On the smoothness of Hölder doubling measures

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Abstract

It is known that in low dimensions supports of Hölder doubling measures are $C^{1,\beta}$ manifolds. In higher dimensions singularities may occur. We provide a full description of such supports by showing that they are $C^{1,\beta}$ manifolds away from a closed set of measure zero and that at singular points they are uniformly far from being flat at every scale.

1 Introduction

In this paper we study the extent to which the doubling character of a measure in $\mathbb{R}^m$ determines the regularity of its support (in a classical sense). This problem was studied in [1] for measures supported on codimension 1 sets under the assumption that the support be flat. In [1], the authors exploited the fact that for Hölder doubling measures the points in the support "almost" satisfy a quadratic equation. More precisely they satisfy a quadratic inequality. The flatness hypothesis was used to rule out the singular solutions to the equation mentioned above. In this paper we consider Hölder doubling measures in $\mathbb{R}^m$ supported on sets of any codimension. While we still know that the points in the support satisfy a quadratic inequality, we face a serious difficulty and new ideas are required: in codimension k, k independent equations are needed to determine a set of codimension k. To overcome this issue we use multi-scale analysis. By choosing the scales appropriately the quadratic inequality unfolds as k independent inequalities. In the choice of scales several compatibility issues need to be addressed. Nevertheless we show a local regularity result: in the neighborhood of a flat point the support of a Hölder doubling measure coincides with a $C^{1,\beta}$ submanifold of $\mathbb{R}^m$. We provide a full description of the support of such measures by showing that the set of flat points is open and its complement has measure zero.

In order to give precise statements we need to introduce some definitions. Fix integer dimensions $0 < n < m$ and a closed set $\Sigma \subset \mathbb{R}^m$. For $x \in \Sigma$ and $r > 0$, set

$$\theta_\Sigma(x,r) = \frac{1}{r} \inf\{D[\Sigma \cap B(x,r), L \cap B(x,r)]; L \text{ is an affine } n\text{-plane through } x\}, \tag{1.1}$$

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where $B(x,r)$ denotes the open ball of center x and radius r in $\mathbb{R}^m$, and where

$$D[E,F] = \sup\{\mathrm{dist}(y,F); y \in E\} + \sup\{\mathrm{dist}(y,E); y \in F\} \tag{1.2}$$

denotes the usual Hausdorff distance between (nonempty) sets. If there is no ambiguity over the set we are considering we write $\theta(x,r)$ rather than $\theta_\Sigma(x,r)$.

Definition 1.1 Let $\delta > 0$ be given. We say that the closed set $\Sigma \subset \mathbb{R}^m$ is *δ-Reifenberg flat* of dimension n if, for all compact sets $K \subset \Sigma$, there is a radius $r_K > 0$ such that

$$\theta(x,r) \le \delta \qquad \text{for all} \qquad x \in K \quad \text{and} \quad 0 < r \le r_K. \tag{1.3}$$

Note that it does not make sense to take δ large (like $\delta \ge 2$), because $\theta(x,r) \le 2$ anyway.

Definition 1.2 We say that the closed set $\Sigma \subset \mathbb{R}^m$ is *Reifenberg flat with vanishing constant* (of dimension n) if, for every compact subset K of Σ,

$$\lim_{r \to 0^+} \theta_K(r) = 0, \tag{1.4}$$

where

$$\theta_K(r) = \sup_{x \in K} \theta(x,r). \tag{1.5}$$

Unless otherwise specified, “measure” here will mean “positive Radon measure”, i.e. “Borel measure which is finite on compact sets.” Let μ be a measure on $\mathbb{R}^m$, set

$$\mathrm{supp}(\mu) = \{x \in \mathbb{R}^m; \mu(B(x,r)) > 0 \text{ for all } r > 0\}. \tag{1.6}$$

For a fixed positive integer n and for a measure μ on $\mathbb{R}^m$, with support $\Sigma = \mathrm{supp}(\mu)$ we define for $x \in \Sigma$, $r > 0$ and $t \in (0,1]$ the quantity

$$R_t(x,r) = \frac{\mu(B(x,tr))}{\mu(B(x,r))} - t^n, \tag{1.7}$$

which encodes the doubling properties of μ.

Definition 1.3 A measure μ supported on Σ is said to be asymptotically optimally doubling if for each compact set $K \subset \Sigma$, $x \in K$, and $t \in [\frac{1}{2}, 1]$

$$\lim_{r \to 0^+} \sup_{x \in K} |R_t(x,r)| = 0. \tag{1.8}$$

The results in this paper can be summarized as follows: first under the appropriate conditions on $\theta(x,r)$ (see (1.1)) the asymptotic behavior of $R_t(x,r)$ as r tends to 0 fully determines the regularity of Σ. Second for asymptotically doubling measures which are Ahlfors regular flatness is an open condition.

We mention the local versions of some of the previous results along these lines.

Theorem 1.4 ([6], [1]) *Let μ be an asymptotically optimally doubling measure supported on $\Sigma \subset \mathbb{R}^m$. If $n = 1, 2$, Σ is Reifenberg flat with vanishing constant. If $n \geq 3$, there exists a constant $\delta(n,m)$ depending only on n and m such that if $x_0 \in \Sigma$ and $\Sigma \cap B(x_0, 2R_0)$ is $\delta(n,m)$-Reifenberg flat, then $\Sigma \cap B(x_0, R_0)$ is Reifenberg flat with vanishing constant.*

The converse is also true.

Theorem 1.5 ([1]) *If Σ is a Reifenberg flat set with vanishing constant there exists a measure μ supported on Σ which satisfies (1.8).*

Precise asymptotic estimates on the quantity $R_t(x,r)$ yield stronger results about the regularity of Σ.

Theorem 1.6 ([1]) *For each constant $\alpha > 0$ we can find $\beta = \beta(\alpha) > 0$ with the following property. Let μ be a measure in $\mathbb{R}^{n+1}$, set $\Sigma = \mathrm{supp}(\mu)$, and suppose that for each compact set $K \subset \Sigma$, there is a constant C_K such that*

$$|R_t(x,r)| \leq C_K r^\alpha \quad \textit{for } r \in (0,1],\ t \in [\frac{1}{2}, 1] \quad \textit{and } x \in K. \tag{1.9}$$

If $n = 1, 2$, Σ is a $C^{1,\beta}$ submanifold of dimension n in $\mathbb{R}^{n+1}$. If $n \geq 3$, for $x_0 \in \Sigma$ if $\Sigma \cap B(x_0, 2R_0)$ is $\frac{1}{4\sqrt{2}}$-Reifenberg flat, then $\Sigma \cap B(x_0, R_0)$ is a $C^{1,\beta}$ submanifold of dimension n in $\mathbb{R}^{n+1}$.

For $n \geq 3$, the preceding theorem fails if one removes the flatness assumption. Indeed, Kowalski and Preiss [5] discovered that the 3-dimensional Hausdorff $\mathcal{H}^3$ measure on the cone $X = \{x \in \mathbb{R}^4 : x_4^2 = x_1^2 + x_2^2 + x_3^2\}$ satisfies $\mathcal{H}^3(B(x,r) \cap X) = Cr^3$ for all $x \in X$ and all $r > 0$. Clearly, (1.9) holds in this case and X is non smooth at the origin.
In this paper we extend Theorem 1.6 to general codimensions in $\mathbb{R}^m$, and moreover we prove that, when $n \geq 3$, if one does not assume Σ to be Reifenberg flat, one still has that Σ is smooth off a small closed set (like in the case of the cone X). The precise statement is the following.

Theorem 1.7 *For each constant $\alpha > 0$ we can find $\beta = \beta(\alpha) > 0$ with the following property. Let μ be a measure in $\mathbb{R}^m$ supported on Σ, and suppose that for each compact set $K \subset \Sigma$, there is a constant C_K such that*

$$|R_t(x,r)| \leq C_K r^\alpha \quad \textit{for } r \in (0,1],\ t \in [\frac{1}{2}, 1] \quad \textit{and } x \in K. \tag{1.10}$$

If $n = 1, 2$, Σ is a $C^{1,\beta}$ submanifold of dimension n in $\mathbb{R}^m$. If $n \geq 3$, Σ is a $C^{1,\beta}$ submanifold of dimension n in $\mathbb{R}^m$ away from a closed set $\mathcal{S}$ such that $\mathcal{H}^n(\mathcal{S}) = 0$.

We would like to point out that condition (1.10) implies an apparently stronger condition, namely that for each compact set $K \subset \Sigma$, there is a constant C_K depending on K, n and α such that

$$|R_t(x,r)| \leq C_K r^\alpha \quad \text{for } r \in (0,1],\ t \in (0,1] \quad \text{and } x \in K. \tag{1.11}$$

In fact assume that (1.10) holds and let $\tau \in (0,1/2)$. There exits $j \in \mathbb{N}$, $j \geq 2$ so that $1/2^j \leq \tau < 1/2^{j-1}$ thus $\tau^{\frac{1}{j}} = t \in [1/2, 1/\sqrt{2})$. For $x \in K$, and $r \in (0,1]$, (1.10) yields

$$\begin{aligned} t^{n(j-1)} \left|\mu(B(x,tr)) - t^n \mu(B(x,r))\right| &\leq C_K r^\alpha t^{n(j-1)} \mu(B(x,r)) \\ t^{n(j-2)} \left|\mu(B(x,t^2 r)) - t^n \mu(B(x,tr))\right| &\leq C_K r^\alpha t^{n(j-2)} \mu(B(x,tr)) \\ \dots &\leq \dots \\ \left|\mu(B(x,t^j r)) - t^n \mu(B(x,t^{j-1} r))\right| &\leq C_K r^\alpha \mu(B(x,t^{j-1} r)). \end{aligned} \tag{1.12}$$

Adding the above inequalities, we obtain that

$$|\mu(B(x,\tau r)) - \tau^n \mu(B(x,r))| \leq C_K r^\alpha \mu(B(x,r)) \sum_{i=0}^{j-1} \left(\frac{1}{(\sqrt{2})^n}\right)^i, \tag{1.13}$$

which implies that for $x \in (0,R)$, $x \in K$ and $\tau \in (0,1/2)$

$$\left|\frac{\mu(B(x,\tau r))}{\mu(B(x,r))} - \tau^n\right| \leq C C_K r^\alpha. \tag{1.14}$$

The constant C depends only on the dimension n.

Definition 1.8 Let μ be a positive Radon measure supported on $\Sigma \subset \mathbb{R}^m$. Let $\alpha \in (0,1]$. We say that the density ratio of μ is locally C^α if, for each compact set $K \subset \Sigma$, there is a constant C_K such that

$$\left|\frac{\mu(B(x,r))}{\omega_n r^n} - 1\right| \leq C_K r^\alpha \tag{1.15}$$

for $x \in K$ and $0 < r < 1$. Here ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Like in [1], a first step in the proof consists in proving that if μ satisfies (1.10), then the restriction μ_0 of $\mathcal{H}^n$ to Σ is locally finite, and $d\mu(x) = D(x) d\mu_0(x)$ for some positive density $D(x)$ such that $\log D(x)$ is (locally) Hölder with exponent $\frac{\alpha}{\alpha+1}$ and, moreover, μ_0 has density ratio locally $C^{\frac{\alpha}{\alpha+1}}$. See Proposition 2.1 below for the precise details. Theorem 1.7 follows then from next results.

Theorem 1.9 *For each $\alpha \in (0,1]$ there exists $\beta = \beta(\alpha) > 0$ with the following property. If μ is a positive Radon measure supported on $\Sigma \subset \mathbb{R}^m$ whose density ratio is locally C^α, then there exists a constant $\delta(n,m)$ depending only on n and m such that if $x_0 \in \Sigma$ and $\Sigma \cap B(x_0, 2R_0)$ is $\delta(n,m)$-Reifenberg flat, then $\Sigma \cap B(x_0, R_0)$ is a $C^{1,\beta}$ submanifold of dimension n in $\mathbb{R}^m$.*

Theorem 1.10 *For each $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ with the following property. If μ is a positive Radon measure supported on $\Sigma \subset \mathbb{R}^m$ whose density ratio is locally C^α, then:*

(i) if $n = 1, 2$, Σ is a $C^{1,\beta}$ submanifold of dimension n in $\mathbb{R}^m$,

(ii) if $n \geq 3$, Σ is a $C^{1,\beta}$ submanifold of dimension n in $\mathbb{R}^m$ away from a closed set $\mathcal{S}$ such that $\mathcal{H}^n(\mathcal{S}) = 0$.

To prove Theorem 1.9, building on ideas from [5] and [1], we first show that if μ is a Radon measure supported on $\Sigma \subset \mathbb{R}^m$, the local behavior of the quantity $\frac{\mu(B(x,r))}{\omega_n r^n}$ for $x \in \Sigma$ and $r \in (0,1]$ determines the regularity of Σ near flat points. This "ε-regularity" type result is contained in Section 3. These ideas, combined with the analysis of uniform measures contained in [8], yield the proof of Theorem 1.10 in Section 4. A key ingredient for the proof is the classification theorem of uniform measures of Preiss (see Theorem 4.1 below).

A careful look at the proof of Theorem 1.9 combined with the fact that in $\mathbb{R}^{n+1}$ the support of an n-uniform measure is either an n-plane or (modulo translation and rotation) $\{x \in \mathbb{R}^4 : x_4^2 = x_1^2 + x_2^2 + x_3^2\} \times \mathbb{R}^{n-3}$ (see [5]) gives the following result in codimension 1.

Corollary 1.11 *For each $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ with the following property. If μ is a positive Radon measure supported on $\Sigma \subset \mathbb{R}^{n+1}$ whose density ratio is locally C^α, then Σ is a $C^{1,\beta}$ submanifold of dimension n in $\mathbb{R}^{n+1}$ away from a closed set $\mathcal{S}$ of dimension at most $n-3$. If $n = 3$, $\mathcal{S}$ is discrete.*

In fact, it is easy to show that when $n \geq 4$ for $x_0 \in \mathcal{S}$, given $\varepsilon > 0$ there exists $r_0 > 0$ such that for $r \in (0, r_0)$, there exists an $(n-3)$-plane $L(x_0, r)$, through x, such that

$$\mathcal{S} \cap B(x_0, r) \subset (L(x_0, r) \cap B(x_0, r); \varepsilon r)\,. \tag{1.16}$$

Lemma 2.4 in [4], together with (1.16), ensures that the Hausdorff dimension of $\mathcal{S}$ is, at most, $n-3$. In the case when $n = 3$, one can show that for $x_0 \in \mathcal{S}$ there exists $r_0 > 0$ so that $(\Sigma \cap B(x_0, r_0) \backslash \{x_0\}) \cap \mathcal{S} = \emptyset$. This combined with the fact that $\mathcal{S}$ is closed implies that $\mathcal{S}$ is discrete.

2 Preliminaries

In this section we state several results which will be used throughout the paper. The codimension one versions appear in [1]. The reader would realize that the proofs given in there do not depend on the codimension. Thus we do not include proofs.

Proposition 2.1 *Let $\alpha > 0$ be given. Let μ be a measure supported on $\Sigma \subset \mathbb{R}^m$ and suppose that for all compact sets $K \subset \Sigma$, there is a constant C_K such that*

$$|R_t(x,r)| \leq C_K r^\alpha \text{ for } x \in K \text{ and } t, r \in (0,1]. \tag{2.1}$$

Then the density

$$D(x) = \lim_{r \to 0^+} \frac{\mu(B(x,r))}{\omega_n r^n} \tag{2.2}$$

(where ω_n denotes the n-dimensional Hausdorff measure of the unit ball in $\mathbb{R}^n$) exists for all $x \in \Sigma$, and

$$0 < D(x) < +\infty \quad \text{for} \quad x \in \Sigma. \tag{2.3}$$

Moreover, $\log D(x)$ is locally Hölder; i.e., for all compact sets $K \subset \Sigma$, we can find C'_K such that

$$|\log D(x) - \log D(y)| \leq C'_K |x-y|^{\frac{\alpha}{1+\alpha}} \quad \text{for} \quad x, y \in K. \tag{2.4}$$

*Finally, denote by μ_0 the restriction of $\mathcal{H}^n$ to Σ, i.e.,
$\mu_0 = \mathcal{H}^n \llcorner \Sigma$. Then μ_0 is finite on compact sets,*

$$d\mu(x) = D(x) d\mu_0(x), \tag{2.5}$$

the density ratio of μ_0 is locally $C^{\frac{\alpha}{1+\alpha}}$.

Remark 2.2 Notice that (2.1) only gives useful information for small values of r (i.e. for $r^\alpha < C_K^{-1}$). Thus, even though we did not say explicitly that $R_t(x,r)$ is controlled for r small enough, this is implicit in (2.1).
Since (2.4) and the fact that the density ratio of μ_0 is locally Hölder continuous contain some amount of large-scale information, we might be forced in some cases to take huge values of C'_K and C''_K, that depend on the large-scale behavior of μ (and not only on the C_K). This problem can easily be fixed by restricting the domain of validity of (2.4) to $|x-y| \leq r_0$, where r_0 depends on C_K, and similarly restricting the definition of locally Hölder continuity to radii $0 < r < r_0$. With this restriction, we may then chose constants C'_K and C''_K that depend only on C_K. Alternatively, we could also fix the problem by requiring that μ be doubling.

Let μ be an n-Ahlfors regular measure supported on $\Sigma \subset \mathbb{R}^m$, i.e suppose that for each compact set $K \subset \Sigma$ there is a constant $C_K > 1$ such that

$$C_K^{-1} < \frac{\mu(B(x,r))}{\omega_n r^n} < C_K \tag{2.6}$$

for $x \in K$ and $0 < r < 1$. We follow [5] and introduce some moments for Ahlfors regular measures. Fix a compact set K and for $x_1 \in K$, define the vector $b = b_{x_1,r}$ by

$$b = \frac{n+2}{2\omega_n r^{n+2}} \int_{B(x_1,r)} (r^2 - |y-x_1|^2)(y-x_1) d\mu(y). \tag{2.7}$$

Also define the quadratic form $Q = Q_{x_1,r}$ on $\mathbb{R}^m$ by

$$Q(x) = \frac{n+2}{\omega_n r^{n+2}} \int_{B(x_1,r)} \langle x, y-x_1 \rangle^2 d\mu(y) \tag{2.8}$$

for $x \in \mathbb{R}^m$. In all our estimates we use the fact that

$$|\mu(B(x,t)) - \omega_n t^n| \leq C_K t^{n+\alpha} \text{ for } x \in \Sigma \cap \overline{B}(x_1, 1) \text{ and } 0 < t < 1, \tag{2.9}$$

which we get by applying (1.15) with $K^* = \{x \in \Sigma; \mathrm{dist}(x, K) \leq 1\}$.
Roughly speaking the following proposition shows that if the density ratio of μ, $\frac{\mu(B(x,r))}{\omega_n r^n}$ approaches 1 as r tends to 0 in a Hölder fashion then the points in the support of μ almost satisfy a quadratic equation.

Proposition 2.3 *Let μ be a measure supported on $\Sigma \subset \mathbb{R}^m$. Assume that the density ratio of μ is locally C^α. Let $K \subset \mathbb{R}^m$ be a compact set. For $x_1 \in K$ and $0 < r < 1$, let*

$$Tr(Q) = \frac{n+2}{\omega_n r^{n+2}} \int_{B(x_1,r)} |y - x_1|^2 d\mu(y) \tag{2.10}$$

denote the trace of Q. Then

$$|Tr(Q) - n| \leq C_K r^\alpha. \tag{2.11}$$

Also, if $0 < r < \frac{1}{2}$, for $x \in \Sigma \cap B\left(x_1, \frac{r}{2}\right)$,

$$\left|2\langle b, x - x_1\rangle + Q(x - x_1) - |x - x_1|^2\right| \leq C\frac{|x - x_1|^3}{r} + C_K r^{2+\alpha}. \tag{2.12}$$

For a measure μ supported on Σ and satisfying (1.15) we introduce the quantity that allows us to measure the local flatness of Σ and prove its regularity. Let $K \subset \Sigma$ be a compact set and let $x_1 \in K$, for small radii ρ consider

$$\beta(x_1, \rho) = \inf_P \{\frac{1}{\rho} \sup\{\mathrm{dist}(y, P); y \in \Sigma \cap B(x_1, \rho)\}\}. \tag{2.13}$$

Here the infimum is taken over all affine n-planes P through x_1. In particular by (1.1)

$$\beta(x_1, \rho) \leq \theta(x_1, \rho).$$

Note that (1.15) implies that μ satisfies the hypothesis of Theorem 1.4 which ensures that if $\Sigma \cap B(x_0, 2R_0)$ is Reifenberg flat for some $x_0 \in \Sigma$ then

$$\Sigma \cap B(x_0, R_0) \text{ is Reifenberg flat with vanishing constant.} \tag{2.14}$$

Hence for $x_1 \in \Sigma \cap B(x_0, R_0)$, $\beta(x_1, \rho)$ converges to 0 as $\rho \to 0$ uniformly on compact sets. The key step in the proof of Theorem 1.9 is to show that if μ satisfies (1.10) then there exists $\gamma > 0$ such that for ρ small $\beta(x_1, \rho) < C_K \rho^\gamma$. This is also the main idea behind the proof of Theorem 1.6. Its implementation in the codimension 1 case is significantly simpler. Once the asymptotic behavior of β has been established we simply apply the following theorem which appears in Section 9 in [1]

Proposition 2.4 *Let $0 < \beta \leq 1$ be given. Suppose $\Sigma \cap B(x_0, 2R_0)$ is a Reifenberg flat set with vanishing constant of dimension n in $\mathbb{R}^m$ and that, for each compact set $K \subset \Sigma$, there is a constant C_K such that*

$$\beta(x, r) \leq C_K r^\beta \text{ for } x \in K \text{ and } r \leq 1. \tag{2.15}$$

Then $\Sigma \cap B(x_0, R_0)$ is a $C^{1,\beta}$ submanifold of dimension n of $\mathbb{R}^m$.

3 The proof of Theorem 1.9

This section is devoted to the proof of Theorem 1.9. So we assume that μ is a measure supported on $\Sigma \subset \mathbb{R}^m$ whose density ratio is locally C^α and we consider a ball $B(x_0, 2R_0)$, with $x_0 \in \Sigma$, such that $\Sigma \cap B(x_0, 2R_0)$ is δ-Reifenberg flat, with δ small enough. Since $\Sigma \cap B(x_0, R_0)$ is Reifenberg flat with vanishing constant (by Theorem 1.4), for any compact set $K \subset \Sigma$ and for each small $\delta_0 > 0$, we can find $r_0 \in (0, 10^{-2}R_0)$ depending on K such that

$$\theta(x, r) \leq \delta \text{ when } x \in \Sigma \cap B(x_0, R_0), \quad \mathrm{dist}(x, K) \leq 1, \text{ and } 0 < r \leq 10r_0. \tag{3.1}$$

By Proposition 2.4, to show that $\Sigma \cap B(x_0, R_0)$ is a $C^{1,\beta}$ submanifold of dimension n, it is enough to show that $\beta(x_1, r) \leq C_K r^\beta$ for $x \in K$ and $r \leq 1$. Our goal is to show that this is the case.
Without loss of generality we may assume that $x_1 = 0 \in \Sigma \cap B(x_0, R_0)$.

3.1 Preliminaries

We recall the main properties of b and Q that will be used in this section. First, $b = b_r \in \mathbb{R}^m$ and

$$\begin{aligned} |b_r| &\leq \frac{n+2}{2\omega_n r^{n+2}} \int_{B(0,r)} r^2 |y| d\mu(y) \leq \frac{(n+2)r}{2\omega_n r^n} \mu(B(0,r)), \\ |b_r| &\leq \frac{(n+2)r}{2} \left\{ 1 + \frac{C_K r^\alpha}{\omega_n} \right\} \leq (n+2)r, \end{aligned} \tag{3.2}$$

by (2.7) and (2.9), provided we assume that $\frac{C_K r_0^\alpha}{\omega_n} \leq 1$. We do not explicitly need (3.2), but the homogeneity is important to keep in mind.
Next, Q is a quadratic form defined on $\mathbb{R}^m$, (2.8) and (2.9) ensure that for $x \in \mathbb{R}^m$

$$\begin{aligned} 0 \leq Q(x) &\leq \frac{n+2}{\omega_n r^{n+2}} \int_{B(0,r)} |x|^2 r^2 d\mu(y) \leq \frac{(n+2)|x|^2}{\omega_n r^n} \mu(B(0,r)) \\ &\leq (n+2)|x|^2 (1 + \omega_n^{-1} C_K r^\alpha) \leq (2n+4)|x|^2, \end{aligned} \tag{3.3}$$

and

$$|Tr(Q) - n| \leq C C_K r^\alpha \tag{3.4}$$

by (2.11). It is convenient to set

$$\widetilde{Q}(x) = |x|^2 - Q(x). \tag{3.5}$$

Then (2.12) yields that

$$|2\langle b_r, x\rangle - \widetilde{Q}(x)| \leq C r^{-1} |x|^3 + C C_K r^{2+\alpha} \text{ for } x \in \Sigma \cap B(0, \frac{r}{2}). \tag{3.6}$$

Initially we use (3.4) and (3.6) to derive more information about Q and b. We work at scales of the form $\rho = r^{1+\gamma}$ smaller than r. Here γ is a positive constant that will assume several different values.
It is important to understand how (3.6) is modified by a change of scale. Set

$$\Sigma_\rho = \frac{1}{\rho}\Sigma, \tag{3.7}$$

and

$$\Sigma'_\rho = \Sigma_\rho \cap B(0, \frac{r}{2\rho}) = \frac{1}{\rho}(\Sigma \cap B(0, \frac{r}{2})). \tag{3.8}$$

Note that (3.1) guarantees that we can choose an n-plane L through the origin such that

$$D[L \cap B(0,\rho), \Sigma \cap B(0,\rho)] \leq \rho\theta(0,\rho) \leq \rho\delta, \tag{3.9}$$

where D denotes the Hausdorff distance between sets, as in (1.2). (See also (1.1) for the definition of $\theta(0,\rho)$). Moreover for $z \in \Sigma'_\rho$ we can apply (3.6) to $x = \rho z$ and get that

$$\begin{aligned} |2\langle \frac{b_r}{\rho}, z\rangle - \widetilde{Q}(z)| &= \rho^{-2}|2\langle b, x\rangle - \widetilde{Q}(x)| \\ &\leq C\rho^{-2}r^{-1}|x|^3 + CC_K\rho^{-2}r^{2+\alpha} \\ &= C\rho r^{-1}|z|^3 + CC_K\rho^{-2}r^{2+\alpha} \\ &= Cr^{\gamma}|z|^3 + CC_K r^{\alpha-2\gamma} \end{aligned} \tag{3.10}$$

because $\rho = r^{1+\gamma}$. In particular,

$$|\langle 2b_r r^{-1-\gamma}, z\rangle - \widetilde{Q}(z)| \leq Cr^{\gamma} + CC_K r^{\alpha-2\gamma} =: \epsilon_0(r,\gamma) \text{ for } z \in \Sigma_{r^{1+\gamma}} \cap B(0, \frac{1}{2}). \tag{3.11}$$

To motivate the argument in the proof of Theorem 1.9 we briefly recall the main ideas in the proof of Theorem 1.6. (3.11) encodes the information required to estimate the quantity $\beta(0,\rho_0)$ defined in (2.13). In the codimension 1 case one needs to consider two cases. Either b in (3.11) is very small, and then one obtains an estimate on the smallest eigenvalue of Q which allows one to say that at the appropriate scale Σ is very close to the plane normal to the corresponding eigenspace. If b is "large" then at the appropriate scale Σ is very close to the plane orthogonal to b. In both cases one produces the normal vector which is orthogonal to the plane Σ is close to. In higher codimensions we need to produce an $m-n$ orthonormal family of vectors whose span is orthogonal to the n-plane Σ is close to, at a given scale. The difficulty lies on the fact that there is only a single equation at hand, namely (3.11). To overcome this problem we are forced to do a multi-scale analysis of (3.11).

3.2 Estimates for $\beta(0,t)$ when b_r is small

Let us assume that

$$|b| \leq r^{1+2\theta} \tag{3.12}$$

for some $\theta > 0$ that will be fixed below. Firs we estimate Q. Notice that (3.11) and (3.12) ensure that for $z \in \Sigma_{r^{1+\gamma}} \cap B(0, \frac{1}{2})$ we have

$$\begin{aligned} |\widetilde{Q}(z)| &\leq |\langle 2br^{-1-\gamma}, z\rangle| + Cr^{\gamma} + CC_K r^{\alpha-2\gamma} \\ &\leq r^{2\theta-\gamma} + Cr^{\gamma} + CC_K r^{\alpha-2\gamma} \\ &=: \epsilon_1(r,\theta,\gamma). \end{aligned} \tag{3.13}$$

Note that (3.13) only provides useful information when γ satisfies

$$0 < \gamma < 2\theta \quad \text{and} \quad 2\gamma < \alpha. \tag{3.14}$$

Choose an orthonormal basis $(e_1, \ldots, e_m)$ of $\mathbb{R}^m$ that diagonalizes Q. Thus

$$Q(z) = \sum_{i=1}^{m} \lambda_i \langle z, e_i\rangle^2 \tag{3.15}$$

for $z \in \mathbb{R}^m$. Without loss of generality we may assume that

$$\lambda_1 \leq \lambda_2 \cdots \leq \lambda_m. \tag{3.16}$$

Note that $\lambda_1 \geq 0$ because $Q(z) \geq 0$ (see (2.8) or (3.3)). Also, by (3.4)

$$\sum_{i=1}^{m} \lambda_i = Tr(Q) \leq n + CC_K r^{\alpha}. \tag{3.17}$$

In particular, by (3.16) if $k = m - n$

$$m\lambda_1 \leq Tr(Q) \leq n + CC_K r^{\alpha} < n + \frac{1}{2}, \tag{3.18}$$

$$(n+1)\lambda_k \leq Tr(Q) \leq n + CC_K r^{\alpha} < n + \frac{1}{2}, \tag{3.19}$$

provided we take r_0 small enough. Thus

$$0 \leq \lambda_1 \leq \frac{2n+1}{2m} \quad \text{and} \quad \lambda_k \leq \frac{2n+1}{2n+2}. \tag{3.20}$$

This is just a crude first step. Our next goal is to obtain more precise estimates on Q, when (3.12) holds, i.e., $|b| \leq r^{1+2\theta}$, under the additional constraint that

$$0 < \theta < \frac{\alpha}{3}. \tag{3.21}$$

Lemma 3.1 *Suppose that (3.12), (3.14) and (3.21) hold. Let* $k = m - n$*. For* r_0 *small enough and* $\epsilon_2(r,\theta,\gamma) = na^{-2}\epsilon_1(r,\theta,\gamma)$*, where* a *is a constant that only depends on* n *and* m*, we have*

$$\begin{aligned} 0 \le \sum_{i=1}^{k} \lambda_i &\le \epsilon_2(r,\theta,\gamma) + CC_K r^{\alpha}, && (3.22)\\ |\lambda_{k+i} - 1| &\le \epsilon_2(r,\theta,\gamma) + r^{\alpha/2} \quad \text{for} \quad 1 \le i \le n, && (3.23) \end{aligned}$$

and

$$|\widetilde{Q}(z) - \sum_{l=1}^{k} \langle z, e_l\rangle^2| \le \big(\epsilon_2(r,\theta,\gamma) + r^{\alpha/2}\big)\, |z|^2 \text{ for } z \in \mathbb{R}^m. \tag{3.24}$$

Note that (3.24) automatically follows from (3.22) and (3.23). In fact if we write $z = \sum\limits_{i=1}^{m} z_i e_i$, then by (3.5) and (3.15) we have

$$\begin{aligned} \widetilde{Q}(z) - \sum_{i=1}^{k} \langle z, e_i\rangle^2 &= |z|^2 - Q(z) - \sum_{i=1}^{k} z_i^2 \\ &= \sum_{i=1}^{n} z_{k+i}^2 - \sum_{i=1}^{m} \lambda_i z_i^2 \\ &= -\sum_{l=1}^{k} \lambda_l z_l^2 + \sum_{i=1}^{n} (1 - \lambda_{i+k}) z_i^2. \end{aligned} \tag{3.25}$$

Note that the choice $\gamma = \theta$ with θ as in (3.21) satisfies (3.14). In this case Lemma 3.1 becomes

Corollary 3.2 *Suppose that (3.12), and (3.21) hold. For* r_0 *small enough*

$$\begin{aligned} 0 \le \sum_{i=1}^{k} \lambda_i &\le Cr^{\theta}, && (3.26)\\ |\lambda_{k+i} - 1| &\le Cr^{\theta} \quad \text{for} \quad 1 \le i \le n, && (3.27) \end{aligned}$$

and

$$|\widetilde{Q}(z) - \sum_{l=1}^{k} \langle z, e_l\rangle^2| \le Cr^{\theta}|z|^2 \text{ for } z \in \mathbb{R}^m, \tag{3.28}$$

where C *is a constant that depends on* K*,* n *and* m*.*

To prove Lemma 3.1 we need some preliminary results. The first one is the following.

Lemma 3.3 *Let L denote an n-plane satisfying (3.9). For $l = 1, \cdots, k$ let v_l denote the orthogonal projection of e_l onto L. If δ and r_0 are chosen small enough,*

$$|\sum_{l=1}^{k} x_l v_l - \sum_{l=1}^{k} x_l e_l| \geq C^{-1}, \tag{3.29}$$

whenever $\sum_{l=1}^{k} |x_l|^2 = 1$

In Lemma 3.3, δ and r_0 depend on n, α, θ and γ. At most $2k$ values of θ and γ, are used depending only on α, and a choice of θ. Thus one can always choose $\delta > 0$ and $r_0 > 0$ to work simultaneously for all our choices. The constant $C > 1$ depends only on n and m.

Proof: To prove Lemma 3.3, we first estimate $\widetilde{Q}(z)$ for $z \in L \cap B(0, \frac{1}{3})$. Since $\rho z \in L \cap B(0, \rho)$, where $\rho = r^{1+\gamma}$, (3.9) guarantees that there is a point $x \in \Sigma \cap B(0, \rho)$ such that $|x - \rho z| \leq 2\rho\delta$. If δ is small enough, $|\rho^{-1}x| < \frac{1}{2}$, and so (3.13) ensures that $|\widetilde{Q}(\rho^{-1}x)| \leq \epsilon_1(r, \theta, \gamma)$. Also, $|\rho^{-1}x - z| = \rho^{-1}|x - \rho z| \leq 2\delta$, and hence (3.3) and (3.5) guarantee that

$$|\widetilde{Q}(\rho^{-1}x) - \widetilde{Q}(z)| \leq C\delta. \tag{3.30}$$

Altogether,

$$|\widetilde{Q}(z)| \leq \epsilon_1(r, \theta, \gamma) + C\delta \text{ for } z \in L \cap B(0, \frac{1}{3}). \tag{3.31}$$

We are now ready to prove (3.29). Let $u = \sum_{l=1}^{k} x_l e_l$ with $|u| = 1$. Then $w = \sum_{l=1}^{k} x_l v_l$ satisfies $|w| \leq 1$, thus (3.31) guarantees that

$$|\widetilde{Q}(u)| \leq |\widetilde{Q}(w)| + |\widetilde{Q}(u) - \widetilde{Q}(w)| \leq \epsilon_1(r, \theta, \gamma) + C\delta + |\widetilde{Q}(u) - \widetilde{Q}(w)|. \tag{3.32}$$

On the other hand since u belongs to the span of the first k eigenvectors of Q we have that

$$\widetilde{Q}(u) = 1 - Q(u) \geq 1 - \lambda_k \geq \frac{1}{2n+2}, \tag{3.33}$$

by (3.5), (3.15), and (3.20). If δ and r_0 are small enough, (3.32) and (3.33) imply that

$$|\widetilde{Q}(u) - \widetilde{Q}(w)| \geq \frac{1}{4n+4}. \tag{3.34}$$

Thus w cannot be too close to u (because of (3.3)), and (3.29) holds. ■

Now we want to use the fact that $\Sigma \cap B(x_0, R_0)$ is Reifenberg flat with vanishing constant to get important topological information on $\Sigma_\rho \cap B(0, \frac{1}{2})$, $\rho = r^{1+\gamma}$. Denote by P the n-plane through 0 which is orthogonal to $e_1, \cdots, e_k$. Thus

$$P = \text{span}^{\perp}(e_1, \cdots, e_k) = \text{span}(e_{k+1}, \ldots, e_m). \tag{3.35}$$

Figure 3.1:

Call π the orthogonal projection onto P. Also denote by $\pi^* : \mathbb{R}^m \to L$ the projection onto L parallel to the direction $\{e_1, \cdots, e_k\}$, i.e.

$$\pi^*(x) = \pi^*(\sum_{l=1}^{m} x_l e_l) = \sum_{l=1}^{n} x_{k+l} e_{k+l} - \sum_{l=1}^{k} y_l e_l,$$

where the orthogonal projection of $\sum_{l=1}^{k} y_l e_l$ into $L^\perp$ coincides with that of $\sum_{l=1}^{n} x_{k+l} e_{k+l}$. Here L is as in (3.9), and $L^\perp$ denotes the $(m-n)$ space orthogonal to L. Denote by π' the orthogonal projection of $\mathbb{R}^m$ onto $L^\perp$. Lemma 3.3 ensures that

$$\begin{aligned} C^{-1} |\sum_{l=1}^{k} y_l e_l| &\le |\sum_{l=1}^{k} y_l e_l - \sum_{l=1}^{k} y_l v_l| = |\pi'(\sum_{l=1}^{k} y_l e_l)| \\ &\le |\pi'(\sum_{l=1}^{n} x_{k+l} e_{k+l})| \le |\sum_{l=1}^{n} x_{k+l} e_{k+l}| \le |x|. \end{aligned} \tag{3.36}$$

Thus

$$|\pi^*(x)| \le C_0 |x| \text{ for } x \in \mathbb{R}^m. \tag{3.37}$$

Here $C_0 = 2C$ where C is as in (3.29), a constant that depends only on n and m.
Set $a = (4C_0)^{-1}$, and recall that $\rho = r^{1+\gamma}$, where γ satisfies (3.14). A degree argument like the one used in the proof of Lemma 8.3 in [1] guarantees that:

Lemma 3.4 *For every* $\xi \in P \cap \overline{B}(0, a)$, *there is a point* $z \in \Sigma_\rho \cap B(0, \frac{1}{2})$ *such that* $\pi(z) = \xi$. [See Figure 8.1]

We have gathered all the information needed to prove Lemma 3.1.

Proof of Lemma 3.1: To prove (3.22) and (3.23), we apply Lemma 3.4 with $\xi = a e_{k+i}$, $1 \le i \le n$. We choose γ so that (3.14) holds. We get that for some $(t_i^1, \cdots, t_i^k) \in \mathbb{R}^k$,

$$z_i = \sum_{l=1}^{k} t_i^l e_l + a e_{k+i} \in \Sigma_\rho \cap B(0, \frac{1}{2}). \tag{3.38}$$

If we take $\gamma = \theta$, (3.13) and (3.21) guarantee that

$$|\widetilde{Q}(z_i)| \le \epsilon_1(r, \theta, \gamma). \tag{3.39}$$

Combining (3.5), (3.15) and (3.38), we obtain that

$$\widetilde{Q}(z_i) = |z_i|^2 - Q(z_i) = \sum_{l=1}^{k} (1 - \lambda_l)(t_i^l)^2 + (1 - \lambda_{k+i}) a^2. \tag{3.40}$$

Since $1-\lambda_l \geq (2n+2)^{-1}$ for $1 \leq l \leq k$ (by (3.20)), we get that

$$(1-\lambda_{k+i})a^2 \leq \widetilde{Q}(z_i) \leq \epsilon_1(r,\theta,\gamma) \tag{3.41}$$

(by (3.39)). Thus

$$\lambda_{k+i} \geq 1 - a^{-2}\epsilon_1(r,\theta,\gamma), \tag{3.42}$$

for $1 \leq i \leq n$, and hence

$$\sum_{i=1}^{n} \lambda_{k+i} \geq n - \epsilon_2(r,\theta,\gamma). \tag{3.43}$$

By (3.17) and (3.43) we have that

$$\sum_{l=1}^{k} \lambda_l = Tr(Q) - \sum_{i=1}^{n} \lambda_i \leq CC_K r^{\alpha} + a^{-2}\epsilon_1(r,\theta,\gamma). \tag{3.44}$$

This proves (3.22), because we already know that $\sum_{l=1}^{k} \lambda_l \geq 0$. To prove (3.23), we proceed by contradiction and suppose that we can find $1 \leq i_0 \leq n$ such that

$$\lambda_{k+i_0} > 1 + \epsilon_2(r,\theta,\gamma) + r^{\alpha/2}. \tag{3.45}$$

Then (3.42) and (3.45) yield

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{n} \lambda_{k+i} \geq \lambda_{k+i_0} + (n-1)(1 - \frac{\epsilon_2(r,\theta,\gamma)}{n}) > n + r^{\alpha/2}. \tag{3.46}$$

This contradicts (3.17), thus (3.45) is impossible and (3.23) holds. We already observed earlier that (3.24) is a consequence of (3.22) and (3.23), and so Lemma 3.1 follows. ∎

Next we use Corollary 3.2 to rewrite (3.11), still under the assumption that (3.12) holds for some $\theta \in (0, \frac{\alpha}{3})$. Combining (3.11) and (3.24) we get that for $z \in \Sigma_\rho \cap B(0, \frac{1}{2})$

$$\begin{aligned} |\langle 2br^{-1-\gamma}, z\rangle - \sum_{l=1}^{k} \langle z, e_l\rangle^2| &\leq Cr^{\gamma} + CC_K r^{\alpha-2\gamma} + |\widetilde{Q}(z) - \sum_{l=1}^{k}\langle z, e_l\rangle^2| \\ &\leq Cr^{\gamma} + CC_K r^{\alpha-2\gamma} + Cr^{\theta} \\ &=: \epsilon_3(r,\theta,\gamma). \end{aligned} \tag{3.47}$$

Note that here $\rho = r^{1+\gamma}$ for any $\gamma > 0$ (as in (3.11)). Of course (3.47) only provides useful information when $0 < \gamma < \frac{\alpha}{2}$.
Next we want to get a better estimate on the "tangential part" of b. This allows us to estimate $\beta(0,s)$ as defined in (2.13) for an appropriately chosen s.

Proposition 3.5 *Suppose that* (3.12) *holds. With the notation above, if* $b = \sum_{i=1}^{m} b_i e_i$, *then*

$$|b_{k+i}| \leq Cr^{1+\eta}\epsilon_3(r,\theta,\eta) + Cr^{1+4\theta-\eta} \text{ for } 1 \leq i \leq n. \tag{3.48}$$

Remark 3.6 The goal is to show that given appropriate choices for θ and η satisfying (3.21) and (3.14) with η in place of γ, (3.48) provides an improvement over (3.12). In the codimension 1 case it was possible to choose $\gamma = \eta = 3\theta/2$. The reader will note that this choice does improve estimate (3.12). Unfortunately in the higher codimension set up, it is premature to choose η at this stage.

Proof: Choose θ and $\gamma = \eta$ such that (3.21) and (3.14) hold. We can then apply Lemma 3.4. Fix $i \in \{1, 2, \ldots, n\}$ and apply Lemma 3.4 to the two points $\xi_\pm = \pm a e_{k+i}$. We get two k-tuples $(t_1^\pm, \cdots, t_k^\pm)$ such that

$$z_\pm = \sum_{l=1}^{k} t_l^\pm e_l \pm a e_{k+i} \in \Sigma_{r^{1+\eta}} \cap B(0, \frac{1}{2}). \tag{3.49}$$

Then (3.47) implies that

$$\sum_{l=1}^{k} 2b_l r^{-1-\eta} t_l^\pm \pm 2b_{k+i} r^{-1-\eta} a - \sum_{l=1}^{k} (t_l^\pm)^2 \geq -\epsilon_3(r,\theta,\eta). \tag{3.50}$$

Set $f_l(t) = 2b_l r^{-1-\eta} t - t^2$ for $1 \leq l \leq k$. Then

$$f_l(t) = (b_l r^{-1-\eta})^2 - (b_l r^{-1-\eta} - t)^2 \leq (b_l r^{-1-\eta})^2 \tag{3.51}$$

for all $t \in \mathbb{R}$. Hence by (3.50) and (3.51) we have that

$$\pm 2b_{k+i} r^{-1-\eta} a \geq -\epsilon_3(r,\theta,\eta) - \sum_{l=1}^{k} f_l(t_l^\pm) \geq -\epsilon_3(r,\theta,\eta) - \sum_{l=1}^{k} (b_l r^{-1-\eta})^2. \tag{3.52}$$

Here we have two inequalities, one for each sign ±. Thus by (3.12)

$$|b_{k+i}| \leq (2a)^{-1} r^{1+\eta}\epsilon_3(r,\theta,\eta) + (2a)^{-1}|b|^2 r^{-1-\eta} \leq Cr^{1+\eta}\epsilon_3(r,\theta,\eta) + Cr^{1+4\theta-\eta}. \tag{3.53}$$

∎

Combining (3.48) and (3.47), we get that for $z \in \Sigma_\rho \cap B(0, \frac{1}{2})$, where $\rho = r^{1+\gamma}$,

(3.54)

$$\begin{aligned} |\langle 2\sum_{l=1}^{k} b_l r^{-1-\gamma}, z\rangle - \sum_{l=1}^{k} \langle z, e_l\rangle^2| &\leq \epsilon_3(r,\theta,\gamma) + |\sum_{i=1}^{n} 2b_{k+i} r^{-1-\gamma}\langle z, e_{k+i}\rangle| \\ &\leq \epsilon_3(r,\theta,\gamma) + Cr^{-1-\gamma} r^{1+\eta}\epsilon_3(r,\theta,\eta) + Cr^{4\theta-\eta-\gamma} \\ &\leq C(r^\gamma + r^{\alpha-2\gamma} + r^\theta + r^{2\eta-\gamma} + r^{\alpha-\eta-\gamma} + r^{\theta+\eta-\gamma} + r^{4\theta-\eta-\gamma}). \end{aligned}$$

This holds for θ as in (3.21), η satisfying

$$0 < \eta < 2\theta \quad \text{and} \quad 2\eta < \alpha, \tag{3.55}$$

and all $\gamma > 0$ as in (3.47). It only provides an interesting estimate for some values of γ. Choose

$$0 < 4\gamma < \alpha, \tag{3.56}$$

and define

$$\epsilon_4(r, \theta, \gamma, \eta) := C(r^{\gamma} + r^{\theta} + r^{2\eta-\gamma} + r^{\theta+\eta-\gamma} + r^{4\theta-\eta-\gamma}). \tag{3.57}$$

Then (3.54) becomes

$$|\langle 2\sum_{l=1}^{k} b_l r^{-1-\gamma}, z\rangle - \sum_{l=1}^{k}\langle z, e_l\rangle^2| \leq \epsilon_4(r, \theta, \gamma, \eta). \tag{3.58}$$

Proposition 3.7 *Suppose that* (3.12) *holds. With the notation above we have that*

$$|\sum_{l=1}^{k}\langle z, e_l\rangle e_l| \leq 3\epsilon_4(r, \theta, \gamma, \eta)^{1/2} \text{ for } z \in \Sigma_\rho \cap B(0, \frac{1}{4}). \tag{3.59}$$

Here $\rho = r^{1+\gamma}$. The exponents θ, γ and η satisfy (3.21), (3.55) and (3.56).

Proof: Set $z^\perp = \sum_{l=1}^{k}\langle z, e_l\rangle e_l$ for $z \in \Sigma_\rho \cap B(0, \frac{1}{2})$. Then (3.54) can be written

$$|\langle z^\perp, z^\perp - d\rangle| \leq \epsilon_4(r, \theta, \gamma, \eta), \tag{3.60}$$

where $d = 2b^\perp r^{-1-\gamma}$. This forces

$$|z^\perp| \leq \epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}} \tag{8.59+}$$

or

$$|z^\perp - d| \leq \epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}}. \tag{8.59-}$$

If $|d| \leq 2\epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}}$, then (3.59) trivially follows from this. So let us assume that $|d| > 2\epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}}$. Denote by $\mathcal{U}$ the connected component of $\Sigma_\rho \cap B(0, \frac{1}{2})$ containing the origin, and set

$$\mathcal{U}_\pm = \{z \in \mathcal{U}; (8.59\pm) \text{ holds}\}. \tag{3.61}$$

Obviously $\mathcal{U}_+$ and $\mathcal{U}_-$ are closed in $\mathcal{U}$, and since $\mathcal{U}$ is the disjoint union of $\mathcal{U}_+$ and $\mathcal{U}_-$ (because $|d| > 2\epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}}$), $\mathcal{U}$ must be equal to $\mathcal{U}_+$. Thus, to prove (3.59), it is enough to show that

$$\Sigma_\rho \cap B(0, \frac{1}{4}) \subset \mathcal{U}. \tag{3.62}$$

Since (3.1) holds, Σ_ρ is locally Reifenberg flat. Thus the same argument used to prove Proposition 8.5 in [1] yields (3.62). Proposition 3.7 follows. ■

Note that Proposition 3.7 is equivalent to

Proposition 3.8 *Suppose that θ, γ and η satisfy (3.21), (3.55) and (3.56). If $|b_r| \le r^{1+2\theta}$, then*

$$\beta(0, \frac{1}{4} r^{1+\gamma}) \le 12 \epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}} =: \epsilon_5(r, \theta, \gamma, \eta), \tag{3.63}$$

where

$$\epsilon_4(r, \theta, \gamma, \eta) = C(r^\gamma + r^\theta + r^{2\eta-\gamma} + r^{\theta+\eta-\gamma} + r^{4\theta-\eta-\gamma}), \tag{3.64}$$

with C depending on n, m and K.

3.3 Estimates for $\beta(0, t)$ when b_r is either big or small

Recall that b depends on r. So far we have not emphasized this dependence as there was no room for confusion. From now on, we need to keep track of it as it will be made clear shortly.

When (3.12) does not hold, i.e.

$$|b_r| > r^{1+2\theta}, \tag{3.65}$$

(3.11) and (3.3) tell us that

$$|\langle 2b_r r^{-1-\gamma}, z\rangle| \le |\widetilde{Q}(z)| + Cr^\gamma + CC_K r^{\alpha-2\gamma} \le C \tag{3.66}$$

for $z \in \Sigma_{r^{1+\gamma}} \cap B(0, \frac{1}{2})$, provided that we choose $0 < \gamma < \frac{\alpha}{2}$, $r < r_0$ and r_0 small enough. Set $\tau = |b_r|^{-1} b_r$. Then

$$|\langle \tau, z\rangle| \le C|b_r|^{-1} r^{1+\gamma} \le Cr^{\gamma-2\theta} \tag{3.67}$$

for $z \in \Sigma_{r^{1+\gamma}} \cap B(0, \frac{1}{2})$. In the codimension 1 case, $|\langle \tau, z\rangle|$ measures the distance from z to the n-plane orthogonal to τ. (3.67) implies that $\beta(0, \frac{1}{4} r^{1+\gamma}) \le Cr^{\gamma-2\theta}$. In this case, choosing η, γ, θ appropriately one can guarantee that $\beta(0, \frac{1}{4} r^{1+\gamma})$ is bounded by a positive power of r. This case is done in [1].

In codimension $k = m - n$, we need to produce a k plane such that $z^\perp$, the orthogonal projection of $z \in \Sigma_{r^{1+\gamma}} \cap B(0, \frac{1}{2})$ onto this plane, is bounded by a positive power on r. To accomplish this, we need to choose $3k$ exponents η_i, γ_i, θ_i and $k+1$ radii r_i with $1 \le i \le k$ satisfying

$$0 < 3\theta_i < \alpha, \quad 0 < 4\gamma_i < \alpha, \quad 0 < \eta_i < 2\theta_i \quad \text{and} \quad 2\eta_i < \alpha, \tag{3.68}$$

and

$$r_1 = r, \qquad r_{i+1} = r_i^{1+\gamma_i}. \tag{3.69}$$

The difficulty lies on the fact that several additional compatibility conditions arise along the proof, and we need to check that they can be satisfied.

Next lemma is a straightforward consequence of Proposition (3.8).

Lemma 3.9 *Let $r_1, \dots, r_{k+1}$ satisfy* (3.68) *and* (3.69). *Suppose that there exists i, $1 \leq i \leq k$, such that*

$$|b_{r_i}| \leq r_i^{1+2\theta_i}. \tag{3.70}$$

Then,

$$\beta(0, \frac{1}{4} r_{i+1}) \leq \epsilon_5(r_i, \theta_i, \gamma_i, \eta_i) \tag{3.71}$$

and

$$\beta(0, \frac{r_{k+1}}{4}) \leq \frac{r_{i+1}}{r_{k+1}} \beta(0, \frac{r_{i+1}}{4}) \leq \frac{r_{i+1}}{r_{k+1}} \epsilon_5(r_i, \theta_i, \gamma_i, \eta_i). \tag{3.72}$$

The case when all the b_{r_i} are big is considered in next lemma.

Lemma 3.10 *Let $r_1, \dots, r_{k+1}$ satisfy* (3.68) *and* (3.69). *Suppose that for all $i = 1, \cdots, k$*

$$|b_{r_i}| \geq r_i^{1+2\theta_i}. \tag{3.73}$$

Then,

$$|\langle \tau_i, x \rangle| \leq C r_{i+1} r_i^{\gamma_i - 2\theta_i} = C r_i^{1+2\gamma_i - 2\theta_i} \quad \text{for} \quad x \in \Sigma \cap B(0, \frac{r_{i+1}}{2}), \tag{3.74}$$

where

$$\tau_i = \frac{b_{r_i}}{|b_{r_i}|}. \tag{3.75}$$

In this case, for $j \geq i+1$,

$$|\langle \tau_i, b_{r_j} \rangle| \leq C r_{i+1} r_i^{\gamma_i - 2\theta_i} \tag{3.76}$$

and

$$|\langle \tau_i, \tau_j \rangle| \leq C r_j^{-1-2\theta_j} r_{i+1} r_i^{\gamma_i - 2\theta_i} = C r_j^{-1-2\theta_j} r_i^{1+2\gamma_i - 2\theta_i} \quad \text{for} \quad j \geq i+1. \tag{3.77}$$

If, moreover, $|\langle \tau_i, \tau_j \rangle| < \frac{1}{2k}$ for $1 \leq i < j \leq k$, then

$$\beta(0, \frac{r_{k+1}}{4}) \leq C r_{k+1}^{-1} \max_{1 \leq i \leq k} r_i^{1+2\gamma_i - 2\theta_i}, \tag{3.78}$$

where C depends on n, m and K.

Proof: Notice that (3.67) guarantees that

$$|\langle \tau_i, z\rangle| \le C r_i^{\gamma_i - 2\theta_i} \quad \text{for} \quad z \in \Sigma_{r_{i+1}} \cap B(0, \frac{1}{2}), \tag{3.79}$$

and so (3.74) follows. On the other hand, inequality (3.76) is deduced from the definition of b_{r_j}, which appears in (2.7), and (3.74). The definition of τ_j combined with (3.73) and (3.76) yields (3.77).

If $|\langle \tau_i, \tau_j\rangle| < \frac{1}{2k}$ for $1 \le i < j \le k$, then (3.74) implies that $x^\perp$, the orthogonal projection of $x \in \Sigma \cap B(0, \frac{r_{k+1}}{2})$ onto the k-plane generated by $\tau_1, \ldots, \tau_k$, satisfies

$$|x^\perp| \le C \max_{1\le i\le k} r_i^{1+2\gamma_i - 2\theta_i}, \tag{3.80}$$

and then (3.78) follows. ■

3.4 Parameter adjustment: the end of the proof of Theorem 1.9

Given $r_1, \ldots, r_{k+1}$ satisfying (3.68) and (3.69), our goal is to show that either under the assumptions of Lemma 3.9 or Lemma 3.10 there exists $\beta > 0$ such that $\beta(0, \frac{1}{4}\, r_{k+1}) \le r_{k+1}^{\beta}$. By Proposition 2.4, this proves Theorem 1.9 because for any t sufficiently small (depending on K, n, m), we can choose $r_1, \ldots, r_{k+1}$ so that $t = r_{k+1}/4$.

In the situation of Lemma 3.9 we need the exponent of r in the right hand side of (3.72) to be positive. In the case of Lemma 3.10, the same concerning the exponent on the right side of (3.78). However, in order to apply (3.78), we first need to ensure that $|\langle \tau_i, \tau_j\rangle| < \frac{1}{2k}$. This will be achieved by showing that the exponent of r that appears in (3.77) can be made positive, so that $|\langle \tau_i, \tau_j\rangle| < \frac{1}{2k}$ for r small enough.

Our immediate task is to show that by choosing θ_i, η_i and γ_i appropriately and satisfying (3.68), the right hand sides of (3.72),(3.77), and (3.78) can be written as positive powers of r.

We first focus on the right hand side of (3.77) for $j \ge i+1$. Recall that

$$r_j = r_{j-1}^{1+\gamma_{j-1}} = r_i^{\prod_{l=i}^{j-1}(1+\gamma_l)}, \tag{3.81}$$

hence

$$r_j^{-1-2\theta_j} r_i^{1+2\gamma_i-2\theta_i} = r_i^{1+2\gamma_i-2\theta_i-(1+2\theta_j)\prod_{l=i}^{j-1}(1+\gamma_l)}. \tag{3.82}$$

Thus for each $i = 1, \cdots, k$ and $j \ge i+1$ we need

$$1 + 2\gamma_i - 2\theta_i - (1+2\theta_j)\prod_{l=i}^{j-1}(1+\gamma_l) > 0. \tag{3.83}$$

Similarly the right hand side of (3.78) yields

$$r_{k+1}^{-1} r_i^{1+2\gamma_i-2\theta_i} = r_i^{1+2\gamma_i-2\theta_i-\prod_{l=i}^{k}(1+\gamma_l)}, \tag{3.84}$$

which leads to the condition

$$1 + 2\gamma_i - 2\theta_i - \prod_{l=i}^{k}(1 + \gamma_l) > 0, \tag{3.85}$$

for all $i = 1, \cdots, k$.
Note that if (3.83) is satisfied for $j = k + 1$ then so is (3.85). Moreover (3.83) applied to $j = i + 1$ requires that for $i = 1, \cdots, k$

$$\gamma_i > 2\theta_i. \tag{3.86}$$

The right hand side of (3.72) produces five conditions for each $i = 1, \cdots, k$. In fact the term

$$r_{k+1}^{-1} r_{i+1} = r_i^{1+\gamma_i - \prod_{l=i}^{k}(1+\gamma_l)} \tag{3.87}$$

is multiplied by each one of the terms in $\epsilon_5(r_i, \theta_i, \gamma_i, \eta_i)$. We obtain:

$$1 + \frac{3}{2}\gamma_i - \prod_{l=i}^{k}(1 + \gamma_l) > 0, \tag{3.88}$$

$$1 + \gamma_i + \frac{\theta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l) > 0, \tag{3.89}$$

$$1 + \frac{\gamma_i}{2} + \eta_i - \prod_{l=i}^{k}(1 + \gamma_l) > 0, \tag{3.90}$$

$$1 + \frac{\gamma_i}{2} + \frac{\theta_i}{2} + \frac{\eta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l) > 0, \tag{3.91}$$

$$1 + \frac{\gamma_i}{2} + 2\theta_i - \frac{\eta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l) > 0. \tag{3.92}$$

Using (3.86) and (3.68), we observe that

$$1 + \frac{3}{2}\gamma_i - \prod_{l=i}^{k}(1 + \gamma_l) \geq 1 + \frac{\gamma_i}{2} + 2\theta_i - \frac{\eta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l), \tag{3.93}$$

and

$$1 + \gamma_i + \frac{\theta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l) \geq 1 + \frac{\gamma_i}{2} + \frac{\theta_i}{2} + \frac{\eta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l). \tag{3.94}$$

Thus (3.88) and (3.89) are satisfied whenever (3.68), (3.85), (3.91) and (3.92) hold.
At this point we are ready to choose the form of the exponents. Let

$$\gamma_{l+1} = \kappa\gamma_l, \quad \theta_{l+1} = \kappa\theta_l, \quad \eta_{l+1} = \kappa\eta_l, \tag{3.95}$$

with

$$0 < \kappa < \frac{1}{16}, \quad 0 < 3\theta_1 < \alpha, \quad 0 < 4\gamma_1 < \alpha, \qquad \text{and} \quad 0 < \eta_1 = \frac{3}{2}\theta_1 < 2\theta_1. \tag{3.96}$$

Note that this implies that $2\eta_1 < \alpha$.
This choice of $\gamma_1,\ \theta_1,\ \eta_1$ and κ ensures that (3.68) is satisfied, that $\eta_i = \frac{3}{2}\theta_i$, and that $\gamma_i < 1/4$.
Note that three of the four remaining conditions (3.83), (3.90), (3.91) and (3.92) contain the term $\prod_{l=i}^{k}(1+\gamma_l)$, or a product term which is bounded by it. Using the fact that for $x \geq 0$ $1 + x \leq e^x$ and that for $x < 1/2$, $e^x \leq 1 + x + x^2$ we have

$$\prod_{l=i}^{k}(1+\gamma_l) \leq e^{\sum_{l=i}^{k}\gamma_l} = e^{\sum_{l=0}^{k-i}\kappa^l\gamma_i} \leq e^{\frac{\gamma_i}{1-\kappa}} \leq 1 + \frac{\gamma_i}{1-\kappa} + \left(\frac{\gamma_i}{1-\kappa}\right)^2. \tag{3.97}$$

Hence (3.83), (3.90), (3.91) and (3.92) become

$$2\gamma_i - 2\theta_i - 2\kappa\theta_i - (1+2\kappa\theta_i)\left(\frac{\gamma_i}{1-\kappa} + \left(\frac{\gamma_i}{1-\kappa}\right)^2\right) > 0, \tag{3.98}$$

where we used the fact for $j \geq i+1$, $\theta_j \leq \kappa\theta_i$.

$$\frac{\gamma_i}{2} + \eta_i - \frac{\gamma_i}{1-\kappa} - \left(\frac{\gamma_i}{1-\kappa}\right)^2 > 0, \tag{3.99}$$

$$\frac{\gamma_i}{2} + \frac{\theta_i}{2} + \frac{\eta_i}{2} - \frac{\gamma_i}{1-\kappa} - \left(\frac{\gamma_i}{1-\kappa}\right)^2 > 0, \tag{3.100}$$

$$\frac{\gamma_i}{2} + 2\theta_i - \frac{\eta_i}{2} - \frac{\gamma_i}{1-\kappa} - \left(\frac{\gamma_i}{1-\kappa}\right)^2 > 0. \tag{3.101}$$

Combining (3.95) and (3.96),(3.98), (3.99), (3.100) and (3.101) become

$$2\gamma_1 - 2\theta_1(1+\kappa) - (1+2\kappa\theta_1)\left(\frac{\gamma_1}{1-\kappa} + \left(\frac{\gamma_1}{1-\kappa}\right)^2\right) > 0, \tag{3.102}$$

$$\frac{\gamma_1}{2} + \frac{3}{2}\theta_1 - \frac{\gamma_1}{1-\kappa} - \left(\frac{\gamma_1}{1-\kappa}\right)^2 > 0, \tag{3.103}$$

$$\frac{\gamma_1}{2} + \frac{5}{4}\theta_1 - \frac{\gamma_1}{1-\kappa} - \left(\frac{\gamma_1}{1-\kappa}\right)^2 > 0. \tag{3.104}$$

Note that if (3.103) is satisfied so is (3.104). Thus we only have two conditions left to satisfy, namely (3.102) and (3.103). At this point we can choose

$$\gamma_1 = \kappa^2(1-\kappa) \quad \text{and} \quad \theta_1 = \frac{\kappa^2(1-\kappa)}{2(1+4\kappa)}, \quad \text{provided} \quad 4\kappa^2(1-\kappa) < \alpha. \tag{3.105}$$

Recalling that $\kappa < \frac{1}{16}$, a straightforward calculation shows that

$$\begin{aligned} 2\gamma_1 \quad &- \quad 2\theta_1(1+\kappa) - (1+2\kappa\theta_1)\left(\frac{\gamma_1}{1-\kappa} + \left(\frac{\gamma_1}{1-\kappa}\right)^2\right) \\ &\geq \quad 2\kappa^2(1-\kappa) - 2\theta_1(1+\kappa) - \kappa^2(1+2\kappa\theta_1)(1+\kappa^2) \\ &\geq \quad \frac{\kappa^3}{1+4\kappa}(2-8\kappa-6\kappa^2) \geq \frac{\kappa^3}{1+4\kappa}, \end{aligned} \tag{3.106}$$

and

$$\begin{aligned} \frac{\gamma_1}{2} + \frac{3}{2}\theta_1 - \frac{\gamma_1}{1-\kappa} - \left(\frac{\gamma_1}{1-\kappa}\right)^2 \quad &\geq \quad \frac{1}{2}\left(\kappa^2(1-\kappa) + 3\theta_1 - 2\kappa^2 - 2\kappa^4\right) \\ &\geq \quad \frac{\kappa^2}{4(1+4\kappa)}\left(1 - 13\kappa - 12\kappa^2 - 16\kappa^3\right) \\ &\geq \quad \frac{\kappa^2}{32(1+4\kappa)}. \end{aligned} \tag{3.107}$$

Inequalities (3.106), (3.106) combined with (3.72), (3.63), (3.64), (3.78), (3.83), (3.90), (3.91) and (3.92) show that for κ such that $4\kappa^2(1-\kappa) < \alpha$

$$\beta(0, \frac{r_{k+1}}{4}) \leq C r^{\frac{\kappa^3}{1+4\kappa}} \quad \text{where} \quad r_{k+1} = r^{\prod_{l=1}^{k}(1+\gamma_l)}. \tag{3.108}$$

Note that (3.105) and (3.106) ensure that

$$\prod_{l=1}^{k}(1+\gamma_l) \leq 1 + \kappa + \kappa^2. \tag{3.109}$$

Therefore for $t = \frac{r_{k+1}}{4}$ (3.108) yields

$$\beta(0,t) \leq C t^{\frac{\kappa^3}{(1+4\kappa)(1+\kappa+\kappa^2)}}, \tag{3.110}$$

where C is a constant that depends on n, m, α and our specific choice of κ. ■

4 Proof of Theorem 1.10 using uniform measures

Statement (i) in Theorem 1.10 is a direct consequence of Theorem 1.9 and Theorem 1.4. So it only remains to prove (ii). To this effect we use "blow up techniques" (i.e. tangent measures) and a classification theorem of uniform measures from [8].

One says that a measure ν on $\mathbb{R}^m$ is n-uniform if there exists some constant $C > 0$ such that $\nu(B(x,r)) = Cr^n$ for all $x \in \operatorname{supp}(\nu)$, $r > 0$. On the other hand, ν is called n-flat if it is of the form $\nu = C\mathcal{H}^n_{|L}$, where L is some n-plane. Recall also that, given a Borel map $T : \mathbb{R}^m \to \mathbb{R}^m$, the image measure $T[\nu]$ is defined by $T[\nu](E) = \nu(T^{-1}(E))$, for $E \subset \mathbb{R}^m$. For $x, y \in \mathbb{R}^m$ and $r > 0$, we denote $T_{x,r}(y) = (y - x)/r$.

In [8, Theorem 3.11] it is shown that if ν is an n-uniform measure on $\mathbb{R}^m$, then there exists another n-uniform measure λ such that

$$\lim_{r\to\infty} \frac{1}{r^n}\, T_{x,r}[\nu] = \lambda \quad \text{weakly for all } x \in \mathbb{R}^m.$$

One says that λ is the tangent measure of ν at ∞. For more information on tangent measures, see [8] or [7, Chapters 14-17], for instance.

The following result is the classification theorem of uniform measures mentioned above.

Theorem 4.1 ([8]) *Let ν be an n-uniform measure on $\mathbb{R}^m$. The following holds:*

(a) If $n = 1$ or 2, then ν is n-flat.

(b) If $n \geq 3$, there exists a constant $\varepsilon_0 > 0$ depending only on n and m such that if ν is normalized so that $\nu(B(x,r)) = r^n$ for all $x \in \operatorname{supp}(\nu)$, $r > 0$, and its tangent measure λ at ∞ satisfies

$$\min_{L\in G(n,m)} \int_{B(0,1)} \operatorname{dist}(x,L)^2 d\lambda(x) \leq \varepsilon_0^2, \tag{4.1}$$

then ν is n-flat. Here $G(n,m)$ stands for the collection of all n-planes in $\mathbb{R}^m$.

For the reader's convenience, let us remark that the statement (a) is in Corollary 3.17 of [8]. Regarding (b), notice that λ satisfies $\lambda(B(x,r)) = r^n$ for all $x \in \operatorname{supp}(\lambda)$, $r > 0$, because it is n-uniform and because of the normalization of ν. Moreover, (b) is not stated explicitly in [8], although it is a straightforward consequence of [8, Theorem 3.14 (1)] (and the arguments in its proof) and [8, Corollary 3.16]. See also [3, Propositions 6.18 and 6.19] for more details.

Now we need to define a smooth version of the usual coefficients β_2 (see [2, Chapter I.1], for example). To this end, let φ be a C_c^∞ radial function with $\chi_{B(0,2)} \leq \varphi \leq \chi_{B(0,3)}$. Let $B = B(x_0, r)$ be a ball with centered at $x_0 \in \operatorname{supp}(\mu)$. We denote

$$\widetilde{\beta}_{2,\mu}(B) = \min_{L\in G(n,m)} \left(\frac{1}{r^{n+2}} \int \varphi\left(\frac{|x - x_0|}{r}\right) \operatorname{dist}(x,L)^2 d\mu(x) \right)^{1/2}. \tag{4.2}$$

The following two theorems are the key tools in the proof of Theorem 1.10 (ii). We postpone their proofs to the end of the section. We first indicate how they are used to prove Theorem 1.10 (ii).

Theorem 4.2 *Let μ be an asymptotically optimally doubling measure supported on $\Sigma \subset \mathbb{R}^m$. Let $K \subset \mathbb{R}^m$ be compact and suppose that*

$$C_0^{-1} r^n \le \mu(B(x,r)) \le C_0 r^n \qquad \text{for } x \in K \cap \Sigma,\ 0 < r \le \operatorname{diam}(K). \tag{4.3}$$

For any $\eta > 0$, there exists $\delta > 0$ depending only on η, n, m, μ, K and C_0 such that if B is a ball contained in K and centered at $K \cap \Sigma$ with $\widetilde{\beta}_{2,\mu}(B) \le \delta$, then $\widetilde{\beta}_{2,\mu}(P) \le \eta$ for any ball $P \subset B$ centered at $K \cap \Sigma$.

Theorem 4.3 *Let μ be an asymptotically optimally doubling measure supported on $\Sigma \subset \mathbb{R}^m$. Assume that $0 \in \Sigma$ Let $K \subset \mathbb{R}^m$ be a compact set such that $B(0,2) \subset K$, and suppose that*

$$C_0^{-1} r^n \le \mu(B(x,r)) \le C_0 r^n \qquad \text{for } x \in K \cap \Sigma,\ 0 < r \le \operatorname{diam}(K). \tag{4.4}$$

Given $\epsilon > 0$, there exists $\delta \in (0, \varepsilon_0)$ depending only on ϵ, n, m, μ, K and C_0 such that if $\widetilde{\beta}_{2,\mu}(B) \le \delta$ for every ball $B \subset B(0,2)$ centered at $K \cap \Sigma$, then there exists $R > 0$ such that $\theta(x,r) < \epsilon$ for all $x \in \Sigma \cap B(0,1)$ and $r < R$.

Corollary 4.4 *Let μ be an asymptotically optimally doubling measure supported on $\Sigma \subset \mathbb{R}^m$ Let $K \subset \mathbb{R}^m$ be compact set and suppose that*

$$C_0^{-1} r^n \le \mu(B(x,r)) \le C_0 r^n \qquad \text{for } x \in K \cap \Sigma,\ 0 < r \le \operatorname{diam}(K). \tag{4.5}$$

Given $\epsilon > 0$, there exists $\delta \in (0, \varepsilon_0)$ depending only on ϵ, n, m, μ, K and C_0 such that if $\widetilde{\beta}_{2,\mu}(B(x_0, 4R_0)) \le \delta$, where $x_0 \in \Sigma$ and $B(x_0, 4R_0) \subset K$, then there exists $R > 0$ such that $\theta(x,r) < \epsilon$ for all $x \in \Sigma \cap B(x_0, 2R_0)$ and $r < R$, i.e. $\Sigma \cap B(x_0, 2R_0)$ is ϵ-Reifenberg flat.

Proof of Theorem 1.10 (ii): First note that (1.15) ensures that condition (4.5) is satisfied. It also implies that the density of μ exists and equals 1 everywhere. Therefore Preiss' work (see [8]) yields that Σ is n-rectifiable. Furthermore $\mu = \mathcal{H}^n \llcorner \Sigma$. Thus, given $\eta \in (0, \varepsilon_0)$, for $\mathcal{H}^n$- a.e $x \in \Sigma$ there exists $\rho > 0$ such that $\theta(x,r) \le \eta$ for $r < \rho$. Let

$$\mathcal{R} = \{x \in \Sigma : \limsup_{r \to 0} \theta(x,r) = 0\}. \tag{4.6}$$

Note that $\mathcal{H}^n(\mathcal{S}) = 0$ where $\mathcal{S} = \Sigma \backslash \mathcal{R}$. For $x_0 \in \mathcal{R}$ there exists R_0 is such that $\theta(x_0, r) \le \eta$ for $r \le 8R_0$. This implies that $\widetilde{\beta}_{2,\mu}(B(x_0, 4R_0)) \le C\eta$, where C only depends on C_0. For $\epsilon \in (0, \delta(n,m))$ where $\delta(n,m)$ is as in Theorem 1.9, by Corollary 4.4 we can find η so that $C\eta \le \delta \le \varepsilon_0$, which ensures that $\Sigma \cap B(x_0, 2R_0)$ is $\delta(n,m)$ Reifenberg flat. We use Theorem 1.9 to conclude that $\Sigma \cap B(x_0, R_0)$ is a $C^{1,\beta}$ n-dimensional submanifold. In particular this implies that $\mathcal{R}$ is open in Σ because, $\Sigma \cap B(x_0, 2R_0) \subset \mathcal{R}$. ∎

To prove Theorem 4.2 we need the following result:

Lemma 4.5 *Let μ be an asymptotically optimally doubling measure on $\mathbb{R}^m$. Let $K \subset \mathbb{R}^m$ be compact and let δ_0 be any positive constant. Suppose that*

$$C_0^{-1} r^n \le \mu(B(x,r)) \le C_0 r^n \qquad \text{for } x \in K \cap \Sigma,\ 0 < r \le \operatorname{diam}(K).$$

There exists some constant ε_1 depending on ε_0 and C_0 (but not on δ_0) and an integer $N > 0$ depending only on μ, K, C_0, and δ_0, such that if B is a ball centered at Σ such that $2^N B \subset K$ and

$$\widetilde{\beta}_{2,\mu}(2^k B) \leq \varepsilon_1 \qquad \textit{for } 1 \leq k \leq N, \quad \textit{then } \ \widetilde{\beta}_{2,\mu}(B) \leq \delta_0. \tag{4.7}$$

Proof: Suppose that the integer N does not exist. Then there exists a sequence of points $\{x_j\} \subset K \cap \Sigma$ and balls $B_j := B(x_j, r_j)$ such that $2^j B_j \subset K$, and

$$\widetilde{\beta}_{2,\mu}(2^k B_j) \leq \varepsilon_1 \qquad \text{for } 1 \leq k \leq j,$$

but $\widetilde{\beta}_{2,\mu}(B_j) > \delta_0$. Clearly, $r_j \to 0$ as $j \to \infty$. For each $j \geq 1$, consider the blow up measure μ_j defined by

$$\mu_j(A) = \frac{\mu(r_j A + x_j)}{\mu(B_j)}.$$

Extracting a subsequence if necessary, we may assume that $\{\mu_j\}$ converges weakly to another measure ν, which by [6] [Theorem 2.2] is n-uniform. We claim that

$$\widetilde{\beta}_{2,\nu}(B(0, 2^k)) \lesssim \varepsilon_1 \qquad \text{for all } k \geq 0 \tag{4.8}$$

and

$$\widetilde{\beta}_{2,\nu}(B(0, 1)) \gtrsim \delta_0. \tag{4.9}$$

Assume the claim for the moment. It is easy to check that (4.8) implies that the tangent measure λ of ν at ∞ satisfies

$$\min_{L \in G(n,m)} \int_{B(0,1)} \operatorname{dist}(x, L)^2 d\lambda(x) \leq \varepsilon_0^2,$$

(assuming $\varepsilon_1 \leq \varepsilon_0$ small enough) and so ν is flat by Theorem 4.1. This contradicts (4.9), and the lemma follows.
Let us prove (4.8). Let $B(0, 2^k)$ be fixed. Extracting a subsequence of $\{\mu_j\}$, we may assume that the n-planes L_j which minimize $\widetilde{\beta}_{2,\mu_j}(B(0, 2^k))$ converge in the Hausdorff metric to another n-plane L, and then it easily follows that

$$\left| \int \varphi\left(\frac{|x|}{2^k}\right) \operatorname{dist}(x, L_j)^2 d\mu_j(x) - \int \varphi\left(\frac{|x|}{2^k}\right) \operatorname{dist}(x, L)^2 d\nu(x) \right| \to 0 \qquad \text{as} \quad j \to \infty. \tag{4.10}$$

Notice also that

$$\begin{aligned} \frac{1}{2^{k(n+2)}} \int \varphi\left(\frac{|x|}{2^k}\right) \operatorname{dist}(x, L_j)^2 d\mu_j(x) &= \frac{1}{2^{k(n+2)} \mu(B_j)} \int \varphi\left(\frac{|x - x_j|}{2^k r_j}\right) \operatorname{dist}\left(\frac{x - x_j}{r_j}, L_j\right)^2 d\mu(x) \\ &= \frac{1}{2^{k(n+2)} r_j^2 \mu(B_j)} \int \varphi\left(\frac{|x - x_j|}{2^k r_j}\right) \operatorname{dist}(x, x_j + r_j L_j)^2 d\mu(x) \\ &\approx \frac{1}{(2^k r_j)^{n+2}} \int \varphi\left(\frac{|x - x_j|}{2^k r_j}\right) \operatorname{dist}(x, x_j + r_j L_j)^2 d\mu(x) \\ &\leq \varepsilon_1^2, \end{aligned} \tag{4.11}$$

since $x_j + r_j L_j$ is the n-plane that minimizes $\widetilde{\beta}_{2,\mu}(B(x_j, 2^k r_j))$. Inequality (4.8) follows from (4.10) and the preceding estimate.
The proof of (4.9) is analogous. Now let L be an arbitrary n-plane. Then we have

$$\begin{aligned}\int \varphi(|x|)\,\mathrm{dist}(x,L)^2 d\nu(x) &= \lim_{j\to\infty} \int \varphi(|x|)\,\mathrm{dist}(x,L)^2 d\mu_j(x)\\ &= \lim_{j\to\infty} \frac{1}{\mu(B_j)} \int \varphi\Big(\frac{|x-x_j|}{r_j}\Big)\,\mathrm{dist}\Big(\frac{x-x_j}{r_j}, L\Big)^2 d\mu(x)\\ &= \lim_{j\to\infty} \frac{1}{r_j^2\mu(B_j)} \int \varphi\Big(\frac{|x-x_j|}{r_j}\Big)\,\mathrm{dist}(x, x_j + r_j L)^2 d\mu(x) \gtrsim \delta_0^2, \end{aligned} \tag{4.12}$$

since $\widetilde{\beta}_{2,\mu}(B_j) > \delta_0$. ■

Proof of Theorem 4.2: Let ε_1 be the constant given by Lemma 4.5, and set $\delta_0 = \min(\varepsilon_1, \eta)$ (recall that ε_1 is independent of δ_0). Let N be the corresponding integer given by the same lemma.
If δ is chosen small enough, then we clearly have $\widetilde{\beta}_{2,\mu}(P) \le \min(\varepsilon_1, \eta)$ for any ball P centered at any point in $B \cap \Sigma$ with $r(P) \ge 2^{-N} r(B)$. By the preceding lemma, by induction on $j \ge 0$, we infer that $\widetilde{\beta}_{2,\mu}(P) \le \min(\varepsilon_1, \eta)$ for any ball P centered at $B \cap \Sigma$ with radius $r(P)$ such that $2^{-j-1} r(B) \le r(P) \le 2^{-j} r(B)$ (where $r(B)$ stands for the radius of B). ■

Proof of Theorem 4.3: We argue by contradiction. Suppose that there exists $\varepsilon_1 > 0$ such that, for each $i \ge i_0$ and each ball $B \subset B(0,2)$ centered in $K \cap \Sigma$, $\widetilde{\beta}_{2,\mu}(B) \le 2^{-i} \le \varepsilon_0$ but there are $x_i \in \Sigma \cap B(0,1)$ and $r_i > 0$ with $\lim_{i\to\infty} r_i = 0$, so that $\theta(x_i, r_i) \ge \varepsilon_1$, i.e $\theta_{\Sigma_i}(0,1) \ge \varepsilon_0$, where $\Sigma_i = \frac{1}{r_i}(\Sigma - x_i)$. Consider the blow up sequence $\{\mu_i\}$ defined by

$$\mu_i(E) = \frac{\mu(r_i E + x_i)}{\mu(B(x_i, r_i))}. \tag{4.13}$$

Modulo passing to a subsequence, Theorem 2.2 in [6] ensures that μ_i converges weakly to a Radon measure μ_∞ which is n-uniform. Moreover Σ_i converges in the Hausdorff distance sense to $\Sigma_\infty = \mathrm{supp}(\mu_\infty)$ uniformly on compact subsets. Therefore $\theta_{\Sigma_\infty}(0,1) \ge \epsilon_0/2$. Statement (4.10) guarantees that for $r > 0$ $\tilde{\beta}_{2,\mu_i}(B(0,r))$ converges to $\tilde{\beta}_{2,\mu_\infty}(B(0,r))$. Since for $r > 0$ there exists i_r so that for $i \ge i_r$ $\tilde{\beta}_{2,\mu_i}(B(0,r)) \le 2^{-i}$, then $\tilde{\beta}_{2,\mu_\infty}(B(0,r)) = 0$ for every $r > 0$. Thus the support of μ_∞, Σ_∞ is contained in an n-plane. Since μ_∞ is n-uniform (and flat at infinity), then Σ_∞ is an n-plane, which contradicts the fact that $\theta_{\Sigma_\infty}(0,1)) \ge \epsilon_0/2$. ■

Remark 4.6 By arguments analogous (and even simpler) to the ones in the last proof, one can show that Theorem 4.3 holds for $n = 1, 2$ without the assumption $\widetilde{\beta}_{2,\mu}(B) \le \delta$ for every ball $B \subset B(0,2)$ centered at $K \cap \Sigma$, because all n-uniform measures for $n = 1$ or 2 are flat. Using this result and Theorem 1.9 one can get an alternative proof of Theorem 1.10 (i) which does rely on Theorem 1.4. However, let us remark that this is only an "apparently alternative" proof, because Theorem 1.4 is proved in [1] using similar techniques.

References

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