) if

$$r^2 + (a-1)r + b = 0. \tag{17}$$

(17) generally has two roots r_1 and r_2, in which case x^{r_1} and x^{r_2} are a fundamental set of solutions for (16) on the intervals $(0, \infty)$ and $(-\infty, 0)$. (Note that $x = 0$ is a singular point of (16).) If there is only one root r_0, then x^{r_0} is a solution, and reduction of order yields the second solution $x^{r_0}\log|x|$. (The case $r_0 = 2$ was worked out in the Example above; the general case is similar.)
Note 1: If r is not an integer, the expression x^r may make you nervous when $x < 0$. But in this case you can write $x^r = (-|x|)^r = (-1)^r|x|^r \cdot |x|^r$ is OK, and $(-1)^r$ is just a constant (probably complex) that can be discarded since a constant multiple of a solution is a solution.

Thus on $(-\infty, 0)$ (as well as on $(0, \infty)$) you can take $|x|^{r_1}$ and $|x|^{r_2}$ as a fundamental set of solutions if (17) has two roots r_1 and r_2, or $|x|^{r_0}$ and $|x|^{r_0} \log |x|$ if (17) has only one root r_0.
Note 2: If the roots of (17) are complex, say $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, the solutions are

$$|x|^{\alpha \pm i\beta} = e^{(\alpha \pm i\beta) \log |x|} = |x|^{\alpha}(\cos[\beta \log |x|] \pm i \sin[\beta \log |x|]).$$

If you want real solutions, you can add and subtract these to get $|x|^{\alpha} \cos[\beta \log |x|]$ and $|x|^{\alpha} \sin[\beta \log |x|]$.
Note 3: The similarity of the solution methods for constant coefficient equations and Euler equations is not just an accident. Each of these types of equations can be transformed into the other by a change of independent variable. See Exercise 36 in Section 18.3 of Salas-Hille.