A generalization of Reifenberg's theorem in $\mathbb{R}^3$

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Abstract

In 1960 Reifenberg proved the topological disc property. He showed that a subset of $\mathbb{R}^n$ which is well approximated by m-dimensional affine spaces at each point and at each (small) scale is locally a bi-Hölder image of the unit ball in $\mathbb{R}^m$. In this paper we prove that a subset of $\mathbb{R}^3$ which is well approximated in the Hausdorff distance sense by one of the three standard area-minimizing cones at each point and at each (small) scale is locally a bi-Hölder deformation of a minimal cone. We also prove an analogous result for more general cones in $\mathbb{R}^n$.

1 Introduction

In 1960, thanks to the development in algebraic topology, E.R. Reifenberg was able to formulate the Plateau problem for m-dimensional surfaces of varying topological type in $\mathbb{R}^k$ (see [R1]). He proved that a given set $\Gamma \subset \mathbb{R}^k$ homeomorphic to $\mathbb{S}^{m-1}$ bounds in his sense a set Σ_0 which minimizes the $\mathcal{H}^m$ Hausdorff measure among all competitors in the appropriate class. Furthermore he showed that for almost every $x \in \Sigma_0$ there exists a neighborhood of x which is a topological disk of dimension m. A remarkable result in [R1] is the Topological Disk Theorem. In general terms it says that if a set is close to an m-plane in the Hausdorff distance sense at all points and at all (small enough) scales, then it is locally biHölder equivalent to a ball of $\mathbb{R}^m$. Using a monotonicity formula for the density, Reifenberg proved that some open subset of full measure of Σ_0 satisfies this condition. In 1964 he proved an Epiperimetric inequality for solutions to the Plateau problem described above. This allowed him to show that the minimizer Σ_0 is locally real analytic (see [R2], and [R3]). Although the Topological Disk Theorem has never again been used as a tool to study the regularity of minimal surfaces it has played a role in understanding their singularities as well as the singular set of energy minimizing harmonic maps (see [HL]). Reifenberg's proof has been adapted to produce biLipschitz, quasi-symmetric and biHölder parameterizations both for subsets of Euclidean space and general metric spaces (under the appropriate flatness assumptions). See [To], [DT], and [CC].

In 1976 J.Taylor [Ta] classified the tangent cones for Almgren almost-minimal sets in $\mathbb{R}^3$. She showed that there are three types of nonempty area-minimizing cones of dimension 2 in

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$\mathbb{R}^3$: the planes, sets that are obtained by taking the product of an equal angle Y in a plane with a line in the orthogonal direction, and sets composed of six angular sectors bounded by four half lines that start from the same point and make (maximal) equal angles. See the more precise Definitions 2.2 and 2.3. By lack of better names, we shall call these sets *minimal cones* and classify them as being of *type 1, type 2*, and *type 3* respectively.

In this paper we generalize Reifenberg's Topological Disk Theorem to the case when the set is close in the Hausdorff distance sense to a set of type 1, 2 or 3 at every point and every (small enough) scale. Namely, if E is a closed set in $\mathbb{R}^3$ which is sufficiently close to a two-dimensional minimal cone at all scales and locations, then there is, at least locally, a bi-Hölder parameterization of E by a minimal cone. Reifenberg's theorem corresponds sets which are similary approximable by just planes, that is, just type 1 cones. Let us first state the main result and comment later.

Theorem 1.1 *For each $\varepsilon \in (0, 10^{-15})$, we can find $\alpha \in (0,1)$ such the following holds. Let $E \subset \mathbb{R}^3$ be a compact set that contains the origin, and assume that for each $x \in E \cap B(0,2)$ and each radius $r > 0$ such that $B(x,r) \subset B(0,2)$, there is a minimal cone $Z(x,r)$ that contains x, such that*

$$D_{x,r}(E, Z(x,r)) \leq \varepsilon, \tag{1.1}$$

where we use the more convenient variant of Hausdorff distance $D_{x,r}$ defined by

$$\begin{aligned} D_{x,r}(E,F) \; = \; \frac{1}{r} \max \Big\{ & \sup \big\{ \operatorname{dist}(z,F)\,;\, z \in E \cap B(x,r) \big\}, \\ & \sup \big\{ \operatorname{dist}(z,E)\,;\, z \in F \cap B(x,r) \big\} \Big\} \end{aligned} \tag{1.2}$$

whenever E, F are closed sets that meet $B(x,r)$. Then there is a minimal cone Z through the origin and an injective mapping $f : B(0,3/2) \to B(0,2)$, with the following properties:

$$B(0,1) \subset f(B(0,3/2)) \subset B(0,2), \tag{1.3}$$

$$E \cap B(0,1) \subset f(Z \cap B(0,3/2)) \subset E \cap B(0,2), \tag{1.4}$$

$$(1+\alpha)^{-1}|x-y|^{1+\alpha} \leq |f(x)-f(y)| \leq (1+\alpha)|x-y|^{1/1+\alpha} \text{ for } x,y \in B(0,3/2), \tag{1.5}$$

$$|f(x)-x| \leq \alpha \text{ for } x \in B(0,3/2). \tag{1.6}$$

We can even take $\alpha \leq C\varepsilon$ when ε is small.

Notice that by (1.4) and (1.5) the restriction of f to $Z \cap B(0,3/2)$ provides a biHölder parameterization a piece of E that contains $E \cap B(0,1)$. This may be enough information, but in some cases it may also be good to know that this parameterization comes from a biHölder local homeomorphism of $\mathbb{R}^3$, as in the statement, because this yields information on the position of E in space. For instance, we can use Theorem 1.1 to construct local retractions of space near E onto E.

Remark 1.1 We shall see that when $Z(0,2)$ is a plane, we can take Z to be a plane; when $Z(0,2)$ is a set of type 2 centered at the origin, we can take Z to be a set of type 2 centered at the origin; when $Z(0,2)$ is a set of type 3 centered at the origin, we can take Z to be a set of type 3 centered at the origin. In addition, our proof will yield that

$$B(0,17/10) \subset f(B(0,18/10)) \subset B(0,2), \tag{1.7}$$

$$E \cap B(0,18/10) \subset f(Z \cap B(0,19/10)) \subset E \cap B(0,2), \tag{1.8}$$

and that (1.5) and (1.6) holds for $x, y \in B(0,18/10)$.

In fact, we shall omit the case of the plane (too easy and well known), and concentrate on the two other special cases. The general case will essentially follow.

Remark 1.2 It would be a little too optimistic to think that f is quasisymmetric. This is already false when $Z(0,2)$ is a plane, because E could be the product of $\mathbb{R}$ with a standard Koch snowflake in $\mathbb{R}^2$ (see [V]).

Remark 1.3 Theorem 1.1 can probably be generalized to a number of situations, where approximation by minimal sets of types 1, 2, and 3 is replaced with approximation by various types of sets with a hierarchical simplex structure. We did not find the optimal way to state this, and probably this would make the proof a little heavier. What we shall do instead is state a slightly more general result in Theorem 2.2, prove Theorem 1.1 essentially as it is, and add a few comments from place to place to explain how to generalize the proof for Theorem 2.2. A more general statement is left out for future investigation.

Our proof will use the hierarchical structure of E. We shall see that $E \cap B(0,2)$ splits naturally into three disjoint subsets E_1, E_2, and E_3, where E_j is the set of points x where E looks like a set of type j in small balls centered at x (more precise definitions will be given in Section 4). If $Z = Z(0,2)$ is a plane, we do not need sets of type 2 or 3 in smaller balls, and we are in the situation of Reifenberg's theorem. Two other main cases will remain, as in Remark 1.2. The case when Z is set of type 2, with its spine passing through the origin (see Definition 2.2 below), and the case when Z is set of type 3 centered at the origin. In the first case, we shall see that $E_3 \cap B(0,199/100)$ is empty and E_2 is locally a Reifenberg-flat one-dimensional set; in the second case, $E_3 \cap B(0,3/2)$ is just composed of one point near the origin, and away from this point E_2 is locally a Reifenberg-flat set of dimension1, while E_1 is locally Reifenberg-flat of dimension 2. See the end of Section 4 for details

The sets E_1 and E_2 will play a special role in the proof. Even though we shall in fact construct f directly as an infinite composition of homeomorphisms in space (that move points at smaller and smaller dyadic scales), we shall pay much more attention to the definition of f on E_2, and then E, just as if we were defining f first from the spine of Z to E_2, then from Z to E, then on the rest of $B(0,3/2)$.

The construction yields that the restriction of f to each of the three or six faces of Z is of class C^1 if the approximation at small scales is sufficiently good (for instance if we can take $\varepsilon = Cr^\beta$ on balls of radius r for suitable $\beta \in (0, 1)$), see Section 10.

Theorem 1.1 will be used in [D1] and [D2] to give a slightly different proof of J. Taylor's regularity result for minimal sets from [Ta].

The plan for the rest of this paper is as follows. In Section 2 we define sets of type 2 and 3 and state a uniform version of our main theorem. We also discuss the more flexible version of Theorem 1.1 mentioned in Remark 1.3. In Section 3 we record some of the simple geometrical facts about minimal sets of types 1, 2, 3, and in particular lower bounds on their relative Hausdorff distances, that will be used in the proof. In Section 4 we show that $E \cap B(0, 2)$ is the disjoint union of E_1, E_2 and E_3, that E_1 is locally Reifenberg-flat, and that E_3 is discrete.

In Section 5 we define the partitions of unity which we use in the construction of the biHölder parameterization. In Section 6 we construct a parameterization of E_2 when E_3 is empty. In Section 7 we extend this to a parameterization of E. In Section 8 we explain how to modify the construction when near a point of E_3. In Section 9 we extend the parameterization to the whole ball. Finally, in Section 10, we prove that our parameterization of E is C^1 on each face of Z when the numbers $D_{x,r}(E, Z(x, r))$ tend to 0 sufficiently fast, uniformly in x, as r tends to 0.

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2 Two other statements

We start with a uniform version of Theorem 1.1.

Definition 2.1 A set $E \subset \mathbb{R}^3$ is ε-Reifenberg flat (of dimension 2) if for each compact set $K \subset \mathbb{R}^3$ there exists $r_K > 0$ such that for $x \in E \cap K$ and $r < r_K$,

$$\inf_{L \ni x} D_{x,r}(E, L) \le \varepsilon, \tag{2.1}$$

where the infimimum is taken over all planes L containing x.

Note that this definition is only meaningful for ε small. Reifenberg's Topological Disk Theorem gives local parameterizations of ε-Reifenberg flat sets. We want to extend this to the ε-minimal sets defined below, but first we give a full description of sets of type 2 and 3. Recall that sets of type 1 are just planes.

Definition 2.2 Define Prop $\subset \mathbb{R}^2$ by

$$\begin{aligned} \text{Prop} &= \{(x_1,x_2) : x_1 \geq 0, x_2 = 0\} \\ & \quad \bigcup \left\{(x_1,x_2) : x_1 \leq 0, x_2 = -\sqrt{3}x_1\right\} \\ & \qquad \bigcup \left\{(x_1,x_2) : x_1 \leq 0, x_2 = \sqrt{3}x_1\right\}. \end{aligned} \tag{2.2}$$

Define $Y_0 \subset \mathbb{R}^3$ by $Y_0 = \text{Prop} \times \mathbb{R}$. The spine of Y_0 is the line $L_0 = \{x_1 = x_2 = 0\}$. A set of type 2 is a set $Y = R(Y_0)$, where R is the composition of a translation and a rotation. The spine of Y is the line $R(L_0)$.

Definition 2.3 Set $A_1 = (1,0,0)$, $A_2 = \left(-\frac{1}{3}, \frac{2\sqrt{2}}{3}, 0\right)$, $A_3 = \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3}\right)$, and $A_4 = \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3}\right)$ (the four vertices of a regular tetrahedron). Denote by T_0 the cone over the union of the six edges $[A_i, A_j]$, $i \neq j$. The spine of T_0 is the union of the four half lines $[0, A_j)$. A set of type 3 is a set $T = R(T_0)$, where R is the composition of a translation and a rotation. The spine of T is the image by R of the spine of T_0.

Definition 2.4 A closed set $E \subset \mathbb{R}^3$ is ε-minimal if for each compact set $K \subset \mathbb{R}^3$ there exists $r_K > 0$ such that for $x \in E \cap K$ and $r < r_K$,

$$\inf_{x \in Z} D_{x,r}(E, Z) \leq \varepsilon, \tag{2.3}$$

where the infimimum is taken over all minimal cones (i.e., sets of type 1, 2, or 3) containing x. That is, there exists a minimal cone $Z(x,r)$ such that $x \in Z(x,r)$ and $D_{x,r}(E, Z(x,r)) \leq \varepsilon$.

Observe that we do not require $Z(x,r)$ to be centered at x, so minimal cones are ε-minimal for every ε. Definition 2.4 is only meaningful for ε small. For the next result, we shall assume that $\varepsilon < 10^{-25}$. Here is a uniform version of our main theorem.

Theorem 2.1 *If E is Reifenberg ϵ-minimal for some $\varepsilon < 10^{-25}$ then for each compact set $K \subset \mathbb{R}^3$ we can find $r_K > 0$ such that for $x \in E \cap K$ and $r < r_K$ there exists a minimal cone Z through the origin and a map $f : B(0, 2r) \to \mathbb{R}^3$, with $f(0) = x$, satisfying*

$$(1 - C\epsilon)\big(r^{-1}|y - z|\big)^{1 + C\epsilon} \leq r^{-1}|f(y) - f(z)| \leq (1 + C\epsilon)\big(r^{-1}|y - z|\big)^{1 - C\epsilon}, \tag{2.4}$$

$$B(x,r) \subset f(B(0, \frac{3r}{2})) \subset B(x, 2r), \tag{2.5}$$

$$E \cap B(x,r) \subset f(Z \cap B(0, \frac{3r}{2})) \subset E \cap B(x, 2r). \tag{2.6}$$

Thus f is a local homeomorphism of the ambient space which sends a minimal cone into the Reifenberg ϵ-minimal set. Theorem 2.1 is an immediate consequence of Theorem 1.1. Also, if we can take ε to tend to 0 in (2.3) (uniformly in $x \in E \cap K$) when r tends to 0, then (2.4) holds with constants that tend to 0 when r tends to 0.

Let us now try to generalize Theorem 1.1. We want to extend the notion of sets of type 1, 2, or 3. Let us fix integers $d \geq 2$ (the dimension of our sets) and $n \geq d+1$ (the ambient dimension).

Sets of type G1 are just d-planes in $\mathbb{R}^n$.

A generalized propeller in $\mathbb{R}^2$ will be any union P of three half lines in $\mathbb{R}^2$ with the same origin, and that meet with angles larger than $\pi/2$ at this point. Incidentally, $\pi/2$ was chosen a little at random here, and we shall not try to see whether it can be replaced by smaller angles.

A set of type G2 is a set $Y = R(P \times \mathbb{R}^{d-1})$, where P is a generalized propeller in $\mathbb{R}^2$ and R is an isometry of $\mathbb{R}^n$. When $n > d+1$, we abused notation slightly, we should have written $Y = R(P \times \mathbb{R}^{d-1} \times \{0\})$ to account for the last $n-d-1$ coordinates. The spine of $R(P \times \mathbb{R}^{d-1} \times \{0\})$ is defined to be $R(\{0\} \times \mathbb{R}^{d-1} \times \{0\})$.

A two-dimensional cone of type G3 in $\mathbb{R}^m$ is a set T that we can construct as follows. We start with a set K in the unit sphere $\partial B(0,1) \subset \mathbb{R}^m$, which is a finite union of great circles (geodesics in $\partial B(0,1)$) and of arcs of great circles. We denote by $\mathcal{C}_j$, $j \in J_1$ the great circles, and by $\mathcal{C}_j$, $j \in J_2$ the arcs of great circles. We also set $J = J_1 \cup J_2$. We first require that the $\mathcal{C}_j$ can only meet when they are arcs of circles, by sets of three, and at a common endpoint. When this happens, we also demand that they make angles larger than $\pi/2$. We also require some uniformity, namely that

$$\text{(2.7)} \qquad \text{the length of each arc } \mathcal{C}_j \text{ is at least } \tau_0$$

for some small constant $\tau_0 > 0$ that will be fixed in this paper, and also that

$$\text{(2.8)} \quad \begin{array}{l} \text{if } i, j \in J,\ i \neq j, \text{ and } x \in \mathcal{C}_i \text{ is such that } \operatorname{dist}(x, \mathcal{C}_j) \leq 10^{-2}\tau_0, \text{ then } \mathcal{C}_i \text{ and } \mathcal{C}_j \\ \text{are both arcs of circles, and they have a common endpoint in } B(x, 2\operatorname{dist}(x, \mathcal{C}_j)). \end{array}$$

Hence, the full great circles do not get close to the other parts of K, and the number of circles and arcs is bounded.

We also exclude the case when K is composed of a single circle (which would lead to a set of type G1) or three arcs of great circles that meet at two opposite points (which would lead to a set of type G2).

A two-dimensional cone of type G3 in $\mathbb{R}^m$ is a set $T = \{tk\,;\, t \geq 0 \text{ and } k \in K\}$ (the cone over K), where K is as above. Thus T is composed of faces F_j (the cones over the $\mathcal{C}_j$) that are either full 2-planes (when $j \in J_1$) or sectors (when $j \in J_2$), and that only meet by sets of three along the half lines that form their boundaries. The spine of T is then the union of these half lines. Put in another way, denote by V the set of all the extremities of arcs $\mathcal{C}_j$, $j \in J_2$. The spine of T is the union of the half lines through the points of V.

It is checked in Section 14 of [D3] that every two-dimensional minimal cone in $\mathbb{R}^m$ is a plane, a set of type G2, or the image by a rotation of a cone of type G3, for some $\tau_0 > 0$ which depends only on m, and with angles between the faces that are equal to $2\pi/3$ (and not just larger than $\pi/2$).

A set of type G3 (of dimension d in R^n) is a set $Z = R(T \times \mathbb{R}^{d-2})$, where R is an isometry of $\mathbb{R}^n$ and T is a two-dimensional cone of type G3 in $\mathbb{R}^{n-d+2}$. The spine of Z is then $R(L \times \mathbb{R}^{d-2})$, where L is the spine of T.

Denote by TGi the collection of sets of of type Gi (of dimension d in $\mathbb{R}^n$), and by TG the union of the three classes TGi. We claim that in Theorem 1.1, we can replace the class of minimal sets with the class TG, as follows.

Theorem 2.2 *Let n and d be as above. Let $E \subset \mathbb{R}^n$ be a compact set that contains the origin, and suppose that for each $x \in E \cap B(0,2)$ and $r > 0$ such that $B(x,r) \subset B(0,2)$, we can find $Z(x,r) \in TG$ that contains x, such that $D_{x,r}(E, Z(x,r)) \leq \varepsilon$. If $\varepsilon > 0$ is small enough, depending only on n, d, and τ_0, there is a set $Z \in TG$ such that (1.3)–(1.6) hold. Here again α depends only on ε, n, d, and τ_0, and we can take $\alpha = C(n,d,\tau_0)\varepsilon$ for ε small. In addition, we can choose Z among the $Z(x,r) \in TG$ that show up in the assumptions.*

The last comment means in particular that if for some reason we only need sets $Z(x,r)$ in a smaller class of TG (for instance, the class of minimal cones of dimension 2), then we can choose Z in this same class.

The reader may be surprised that we do not require any coherence between the various sets $Z(x,r)$, but this coherence will follow automatically from the fact that $D_{x,r}(E, Z(x,r)) \leq \varepsilon$ for many x and r. For instance, if $d = 2$ and $Z(0,2)$ is of type G3, with a center at the origin, our proof will show that for r small, there is a point $x_r \in E$, close to 0, such that $Z(x_r, r)$ is of type G3, and the number k of half-lines in the spine of $Z(x_r, r)$ is the same as for $Z(0,2)$. The point is that angles between the faces may vary little by little when x and r vary, but things like the type of $Z(x,r)$ and the number k cannot jump randomly.

Our main excuse for Theorem 2.2 is that it is used in [D3] to prove that if E is a reduced almost minimal set of dimension 2 in $\mathbb{R}^n$, then every point of E has a neighborhood where E is biHölder-equivalent to a minimal cone.

There is an analogue of Theorem 2.1 in the context of Theorem 2.2, which the reader may easily state.

The proof of Theorem 2.2 will almost be the same as for Theorem 1.1; what we shall do is proceed with the proof of Theorem 1.1, and indicate from time to time which modifications are required for Theorem 2.2.

3 Simple geometrical facts

We shall record in this section a few geometrical properties of the minimal cones that will be used in the construction. The statements will come with various numbers, like 1/3 or 1/4 in the next lemma. Throughout the paper, we have tried to provide correct specific constants which are frequently being readjusted in accord with our lemmas and constructions. In case some adjustment of these constants is necessary, it will not affect the statement of the main result Theorem 2.2. The geometric facts below will be used extensively in Section 4, to establish the hierarchical structure of E.

Our first lemma says that the Hausdorff distances between sets of type 1, 2, and 3, are not too small.

Lemma 3.1 *Let Z be a minimal cone of type 3 centered at x. Then*

$$D_{x,r}(Z,Y) \geq 1/3 \text{ for } r > 0 \tag{3.1}$$

whenever Y is a set of type 1 or 2. Similarly,

$$D_{x,r}(Y,P) > 1/3 \text{ for } r > 0 \tag{3.2}$$

when Y is a set of type 2 centered at x and P is a plane, and

$$D_{x,r}(Y,P) > 1/4 \text{ for } r > 0 \tag{3.3}$$

when Y is a set of type 2 or 3 whose spine contains x and P is a plane.

Proof. We start with the proof of (3.1). By translation, dilation, and rotation invariance, we may assume that $x = 0$, $r = 1$, and Z is the set T_0 of Definition 2.3. It will be good to know that if P is any plane through the origin and π denotes the orthogonal projection onto P, then

$$B(0,1/3) \cap P \subset \pi(T_0 \cap B(0,1)). \tag{3.4}$$

Let Q denote the solid tetrahedron with vertices A_j, $1 \leq j \leq 4$, (i.e., the convex hull of the A_j), where the A_j are as in Definition 2.3. It is clear that $B(0,1/3)$ lies on the right of the left face of Q (where the first coordinate is $-1/3$). By symmetry, $B(0,1/3)$ lies in Q, because it lies on the right side of each face. Also observe that $T_0 \cap Q$ separates the (open) faces of Q from each other inside Q. For instance, the component in $Q \setminus T_0$ of the lower face is a pyramidal domain bounded by three faces of T_0.

Let $z \in B(0,1/3) \cap P$ be given, and let ℓ denote the line through z perpendicular to P. Then ℓ meets Q because $B(0,1/3) \subset Q$. First suppose that it does not touch any edge of Q. Then it enters Q through one (open) face and leaves it through another one, so it meets $T_0 \cap Q$. If ℓ touches a edge of Q, it meets $T_0 \cap Q$ trivially (because the edges are contained in T_0). Thus ℓ meets $T_0 \cap B(0,1)$ (except perhaps if ℓ contains one of the A_j), and hence $z \in \pi(T_0 \cap B(0,1))$. The remaining case when ℓ contains some A_j is easily obtained by density (and anyway we don't need it); (3.4) follows.

Return to (3.1). If Y is of type 2, choose P orthogonal to the spine of Y; then $\pi(Y)$ is a propeller (a set like Prop in (2.2)), and we can find $z \in \partial B(0,1/3) \cap P$ such that $\operatorname{dist}(z,\pi(Y)) \geq 1/3$. Set $z_n = (1-2^{-n})\,z$ for $n > 1$. By (3.4) we can find $y_n \in T_0 \cap B(0,1)$

such that $\pi(y_n) = z_n$. Then $\mathrm{dist}\,(y_n, Y) = \mathrm{dist}\,(z_n, \pi(Y)) \geq 1/3 - 2^{-n}$, and (3.1) follows from the definition (1.2) The case when Y is a plane is even easier; we choose P perpendicular to Y, so that $\pi(Y)$ is a line. So (3.1) holds in all cases.

Let us now prove (3.2). We may assume that $x = 0$ and Y is the set $Y_0 = \mathrm{Prop} \times \mathbb{R}$ in (2.2). Let P be a plane, suppose that $D_{x,r}(Y, P) \leq 1/3$, and let us find a contradiction.

First we want to show that P is almost horizontal. Call P_0 the horizontal plane $\mathbb{R}^2 \times \{0\}$, and denote by b_1, b_2, and b_3 the three points of $\partial B(0, \frac{9}{10}) \cap Y \cap P_0 = \partial B(0, \frac{9}{10}) \cap \big[\mathrm{Prop} \times \{0\}\big]$. Each b_j lies in $Y \cap B(0,1)$, so we can find b_j' in $P \cap \overline{B}(b_j, 1/3)$ because $D_{x,r}(Y, P) \leq 1/3$. Denote by π the orthogonal projection on P_0, and set $p_j = \pi(b_j')$. The p_j lie in the disks $D_j = P_0 \cap \overline{B}(b_j, 1/3)$.

Call d the smallest possible distance from a p_j to the line through the other two p_l. Then d corresponds to the case when, for instance, p_j lies at the extreme right of the left-most disk D_j and the two other p_l lie at the extreme left of the D_l. We get that $d = \frac{9}{10}\frac{3}{2} - 2 \cdot \frac{1}{3} = \frac{41}{60}$.

Recall that P goes through the b_j'. The fact that $d > 0$ (or that the p_j are not aligned) already implies that P is a graph over P_0. We want to show that it is a 1-Lipschitz graph, or equivalently that

$$|v_3| \leq |\pi(v)| \tag{3.5}$$

when v is a unit vector in the (direction of) the plane P, and v_3 denotes its last coordinate. Since the triangle with vertices b_j' is nondegenerate, we can write $v = \alpha(b_j' - c_j)$, where $\alpha \in \mathbb{R} \setminus \{0\}$, $1 \leq j \leq 3$, and c_j lies in the opposite side $[b_k', b_l']$. Since c_j is a convex combination of b_k' and b_l', the size of its last coordinate is at most $\frac{1}{3}$; then the size of the last coordinate of $b_j' - c_j$ is at most $\frac{2}{3}$. On the other hand, $|\pi(b_j' - c_j)| \geq d = \frac{41}{60}$ by definition of d. Now (3.5) follows because $\frac{2}{3} < \frac{41}{60}$.

Finally consider the points $b_4 = (0, 0, -\frac{9}{10})$ and $b_5 = (0, 0, \frac{9}{10})$. Since they lie in $Y \cap B(0,1)$, we can find b_4' and b_5' in P, with $|b_j' - b_j| \leq 1/3$. Set $v = b_5' - b_4'$. Then $v_3 \geq \frac{9}{5} - \frac{2}{3} = \frac{17}{15}$, while $|\pi(v)| \leq \frac{2}{3}$, so v does not satisfy (3.5). This contradiction completes our proof of (3.2).

For the proof of (3.3), we may assume that $x = 0$ and that the spine of Y contains the positive third axis $\{x_1 = x_2 = 0 \text{ and } x_3 \geq 0\}$. Then Y coincides in $\{x_3 \geq 0\}$ with a set of type 2 whose spine is the third axis, and we may even assume that (after a rotation around the third axis) this set is P_0. We shall only need to know that Y contains $Y_0 \cap \{x_3 \geq 0\}$.

This time we assume that $D_{x,r}(Y, P) \leq 1/4$. We can follow the proof of (3.5), except that now our disks D_j are smaller and $d = \frac{9}{10}\frac{3}{2} - 2 \cdot \frac{1}{4} = \frac{17}{20}$. We get that $|v_3| \leq \frac{2}{3} \cdot \frac{20}{17}\,|\pi(v)| = \frac{40}{54}\,|\pi(v)|$ instead of (3.5).

Now we take $b_4 = 0$ and $b_5 = (0, 0, \frac{99}{100})$. As before, we can find b_4' and b_5' in P, with $|b_j' - b_j| \leq 1/4$ (because $D_{x,r}(Y, P) \leq 1/4$). Set $v = b_5' - b_4'$. Then $v_3 \geq \frac{99}{100} - \frac{1}{2} = \frac{49}{100}$, and $|\pi(v)| \leq \frac{1}{2}$, so $v_3 \geq \frac{98}{100}|\pi(v)| > \frac{40}{54}\,|\pi(v)|$, a contradiction; (3.3) and Lemma 3.1 follow. ■

Notice that even for sets of type Gi, Lemma 3.1 is very easy to prove by compactness, when we allow very small constants instead of 1/3 and 1/4.

The next lemmas will make it easier to apply Lemma 3.1.

Lemma 3.2 *Let T be a minimal cone of type 3, and let B be a ball such that $\frac{5}{4}B$ (the ball with the same center, and $\frac{5}{4}$ times the radius) does not contain the center of T. Then there is a minimal cone Y of type 1 or 2 such that $Y \cap B = T \cap B$.*

Proof. Again this would be very easy, even for sets of type TGi, if we allowed a very large constant instead of $\frac{5}{4}$. We shall even prove the lemma with $\frac{5}{4}$ replaced with the better constant α^{-1}, where $\alpha = (2/3)^{1/2}$ (observe that $\frac{5}{4} > \alpha^{-1} = (3/2)^{1/2}$). By translation and rotation invariance, we can assume that T is the set T_0 from Definition 2.3.

Let P denote the plane through 0, A_2, and A_3, and set $x_t = (-t, 0, 0)$ for $t > 0$; we claim that $\operatorname{dist}(x_t, P) = \alpha t$. Indeed, an equation of P is $2\sqrt{2}x + y + \sqrt{3}z = 0$ (just check that 0, A_2, and A_3 satisfy the equation). Then $\operatorname{dist}(x_t, P) = (2\sqrt{2}t)/\sqrt{12} = \alpha t$, as announced.

Let $x \neq 0$ be given, and set $B_x = B(x, \alpha|x|)$. We just checked that when $x = x_t$, B_x does not meet the three faces on the left of T_0. For a general x, B_x cannot meet the three faces at the same time, because the optimal position for this to happen would precisely be when x lies on the negative first axis. By symmetry, B_x never meets the three faces of T that bound a given connected component of $\mathbb{R}^3 \setminus T$. So the worse that it can do is meet the three faces of T that share a single edge $[0, A_j)$ (if B_x meets two opposite faces, it also meets the three faces that bound some connected component).

Suppose first that B_x meets three such faces, and let Y denote the set of type 2 that contains these faces. We claim that

$$T \cap B_x = Y \cap B_x. \tag{3.6}$$

The direct inclusion is clear, because B_x only meets the three faces of T that are contained in Y. For the converse inclusion, we just need to check that $F \cap B_x \subset F' \cap B_x$ when F is a face of Y and F' is the face of T that is contained in F. Let $y \in F \cap B_x$ be given. By assumption, B_x meets F', so we can pick $z \in F' \cap B_x$. Since B_x is open, we can even pick z in the interior of F'. Notice that the segment $[y, z]$ lies in $F \cap B_x$ by convexity. In addition, (y, z) does not meet the spine of Y (because z lies in the interior of F). Now B_x does not meet the other edges of T (those that are not contained in the spine of Y), because if it meets an edge e, it also meets the three faces of T that touch e. So (y, z) does not meet the boundary of F', and $y \in F'$. Our claim follows.

Next suppose that B_x only meets two faces of T. Still denote by Y the set of type 2 that contains them. As before, $T \cap B_x \subset Y \cap B_x$ trivially. To prove the converse inclusion, consider a face F of Y, and first assume that the face F' of T that it contains meets B_x. Pick $z \in F' \cap B_x$. For each $y \in F \cap B_x$, the segment $[x, y]$ is contained in B_x, hence it does not meet the boundary of F' (because if B_x meets any edge of T, it meets at least three faces), so it is contained in F', and $y \in F'$. We just proved that $F \cap B_x \subset F' \cap B_x$.

Call F_1, F_2, and F_3 the three face of Y, and denote by F'_j the face of T which is contained in F_j. Exactly two of the F'_j meet B_x, and the third one does not; let us assume that this last one is F'_3. We just need to check that F_3 does not meet B_x either. Suppose it does. Let y_j be a point of $F_j \cap B_x$, $j \in \{1, 2, 3\}$. Call y'_j the orthogonal projection of y_j onto the plane P through the center of B_x and perpendicular to the spine L of Y; then $y'_j \in F_j \cap B_x$ as well.

Call p the point of $L \cap P$; by convexity of B_x and because p is a convex combination of the three y'_j, B_x contains p. We reached the desired contradiction, because p lies on an edge of P and B_x does not meet edges in the present case. So we proved (3.6) in this second case.

Finally assume that B_x only meets one face F' of T. Then it does not meet any edge of T, so if Y denotes the plane that contains F', $T \cap B_x$ contains $Y \cap B_x$ (as before, B_x does not meet the boundary of F' in Y). In this case also we have (3.6).

We just proved that for $x \neq 0$, we can find a set Y of type 1 or 2 such that (3.6) holds. Now let B be as in the statement, and call x its center and r its radius. We know that $B(x, \frac{5r}{4})$ does not contain the origin, so $r \leq 4|x|/5 \leq \alpha|x|$, $B \subset B_x$, and the analogue of (3.6) holds for B. This completes our proof of Lemma 3.2. ■

Lemma 3.3 *Let Z be a minimal cone of type 3 centered at z, and let T be a minimal cone. If $D_{x,r}(T, Z) < 1/3$, then T is of type 3 and its center lies in $B(z, 5r/3)$.*

Proof. Let Z and T be as in the statement. If T is of type 1 or 2, we can apply Lemma 3.1 directly to get a contradiction. So T is of type 3, and let t denote its center. By Lemma 3.2, we can find a set Y of type 1 or 2 that coincides with T in $B = B(z, \frac{4}{5}|z-t|)$. If $|z-t| \geq 5r/3$, B contains $B(z, \frac{4}{3}r)$, Y coincides with T in $B(z, \frac{4}{3}r)$, and it is easy to see that $D_{x,r}(Y, Z) < 1/3$ because $D_{x,r}(T, Z) < 1/3$. But this is impossible, by Lemma 3.1, so $|z - t| < 5r/3$. Lemma 3.3 follows. ■

Lemma 3.4 *Let Z be a set of type 2 or 3. Suppose that the ball $B(x, r)$ meets at least two faces of Z, or that Z does not coincide with any plane in $B(x, r)$. Then the distance from x to the spine of Z is at most $6r/5$.*

Proof. For sets of type G2 or G3 (and with 6/5 replaced with a large constant), this is a simple consequence of (2.8). We now return to the standard case. Let L denote the spine of Z. We can assume that $B = B(x, r)$ does not meet L. If Z does not coincide with any plane in B, it meets at least one face F; call P the plane that contains F. Since B does not meet L, it does not meet the boundary of F in P, so $P \cap B \subset Z$. Then B meets some other face of F' of Z. Call P' the plane that contains F', and set $D = P \cap P'$ and $d = \operatorname{dist}(x, D)$. Then $r \geq d \cos(30°) = d\sqrt{3}/2$ (the worse case is when x lies in the middle of the 120° sector bounded by P and P', and B is tangent to both planes).

If Z is of type 2, we get the result directly, because $L = D$ and $\sqrt{3}/2 \geq 5/6$. So we can assume that Z is of type 3. If $B(x, d)$ meets L, then $\operatorname{dist}(x, L) \leq d \leq 6r/5$ as needed. Otherwise, $P \cap B(x, d) \subset F$, because $B(x, d)$ meets F and does not meet the boundary of F in P. But $P \cap B(x, d)$ goes all the way to the point of D that minimizes the distance to x, so this point lies in L and $\operatorname{dist}(x, L) \leq d$. Lemma 3.4 follows. ■

4 The decomposition of E

From now on, E is a compact set in $\mathbb{R}^3$ that satisfies the assumptions of Theorem 1.1. In particular, (1.1) holds for some $\varepsilon < 10^{-25}$. The following definitions will make it easier to define the E_j's and work with them. For $x \in E$ and $r > 0$, set

$$a(x,r) = \inf \big\{ D_{x,r}(E,P)\,;\, P \text{ is a plane through } x \big\}, \tag{4.1}$$

$$b(x,r) = \inf \big\{ D_{x,r}(E,Y)\,;\;\; Y \text{ is a set of type 2 whose spine contains } x \big\}, \tag{4.2}$$

and

$$c(x,r) = \inf \big\{ D_{x,r}(E,T)\,;\, T \text{ is a set of type 3 centered at } x \big\}. \tag{4.3}$$

It is not always true that either $a(x,r)$, $b(x,r)$, or $c(x,r)$ is small (when $x \in E$ and $B(x,r) \subset B(0,2)$), because, for instance $Z(x,r)$ may be a minimal cone of type 3 centered anywhere in $B(x,r)$. The pairs where either $a(x,r)$, $b(x,r)$, or $c(x,r)$ is small are interesting to study.

Definition 4.1 Let $x \in E \cap B(0,2)$ be given. We say that:

- x is of type 1 when $a(x,r) \leq 2\varepsilon$ for all r small,
- x is of type 2 when $b(x,r) \leq 1500\varepsilon$ for all r small,
- x is of type 3 when $c(x,r) \leq 150\varepsilon$ for all r small.

For $j = 1, 2, 3$, we denote by E_j the set of points $x \in E \cap B(0,2)$ that are of type j.

We should say something now about the choice of constants. The constants 2, 1500, and 150 are just chosen to make later lemmas easier to use. They depend on the values of the various constants in Section 3, but we could reorganize things with different constants. Other strange constants will appear in the proof; the reader should not pay too much attention to their precise values, hopefully this will make them easier to track. For Theorem 2.2, all these constants need to be replaced with other ones; let us indicate in advance how they need to be chosen, to make it easier for the reader to check things out.

Our first constants are $a_1 = 2$, $a_2 = 1500$, and $a_3 = 150$ in Definition 4.1; a_1 comes from the proof of Lemma 4.1, and we can keep $a_1 = 2$; the constraints on (the new values of) a_2 and a_3 will be explained soon.

The constant $a_4 = 10^{-3}$ in Lemma 4.1 just needs to be small, depending on a_1. In Lemma 4.2, $a_5 = 5$ and $a_6 = 57$ just come from the geometry, $a_7 = 10$ is just $2a_5$, we can keep $\frac{1}{2}$ as it is, and the value of $a_8 = 7$ follows from the geometry. Then $a_9 = 10^{-3}$ just needs to be small, depending on a_5 and a_6. Also, $a_{10} = 11$ just needs to be a little larger than a_7. The constant $a_{11} = 15$ in Corollary 4.1 just needs to be larger than $2a_8$.

The constants $a_{12} = 10^{-3}$, $a_{13} = 10^{-3}$, and $a_{14} = 132$ in Lemma 4.3 are just geometric constants that come from Lemma 3.3. Our first constraint on a_3 comes from Lemma 4.3 and is that $a_3 \geq a_{14}$.

No constraint comes from Lemma 4.4; the constant in (4.12) is just the same as a_{11} in Corollary 4.1.

In Lemma 4.5, we just need $a_{15} = 25$ needs to be large enough (it needs to be larger than 10 in our proof, because we want the pair $(z_1, r/10)$ to satisfy the assumptions in the lemma, a little above (4.19)); other constraints will come later. The value of $a_{16} = 600$ follows by geometric computations, and so does $a_{17} = 150$ in (4.15). We need to pick $a_3 \geq a_{17}$ in Definition 4.1.

The first constraint on a_2 in Definition 4.1 is that a_2 be larger than the constant in (4.20).

The constant $a_{18} = 17$ in Lemma 4.6 is just a geometric constant. In (4.24) and the few lines above, 10^{-3} needs to be replaced with a_9 from Lemma 4.2. A few lines later, we only get that $b(x,r) \leq a_{19}\varepsilon$, instead of 500ε; this gives a new constraint on a_2, namely that $a_2 \geq a_{19}$ (so that we can deduce that $x \in E_2$). The rest of the proof of (4.23) goes smoothly.

In Lemma 4.7, $a_{20} = 24$ can be replaced with $\frac{2}{3}(2a_{18} + 2)$ and $a_{21} = 600$ with the geometric constant a_{16}; in the proof, we need to replace 198/100 with a geometric constant C, just below (4.27); the rest of the proof is the same, but gives a second constraint on a_{15} in Lemma 4.5, i.e., it needs to be larger than a_{20}.

For Lemma 4.8, we need to replace 15 with a_{11}, as in Corollary 4.1, and 200 with some geometric constant a_{23} that depends on Lemmas 3.2 and 3.4. We also get the last constraint that a_{15} be large enough, depending on a_{23}.

This completes our list of constraints on constants for the results of this section; the reader may check that they are compatible.

The main point of this section is to establish a few properties of the sets E_j. We shall prove that $E \cap B(0,2) = E_1 \cup E_2 \cup E_3$ (a disjoint union), that $E_3 \cap B(0, 199/100)$ has at most one point, and that E_2 is locally Reifenberg-flat of dimension 1 (away from E_3 and $\partial B(0,2)$).

The results of this section are also valid with minor modifications for Reifenberg ε-minimal sets (as in the previous section); most of the statement in this section can also be easily deduced from Theorem 2.1.

Let us first check that

$$E_1,\ E_2, \text{ and } E_3 \text{ are disjoint.} \tag{4.4}$$

Suppose for instance that $x \in E_2 \cap E_3$. For r small enough, we can find a set Y of type 2 and a set Z of type 3 centered at x, such that $D_{x,2r}(E,Y) \leq 1500\varepsilon$ and $D_{x,2r}(E,Z) \leq 150\varepsilon$. If $y \in Y \cap B(x,r)$, we can find $e \in E$ such that $|e-y| \leq 3000\varepsilon r$. Then $e \in E \cap B(x,2r)$, so we can find $z \in Z$ such that $|z-e| \leq 300\varepsilon r$. Altogether, $|z-y| \leq 3300\varepsilon r$. Similarly, if $z \in Z \cap B(x,r)$, we can find $e \in E$ such that $|e-y| \leq 300\varepsilon r$, then $e \in E \cap B(x,2r)$, we can find $y \in Y$ such that $|y-e| \leq 2000\varepsilon r$, and $|z-y| \leq 3300\varepsilon r$. So $D_{x,r}(Y,Z) \leq 3300\varepsilon < 1/3$. Lemma 3.1 says that this is impossible. The two other cases are just as easy.

Our first lemma shows that E_1 is a large open set.

Lemma 4.1 *Let $x \in E$ and $r > 0$ be such that $B(x,r) \subset B(0,2)$ and $a(x,r) \leq 10^{-3}$. Then $a(y,t) \leq 2\varepsilon$ for $y \in B(x,2r/3)$ and $0 < t < r/4$. In particular, $E \cap B(x,2r/3) \subset E_1$.*

Proof. Let x and r be as in the statement. Let P_0 be a plane through x such that $D_{x,r}(E, P_0) \leq 10^{-3}$. By (1.1), there is a minimal cone $Z = Z(x, r)$ through x, such that $D_{x,r}(E, Z) \leq \varepsilon$. It could be that Z is of type 2 or 3, but even so let us check that

$$\text{there is a plane } P \text{ through } x \text{ such that } Z \cap B(x, 97r/100) = P \cap B(x, 97r/100). \tag{4.5}$$

First suppose that Z is a set of type 3 centered at some $y \in B(x, 99r/100)$. Set $\rho = r/200$, and let us check that $D_{y,\rho}(Z, P_0) < 1/3$. If $z \in Z \cap B(y, \rho)$, we can find $e \in E$ such that $|z - e| \leq \varepsilon r$. Obviously, $e \in B(x, r)$, so we can find $p \in P_0$ such that $|p - e| \leq 10^{-3}r$; then $|p - z| \leq \varepsilon r + 10^{-3}r < \rho/3$. Similarly, if $p \in P_0 \cap B(y, \rho)$ we can find $e \in E$ such that $|p - e| \leq 10^{-3}r$. Then $e \in B(x, r)$, so we can find $z \in Z$ such that $|z - e| \leq \varepsilon r$; altogether $|p - z| \leq \varepsilon r + 10^{-3}r < \rho/3$, and $D_{y,\rho}(Z, P_0) < 1/3$. This is impossible, by Lemma 3.1.

Next suppose that Z is a set of type 3, whose center $z \notin B(x, 99r/100)$, but whose spine meets $B(x, 98r/100)$ at some point y. As before, $D_{y,\rho}(Z, P_0) < 1/3$ for $\rho = r/200$. Since $|z - y| \geq r/100$, Lemma 3.2 says that Z coincides with a set Y of type 2 in $B(y, 4r/500) = B(y, 8\rho/5)$. Then $D_{y,\rho}(Y, P_0) < 1/3$ too. But the spine of Y goes through y, so Lemma 3.1 says that this is impossible. The same argument excludes the case when Z is a set of type 2 whose spine meets $B(x, 98r/100)$.

We are left with the case when Z is of type 1, or else its spine does not meet $B(x, 98r/100)$. Let F denote the face of Z that contains x, and P the plane that contains F. The boundary of F is contained in the spine of Z, so it does not meet $B(x, 98r/100)$; hence

$$F \cap B(x, 98r/100) = P \cap B(x, 98r/100). \tag{4.6}$$

Every point of $P \cap B(x, 98r/100)$ lies in Z by (4.6), so it is εr-close to E, and then $(\varepsilon + 10^{-3})r$-close to P_0. This forces every point of $P_0 \cap B(x, r)$ to be $2 \cdot 10^{-3}r$-close to P. This stays true (with a constant larger than 2) when we deal with sets of type Gi and approximations with d-planes in $\mathbb{R}^n$.

If (4.5) fails, (4.6) says that there is another face F_1 of Z that meets $B(x, 97r/100)$ at some point y. We know that y is εr-close to E, hence $(\varepsilon + 10^{-3})\, r$-close to P_0. Hence, $\operatorname{dist}(y, P) \leq (\varepsilon + 3 \cdot 10^{-3})r$. [In all these estimates, we use the fact that $y \in B(x, 99r/100)$, so the successive points that we implicitly use never lie out of $B(x, r)$.]

On the other hand, y lies in some other face of Z, and the angles between faces of Z are not too small, so the distance from y to the spine of Z is less than $10^{-2}r$. For sets of type Gi, we deduce this from (2.8) (but we need different constants). This is impossible, because the spine of Z does not meet $B(x, 98r/100)$. So (4.5) holds.

We now return to the lemma, and let $y \in B(x, 2r/3)$ be as in the statement. Let us first estimate $a(y, t)$ for $t \in (r/10, 4r/10)$. First observe that $B(y, t) \subset B(x, 967r/1000)$. By (1.1) and (4.5), we can find $q_y \in P$ such that $|y - q_y| \leq \varepsilon r$. Set $P' = P + (y - q_y)$. We can use P' to compute $a(y, t)$, because it goes through y, so $a(y, t) \leq D_{y,t}(E, P')$. If $e \in E \cap B(y, t)$, $\operatorname{dist}(e, P') \leq \operatorname{dist}(e, P) + |y - q_y| \leq \operatorname{dist}(e, P) + \varepsilon r \leq 2\varepsilon r$, by (1.1) and (4.5). If $p' \in P' \cap B(y, t)$, $p = p' - (y - q_y)$ lies in $Z \cap B(x, 97r/100)$ by (4.5), and $\operatorname{dist}(p', E) \leq \operatorname{dist}(p, E) + \varepsilon r \leq 2\varepsilon r$ by (1.1). So $a(y, t) \leq D_{y,t}(E, P') \leq 2\varepsilon r/t \leq 20\varepsilon$.

The pair (y,t) satisfies the hypothesis of the lemma, namely, $a(y,t) \leq 10^{-3}$, so we can iterate the previous argument (this time, keeping y at the center). This yields that $a(y,s) \leq 20\varepsilon$ for $r/100 \leq s \leq 4r/10$, and (after many iterations of the argument) for every $s < 4r/10$.

Finally, for each $s < 4r/10$ there is a set $Z(y,s)$ as in (1.1), and since $a(y,s) \leq 20\varepsilon$ the proof of (4.5) shows that $Z(y,s)$ coincides with a plane P on $B(y, 97s/100)$. We can use P in the definition of $a(y, 96s/100)$, and we get that $a(y, 96s/100) \leq 100\varepsilon/96 \leq 2\varepsilon$. Since this holds for $s < 4r/10$, Lemma 4.1 follows. ■

Next we focus on E_3 and $c(x,r)$. We start by showing how being close to a tetrahedron at a small scale determines the behavior of the set at larger scales.

Lemma 4.2 *Let $x \in E$ and $r > 0$ be such that $B(x, 11r) \subset B(0,2)$ and $c(x,r) < 10^{-3}$. Let $Z_{11} = Z(x, 11r)$ be a minimal set through x such that $D_{x,11r}(E, Z_{11}) \leq \varepsilon$ (as in (1.1)). Then Z_{11} is of type 3, and its center x_0 is such that $|x - x_0| \leq 5c(x,r)r + 57\varepsilon r$. In addition,*

$$c(x, 10r) \leq \frac{1}{2}\, c(x,r) + 7\varepsilon. \tag{4.7}$$

Proof. The reader should not pay too much attention to the value of the various constants. Nevertheless the fact that $\frac{1}{2} < 1$ will be useful. By assumption, there is a minimal set Z of type 3, centered at x, and such that $D_{x,r}(E, Z) \leq c(x,r) \leq 10^{-3}$. Set $\rho = 3c(x,r)r + 34\varepsilon r$, and let us check that

$$D_{x,\rho}(Z, Z_{11}) < 1/3. \tag{4.8}$$

If $z \in Z \cap B(x,\rho)$, we can find $e \in E$ such that $|e - z| \leq c(x,r)r$. Obviously $e \in B(x, 11r)$, so we can find $z' \in Z_{11}$ such that $|z' - e| \leq 11\varepsilon r$; altogether $\operatorname{dist}(z, Z_{11}) \leq (c(x,r) + 11\varepsilon)r < \rho/3$. Similarly, if $z' \in Z_{11} \cap B(x,\rho)$, we can find $e \in E$ such that $|e - z'| \leq 11\varepsilon r$, and then $z \in Z$ such that $|z - e| \leq c(x,r)r$, so $\operatorname{dist}(z', Z) \leq (c(x,r) + 11\varepsilon)r < \rho/3$. This proves (4.8).

By Lemma 3.3, Z_{11} is of type 3 and its center x_0 lies in $B(x, 5\rho/3)$. That is, $|x - x_0| \leq 5c(x,r)r + 170\varepsilon r/3 \leq 5c(x,r)r + 57\varepsilon r$.

We cannot use Z_{11} directly to estimate $c(x, 10r)$, because it is not centered at x, but we may use $Z' = Z_{11} + (x - x_0)$. If $y \in E \cap B(x, 10r)$, we can find $z \in Z_{11}$ such that $|z - y| \leq 11\varepsilon r$, and then $z' = z + (x - x_0) \in Z'$ is such that $|z' - y| \leq 5c(x,r)r + 68\varepsilon r$. Similarly, if $z' \in Z' \cap B(x, 10r)$, then $z = z' - (x - x_0) \in Z \cap B(x, 11r)$, so we can find $y \in E$ such that $|z - y| \leq 11\varepsilon r$, and then $|z' - y| \leq 5c(x,r)r + 68\varepsilon r$. Altogether, $D_{x,10r}(E, Z') \leq (10r)^{-1}(5c(x,r)r + 68\varepsilon r) \leq \frac{1}{2}c(x,r) + 7\varepsilon$. Lemma 4.2 follows. ■

Corollary 4.1 *If $x \in E_3$, then $c(x,r) \leq 15\varepsilon$ for every $r > 0$ such that $B(x, 11r/10) \subset B(0,2)$.*

Proof. Indeed for k large, $c(x, 10^{-k}r) \leq 150\varepsilon$ by definition of E_3. Lemma 4.2 says that $c(x, 10^{-k}r) \leq \frac{1}{2}\, c(x, 10^{-k+1}r) + 7\varepsilon \leq 157\varepsilon$ for k large. Multiple iterations of Lemma 4.2

eventually lead to the result. In fact define a sequence λ_n by $\lambda_0 = 150\varepsilon$ and $\lambda_{n+1} = \frac{1}{2}\lambda_n + 7\varepsilon$, note that it converges to $14\varepsilon < 15\varepsilon$. ■

The influence of tetrahedral points also transmits to smaller scales.

Lemma 4.3 *Let $x \in E$ and $r > 0$ be such that $B(x,r) \subset B(0,2)$ and $c(x,r) \leq 10^{-4}$. Then there is a point $\xi \in E \cap B(x, 10^{-3}r)$ such that $c(\xi,\rho) \leq 132\varepsilon$ for $0 \leq \rho \leq r/3$. In particular, $\xi \in E_3$.*

Proof. Pick $Z = Z(x,r)$, so that $D_{x,r}(E,Z) \leq \varepsilon$. Also choose a minimal cone Y of type 3 centered at x and such that $D_{x,r}(E,Y) \leq c(x,r) \leq 10^{-3}$. Set $\rho_0 = 3c(x,r)r + 4\varepsilon r$; then $D_{x,\rho_0}(Y,Z) < 1/3$, by the proof of (4.8). As before, Lemma 3.3 says that Z is of type 3, with a center z_0 in $B(x, 5\rho_0/3)$. That is, $|z_0 - x| \leq 5c(x,r)r + 20\varepsilon r/3$. By definition of Z, we can find $x_1 \in E$ so that $|x_1 - z_0| \leq \varepsilon r$, and hence

$$|x_1 - x| \leq 5c(x,r)r + 8\varepsilon r. \tag{4.9}$$

The cone $Z' = Z + (x_1 - z_0)$ is centered at x_1, so we can use it to compute $c(x_1, r/2)$. The same computation as in Lemma 4.2 yields

$$c(x_1, r/2) \leq D_{x_1,r/2}(E,Z') \leq (2/r)\big(|x_1 - z_0| + \varepsilon r\big) \leq 4\varepsilon. \tag{4.10}$$

Let us iterate the construction with the ball $B(x_1, r/2)$. We get a second point $x_2 \in E$, with $|x_2 - x_1| \leq [5c(x_1,r/2) + 8\varepsilon]r/2 \leq 28\varepsilon r/2$, and such that $c(x_2, r/4) \leq 4\varepsilon$. By iteration we can find points x_k, $k \geq 1$, such that $c(x_k, 2^{-k}r) \leq 4\varepsilon$ and $|x_{k+1} - x_k| \leq 28\varepsilon 2^{-k}r$ for $k \geq 1$.

Call ξ the limit of the x_k; thus

$$\begin{aligned} |x - \xi| &\leq |x - x_1| + \sum_{k\geq 1} |x_{k+1} - x_k| \\ &\leq [5c(x,r) + 8\varepsilon]r + \sum_{k\geq 1} 28\varepsilon 2^{-k}r \leq [5c(x,r) + 36\varepsilon]r. \end{aligned} \tag{4.11}$$

We still need to check that $c(\xi,\rho) \leq 132\varepsilon$ for $0 \leq \rho \leq r/3$. Pick k such that $2^{-k-1}r \leq 11\rho/10 \leq 2^{-k}r$; thus $k \geq 1$ and hence $c(x_k, 2^{-k}r) \leq 4\varepsilon$. Also, $|\xi - x_k| \leq \sum_{l\geq k} 28\varepsilon 2^{-l}r \leq 56\varepsilon 2^{-k}r$. Call W a set of type 3 centered at x_k and which is $4\varepsilon 2^{-k}r$-close to E in $B(x_k, 2^{-k}r)$; then the translation of W by $\xi - x_k$ is $60\varepsilon 2^{-k}r$-close to E in $B(\xi,\rho)$, which leads to $c(\xi,\rho) \leq 60\varepsilon 2^{-k}r/\rho \leq (60\varepsilon)(22/10) < 132\varepsilon$, as needed. The fact that $\xi \in E_3$ is just the definition of E_3 ■

Lemma 4.4 *There is at most one point in $E_3 \cap B(0, 199/100)$.*

Proof. Suppose that $E_3 \cap B(0,199/100)$ contains two different points x and y. Notice that

$$c(x,r) \leq 15\varepsilon \quad \text{for } 0 < r \leq 2 \cdot 10^{-3}, \tag{4.12}$$

by Corollary 4.1 and because $B(x, 11r/10) \subset B(0,2)$.

Set $\rho = |x-y|$, and first assume that $\rho < 10^{-3}$. We take $r = 2\rho$ in (4.12) and get that $c(x, 2\rho) \leq 15\varepsilon$. Thus there is a set X of type 3, centered at x, such that $D_{x,2\rho}(X, E) \leq 15\varepsilon$. Similarly, there is a set Y of type 3, centered at y, such that $D_{y,2\rho}(Y, E) \leq 15\varepsilon$. Notice that $B(y, 2\rho)$ contains $B(x, \rho)$, so $D_{x,\rho}(X, Y) \leq 60\varepsilon$ by the proof of (4.4). By Lemma 3.3, the center of Y lies in $B(x, 5\rho/6)$, a contradiction.

We are left with the case when $\rho = |x-y| \geq 10^{-3}$. Since $c(x, 10^{-3}) \leq 15\varepsilon$ by (4.12), we can pick X of type 3, centered at x, and such that $D_{x,10^{-3}}(X, E) \leq 15\varepsilon$. Also set $Z = Z(0,2)$; thus $D_{0,2}(Z, E) \leq \varepsilon$ by (1.1), and the proof of (4.4) yields $D_{x,10^{-4}}(X, Z) \leq 3 \cdot 10^4\varepsilon$. Then Lemma 3.3 says that Z is of type 3, with a center in $B(x, 2 \cdot 10^{-4})$. By the same argument, the center of Z lies in $B(y, 2 \cdot 10^{-4})$. But these balls are disjoint because $|x-y| \geq 10^{-3}$, our last case is impossible, and Lemma 4.4 follows. ∎

The situation with respect to E_3 is reasonably clear now: it contains at most one point x_0 in $B(0, 199/100)$, and Corollary 4.1 says that $c(x_0, r) \leq 15\varepsilon$ for all r such that $B(x_0, 11r/10) \subset B(0,2)$. Furthermore, Lemma 4.3 essentially forbids E to look like a set of type 3 away from x_0, because this would create a new point of E_3. Next we focus on $b(x,r)$ and E_2.

Lemma 4.5 *Let $x \in E$ and $r > 0$ be such that $B(x,r) \subset B(0,2)$, and assume that we can find a minimal cone Y of type 2, whose spine L contains x, and such that $D_{x,r}(E, Y) \leq 25\varepsilon$. Then $B(x, 2r/3)$ does not meet E_3 and $D_{x,2r/3}(L, E_2) \leq 600\varepsilon$.*

Proof. Let x and $r > 0$ be as in the lemma. We shall need to check that

$$z \in E_1 \text{ for } z \in E \cap B(x, 2r/3) \text{ such that } \operatorname{dist}(z, L) \geq 400\varepsilon r. \tag{4.13}$$

Let $z \in E \cap B(x, 2r/3)$ be given, with $\operatorname{dist}(z, L) \geq 400\varepsilon r$. Set $\rho = \min(\operatorname{dist}(z,L)/2, r/5)$; in particular, $16\rho/10 \leq 16r/50 < r/3$, so $B(z, 16\rho/10) \subset B(x,r) \subset B(0,2)$. Notice that $\operatorname{dist}(z, Y) \leq 25\varepsilon r$ because $D_{x,r}(E,Y) \leq 25\varepsilon$. Call P the plane that contains the face of Y that gets close to z; then $Y \cap B(z, 16\rho/10) = P \cap B(z, 16\rho/10)$ because $16\rho/10 \leq 8 \operatorname{dist}(z, L)/10$. Also recall that $25\varepsilon r \leq 25\rho/200$, either trivially because $\rho = r/5$ or else because $\rho = \operatorname{dist}(z,L)/2 \geq 200\varepsilon r$.

Now set $\widetilde{Z} = Z(z, 16\rho/10)$. Let us check that

$$\operatorname{dist}(w, P) \leq 16\varepsilon\rho/10 + 25\rho/200 \leq 13\rho/100 \text{ for } w \in \widetilde{Z} \cap B(z, 14\rho/10). \tag{4.14}$$

Indeed we can find $e \in E$ such that $|e - w| \leq 16\varepsilon\rho/10$ (by (1.1)); then $e \in B(z, 15\rho/10) \subset B(x,r)$, so we can find $y \in Y$ such that $|y-e| \leq 25\varepsilon r \leq 25\rho/200$. Thus $|y - w| \leq 16\varepsilon\rho/10 + 25\rho/200 < 2\rho/10$, so $y \in Y \cap B(z, 16\rho/10) = P \cap B(z, 16\rho/10)$, and (4.14) holds.

We deduce from (4.14) and elementary geometry that $\widetilde{Z}$ coincides with a plane in $B(z, 11\rho/10)$.

In the case of sets of type Gi, we need to replace 400 in (4.13) with a much larger constant, so that $\operatorname{dist}(w, P) \leq \alpha\rho$ for $w \in \widetilde{Z} \cap B(z, 14\rho/10)$, with a very small constant α. If $\widetilde{Z}$ is of type $G2$ or $G3$, this prevents its spine from meeting $B(z, 13\rho/10)$ (by direct inspection: sets of type $G2$ and $G3$ do not stay close to planes near their spine); then Lemma 3.4 says that Z coincides with a plane in $B(z, C^{-1}\rho)$. This is enough to continue the argument (but we get worse constants).

Next, $\widetilde{Z}$ goes through z, so we can use this plane to estimate $a(z,\rho)$. We get that $a(z,\rho) \leq \frac{16}{10} D_{z,16\rho/10}(E,\widetilde{Z}) \leq \frac{16}{10}\,\varepsilon$. Since we know that $B(z,\rho) \subset B(x,r) \subset B(0,2)$, we can apply Lemma 4.1 and get that $a(z,t) \leq 2\varepsilon$ for $t < \rho/4$; hence $z \in E_1$, and (4.13) holds.

Now we want to show that

$$\operatorname{dist}(\ell, E_2) \leq 150\varepsilon r \ \text{ for } \ell \in L \cap B(x, 2r/3). \tag{4.15}$$

Since $\ell \in L \subset Y$, we can find $z_0 \in E$ such that $|z_0 - \ell| \leq 25\varepsilon r$. Set $Z = Z(z_0, r/2)$. Let us first use the fact that Z and Y are both quite close to E to show that

$$D_{z_0, r/4}(Y, Z) \leq 102\varepsilon. \tag{4.16}$$

First let $y \in Y \cap B(z_0, r/4)$ be given. Observe that $|y - x| \leq |y - z_0| + |z_0 - \ell| + |\ell - x| \leq r/4 + 25\varepsilon r + 2r/3 = 11r/12 + 25\varepsilon r$, so $y \in B(x,r)$ and (by definition of Y) we can find $y' \in E$ such that $|y' - y| \leq 25\varepsilon r$. Then $y' \in B(z_0, r/2)$, so we can find $y'' \in Z$ such that $|y'' - y'| \leq \varepsilon r/2$. Thus $\operatorname{dist}(y, Z) \leq 51\varepsilon r/2$. Similarly, if $y \in Z \cap B(z_0, r/4)$ we can find $y' \in E$ such that $|y' - y| \leq \varepsilon r/2$, then $y' \in B(x,r)$, so there exists $y'' \in Y$ such that $|y'' - y'| \leq 25\varepsilon r$. Thus $\operatorname{dist}(y, Y) \leq 51\varepsilon r/2$; (4.16) follows.

Recall that $|z_0 - \ell| \leq 25\varepsilon r$, with $\ell \in L \cap B(x, 2r/3)$ (and L is the spine of Y). By (4.16) and the second half of Lemma 3.1, Z is of type 2 or 3. We claim that

$$Z \text{ coincides with a set of type 2 in } B(z_0, r/6). \tag{4.17}$$

This is clear if Z is of type 2. Otherwise, denote its center by z_1. If $B(z_1, r/500) \subset B(z_0, r/4)$, (4.16) says that $D_{z_1, 10^{-3}r}(Y, Z) \leq 10^5\varepsilon < 1/3$. This is impossible, by Lemma 3.2. So z_1 lies out of $B(z_0, t)$, with $t = r/4 - r/500$, and Lemma 3.2 says Z coincides with a set of type 2 in $B(z_0, 4t/5)$; (4.17) follows.

Once we know (4.17), it is easy to deduce from (4.16) that every point of $L \cap B(z_0, r/10)$ lies within $100\varepsilon r$ of the spine L_Z of Z. This uses the fact that for sets of type 2, the Hausdorff distance between spines is controlled by the Hausdorff distance between the sets. For sets of type $G2$, we can check this directly, or deduce it from Lemmas 3.1 and 3.4, but with worse constants. We apply this to our point $\ell \in L$, and we find $\ell_1 \in L_Z$ such that $|\ell_1 - \ell| \leq 100\varepsilon r$. By definition of Z, we can then find $z_1 \in E$ such that $|z_1 - \ell_1| \leq \varepsilon r/2$ (see (1.1)). Thus

$$|z_1 - z_0| \leq |z_1 - \ell_1| + |\ell_1 - \ell| + |\ell - z_0| \leq 126\varepsilon r. \tag{4.18}$$

Call Y' the set of type 2 provided by (4.17), and then set $Y_1 = Y' + (z_1 - \ell_1)$. Notice that the spine L_1 of Y_1 goes through z_1 (because (4.17) says that ℓ_1 lies in the spine of Y'), so we can use it to estimate $b(z_1, r/10)$. By (4.17), Y' is $\varepsilon r/2$-close to E in $B(z_1, r/7)$, hence Y_1 is εr-close to E in $B(z_1, r/8)$, so $b(z_1, r/10) \leq 10\varepsilon$.

The pair $(z_1, r/10)$ satisfies the assumptions of Lemma 4.5; thus we can apply the argument above, with x, ℓ, and z_0 all replaced with z_1, and r replaced with $r/10$. We get a new point $z_2 \in E$, with $|z_2 - z_1| \leq 101\varepsilon r/10$ and $b(z_2, r/100) \leq 10\varepsilon$ (we can drop the $25\varepsilon r$ that come from $|\ell - z_0|$ in (4.18)).

Then we iterate and find a sequence $\{z_k\}$ in E, with

$$b(z_k, 10^{-k}r) \leq 10\varepsilon \text{ and } |z_{k+1} - z_k| \leq 101 \cdot 10^{-k}\varepsilon r. \tag{4.19}$$

Set $w = \lim_{k\to\infty} z_k$. Then $|w - z_k| \leq 120 \cdot 10^{-k}\varepsilon r$ for $k \geq 1$. Let us check that

$$b(w, t) \leq 1500\varepsilon \text{ for } t \text{ small.} \tag{4.20}$$

Choose k such that $\frac{9}{10}10^{-k-1}r \leq t \leq \frac{9}{10}10^{-k}r$. By (4.19), there is a set T of type 2, with a spine through z_k, which is $10^{-k+1}\varepsilon r$-close to E in $B(z_k, 10^{-k}r)$. We can use $T + (w - z_k)$ to estimate $b(w, t)$. Since $|w - z_k| \leq 120 \cdot 10^{-k}\varepsilon r$, we get that $b(w, t) \leq t^{-1}(130 \cdot 10^{-k}\varepsilon r) \leq 1500\varepsilon$, as needed for (4.20).

By (4.20), $w \in E_2$. Observe that $|w - \ell| \leq |w - z_1| + |z_1 - \ell| \leq 120\varepsilon r/10 + |z_1 - \ell_1| + |\ell_1 - \ell| \leq 113\varepsilon r$, by (4.19) and the estimates above (4.18). So (4.15) holds.

We are ready to estimate $D_{x,2r/3}(L, E_2)$. We just checked that $\operatorname{dist}(\ell, E_2) \leq 150\varepsilon r$ for $\ell \in L \cap B(x, 2r/3)$. Conversely, if $z \in E_2 \cap B(x, 2r/3)$, (4.13) forbids $\operatorname{dist}(z, L)$ to be more than $400\varepsilon r$ (because E_1 is disjoint from E_2, see (4.4)). Hence $D_{x,2r/3}(L, E_2) \leq 600\varepsilon$, as needed.

We still need to check that E_3 does not meet $B(x, 2r/3)$. If $z \in E_3 \cap B(x, 2r/3)$, Corollary 4.1 says that $c(z, \rho) \leq 15\varepsilon$ for $\rho > 0$ such that $B(z, 11\rho/10) \subset B(0, 2)$. We pick $\rho = r/4$, and find a set Z' of type 3, centered at z, and such that $D_{z,r/4}(Z', E) \leq 15\varepsilon$. But $D_{x,r}(E, Y) \leq 25\varepsilon$, hence $D_{z,r/5}(Y, Z) \leq 140\varepsilon$ by the usual comparison argument. This is impossible, by Lemma 3.1, so E_3 does not meet $B(x, 2r/3)$. This completes our proof of Lemma 4.5. ■

Next we study the local Reifenberg-flatness of E_2.

Lemma 4.6 *If $x \in E \cap B(0, 2) \setminus E_1$, then for every $r > 0$ such that $B(x, r) \subset B(0, 2)$, $Z(x, r)$ is a set type 2 or 3, whose spine passes at distance at most $17\varepsilon r$ from x.*

Proof. For $x \in E$ and $r > 0$ such that $B(x, r) \subset B(0, 2)$, we have a minimal cone $Z(x, r)$. Set $\delta(x, r) = +\infty$ when $Z(x, r)$ is of type 1. Otherwise, denote by $L(x, r)$ the spine of $Z(x, r)$, and set $\delta(x, r) = r^{-1} \operatorname{dist}(x, L(x, r))$. First observe that

$$\delta(x, r) < 2/3 \text{ when } a(x, r/2) \geq 10^{-4}. \tag{4.21}$$

Indeed, if $\delta(x, r) \geq 2/3$, $Z(x, r)$ is of type 1 or $L(x, r)$ does not meet $B(x, 2r/3)$, hence Lemma 3.4 says that $Z(x, r)$ coincides with a plane P in $B(x, 10r/18)$. Then we can use P to check that $a(x, r/2) < 10^{-4}$. So (4.21) holds. If we were dealing with sets of type Gi, we would prove (instead of (4.21)) that $\delta(x, r) < 2/3$ when $a(x, r/C) \geq a_4/10$, where a_4 comes from Lemma 4.1.

Next we check that

$$\delta(x, 3r) \leq \frac{1}{3}\,\delta(x, r) + 11\varepsilon. \tag{4.22}$$

when $a(x, r/2) > 10^{-4}$ and $B(x, 3r) \subset B(0, 2)$. For sets of type Gi, we would write $\delta(x, 2Cr) \leq \frac{1}{2C}\,\delta(x, r) + C'\varepsilon$ when $a(x, r/C) > a_4/10$ and $B(x, 2Cr) \subset B(0, 2)$, with C as above.

By (4.21), we can find $z_0 \in L(x, r)$, with $|x - z_0| \leq r\delta(x, r) \leq 2r/3$. Set $\rho = 16\varepsilon r$; trivially, $B(z_0, \rho) \subset B(x, 9r/10)$. For $z \in Z(x, r) \cap B(z_0, \rho)$, there is a point $z_1 \in E$ such that $|z_1 - z| \leq \varepsilon r$, and then a point $z_2 \in Z(x, 3r)$ such that $|z_2 - z_1| \leq 3\varepsilon r$, so $\operatorname{dist}(z, Z(x, 3r)) \leq 4\varepsilon r$. Similarly, $\operatorname{dist}(z, Z(x, r)) \leq 4\varepsilon r$ for $z \in Z(x, 3r) \cap B(z_0, \rho)$. So $D_{z_0,\rho}(Z(x, r), Z(x, 3r)) \leq 1/4$, and Lemma 3.1 says that $Z(x, 3r)$ cannot coincide with a plane in $B(z_0, \rho)$. Hence, $Z(x, 3r)$ is of type 2 or 3, and Lemma 3.4 says that its spine meets $B(z_0, 2\rho)$. Then $\operatorname{dist}(x, L(x, 3r)) \leq |x - z_0| + 2\rho \leq r\delta(x, r) + 32\varepsilon r$, and $\delta(x, 3r) \leq \frac{1}{3}\delta(x, r) + 11\varepsilon$, as needed for (4.22).

Let us also check that $a(x, 3r/2) > 10^{-4}$ in the proof of (4.22). Indeed, otherwise there is a plane P such that $D_{x,3r/2}(P, E) \leq 10^{-4}$. At the same time, $D_{x,r}(Z(x, r), E) \leq \varepsilon$ and $z_0 \in B(x, 2r/3)$, so $D_{z_0,r/10}(Z(x, r), P) \leq 16 \cdot 10^{-4}$ by the usual argument, and this is impossible because the spine of $Z(x, r)$ goes through z_0 and by Lemma 3.1. So we can apply (4.22) again to the pair $(x, 3r)$, at least if $B(x, 9r) \subset B(0, 2)$.

Now let x and r be as in the statement of Lemma 4.6. Since $x \notin E_1$, Lemma 4.1 says that $a(x, \rho) \geq 10^{-3}$ for ρ small. Multiple applications of (4.22) (starting from $\delta(x, 3^{-k}r)$, with k very large) show that $\delta(x, r) < 17\varepsilon$ (because $\frac{3}{2} \cdot 11\varepsilon < 17\varepsilon$). Lemma 4.6 follows. ∎

We are now ready to prove that

$$E \cap B(0, 2) = E_1 \cup E_2 \cup E_3 \quad \text{(a disjoint union).} \tag{4.23}$$

We already know from (4.4) that the union is disjoint. Let $x \in E \cap B(0, 2)$ be given, assume that $x \notin E_1$, and let us check that x lies in E_2 or E_3.

Suppose first that there are arbitrarily small r such that $c(x, r) \leq 10^{-3}$; then multiple applications of Lemma 4.2 give arbitrarily small ρ such that $c(x, 10^k\rho) \leq 15\varepsilon$ for every $k \geq 0$ such that $B(x, \frac{11}{10} \cdot 10^k\rho) \subset B(0, 2)$. [The proof is the same as for Corollary 4.1.] Then we also have that $c(x, r) \leq 150\varepsilon$ for every $r > 0$ such that $B(x, 11\rho) \subset B(0, 2)$. [Use any $\rho < r$ and pick k such that $10^{k-1}\rho \leq r < 10^k\rho$.] In this case $x \in E_3$ and we are happy. So may assume that

$$c(x, r) > 10^{-3} \quad \text{for } r \text{ small enough.} \tag{4.24}$$

Let r be small, and set $Z = Z(x, 10r)$. Since $x \notin E_1$, Lemma 4.6 says that Z is of type 2 or 3, with a spine L that goes through $\overline{B}(x, 170\varepsilon r)$. Pick $\ell \in L \cap \overline{B}(x, 170\varepsilon r)$. If Z coincides with a set Y of type 2 in $B(x, 2r)$, then $Y' = Y + (x - \ell)$ is a set of type 2 whose spine goes through x, we can use it to evaluate $b(x, r)$, and we get that $b(x, r) \leq 10\varepsilon + 170\varepsilon < 500\varepsilon$. If this is the case for every small r, then $x \in E_2$ and we are happy.

So it is enough to show that for r small, Z coincides with a set of type 2 in $B(x,2r)$. Recall that Z is of type 2 or 3; if it is of type 2, we are happy. If it is of type 3, with a center $z_0 \notin B(x,5r/2)$, Lemma 3.2 says that Z coincides with a set of type 1 or 2 in $B(x,2r)$, and we are happy too (type 1 is excluded because L goes through $\overline{B}(x,170\varepsilon r)$). So we can assume that $z_0 \in B(x,5r/2)$. [With sets of type Gi, we only get that $z_0 \in B(x,Cr)$, but this is enough.]

Pick $\xi \in E$ such that $|\xi - z_0| \leq 10\varepsilon r$; we can use $Z+(\xi - z_0)$ to evaluate $c(\xi,r)$, and we get that $c(\xi,r) \leq 20\varepsilon$. If r is small enough, $B(x,10^5 r) \subset B(0,2)$, we can apply Lemma 4.2 four times, and we get that $c(\xi,10^4 r) \leq 20\varepsilon$. This means that there is a set T of type 3, centered at ξ, and such that $D_{\xi,10^4 r}(T,E) \leq 20\varepsilon$. We can use $T+(x-\xi)$ to evaluate $c(x,5000r)$, and we get that $c(x,5000r) \leq (5000r)^{-1}(3r + 2\cdot 10^5\varepsilon r) < 10^{-3}$. This is impossible, by (4.24), so we proved that $x \in E_2$, and our proof of (4.23) is complete.

Before we return to the local Reifenberg-flatness of E_2 away from E_3 (if $E_3 \neq \emptyset$), observe that

$$E_2 \cup E_3 \text{ is closed in } B(0,2). \tag{4.25}$$

Indeed, let $x \in B(0,2)$ be the limit of a sequence in $E_2 \cup E_3$. If $x \in E_1$, $a(x,r) \leq 10\varepsilon$ for r small, and Lemma 4.1 says that $E \cap B(x,2r/3) \subset E_1$. This is impossible, by definition of x and because the sets E_i are disjoint. Recall that E is closed, so $x \in E \setminus E_1$, and (4.23) says that $x \in E_2 \cup E_3$; (4.25) follows.

Lemma 4.7 *If $x \in E_2$ and $0 < r \leq \frac{1}{2}\min\big[\operatorname{dist}(x,E_3), \operatorname{dist}(x,\partial B(0,2))\big]$, then there is a set $Y = Y(x,r)$ of type 2, whose spine $L = L(x,r)$ goes through x, such that*

$$D_{x,3r/2}(E,Y) \leq 24\varepsilon \quad \text{and} \quad D_{x,r}(E_2,L) \leq 600\varepsilon. \tag{4.26}$$

Proof. We already know from Lemma 4.6 that $Z = Z(x,2r)$ is a set type 2 or 3, whose spine L passes at distance at most $34\varepsilon r$ from x. Let us check that

$$\text{there is a set } Y_1 \text{ of type 2 such that } Y_1 \cap B(x,155r/100) = Z \cap B(x,155r/100). \tag{4.27}$$

This is clear if Z is of type 2. If Z is of type 3, and its center lies out of $B(x,198r/100)$, Lemma 3.2 says that there is a set Y_1 of type 1 or 2 that coincides with Z in $B(x,\frac{4}{5}\cdot\frac{198r}{100})$. Obviously, Y_1 is of type 2 because the spine of Z passes through $B(x,r)$, and (4.27) follows because $\frac{155r}{100} < \frac{4}{5}\cdot\frac{198r}{100}$. Finally suppose that Z is of type 3 and its center z lies in $B(x,198r/100)$. By definition of Z, we can find $y \in E \cap \overline{B}(z,2\varepsilon r)$; then we can use $Z+(y-z)$ to compute $c(y,r/200)$, and we get that $c(y,r/200) \leq 200(2\varepsilon + 2\varepsilon) < 10^{-4}$. Lemma 4.3 gives a point $\xi \in E_3 \cap B(y,r/200)$. This is impossible because $\operatorname{dist}(x,E_3) \geq 2r$ so (4.27) holds.

Recall that $\operatorname{dist}(x,L) \leq 34\varepsilon r$. Pick $\ell \in L$ such that $|x-\ell| \leq 34\varepsilon r$ and set $Y = Y_1 + (x-\ell)$; then Y is a set of type 2 whose spine goes through x. Let us check that

$$D_{x,3r/2}(E,Y) \leq 24\varepsilon. \tag{4.28}$$

If $e \in E \cap B(x, 3r/2)$, we can find $y \in Z$ such that $|y-e| \leq 2\varepsilon r$, then $y \in B(x, 155r/100)$, (4.27) says that $y \in Y_1$ and $y' = y-(x-\ell) \in Y$. So $\text{dist}\,(e, Y) \leq |y'-e| \leq |y'-y|+|y-e| = |x-\ell|+|y-e| \leq 36\varepsilon r$. Similarly, if $y' \in Y \cap B(x, 3r/2)$, then $y = y'+(x-\ell)$ lies in $Y_1 \cap B(x, 155r/100)$, so we can find $y \in E$ such that $|y-e| \leq 2\varepsilon r$. Then $\text{dist}\,(y', E) \leq |y'-e| \leq 36\varepsilon r$; (4.28) follows.

Now the ball $B(x, 3r/2)$ satisfies the hypotheses of Lemma 4.5, and this lemma says that $D_{x,r}(E_2, L) \leq 600\varepsilon$. Since (4.28) gives the first part of (4.26), Lemma 4.7 follows. ■

The second part of (4.26) gives the local Reifenberg flatness of E_2 inside of $B(0, 2)$ and away from E_3. The next lemma says that near a point of E_3, E_2 looks like the spine of a set of type 3 (that is, four half lines).

Lemma 4.8 *Let $x \in E_3$ and $r > 0$ be such that $B(x, 2r) \subset B(0, 199/100)$. Then there is a set $T = T(x, r)$ of type 3 centered at x, such that*

$$D_{x,4r/3}(E, T) \leq 15\varepsilon \quad \textit{and} \quad D_{x,r}(E_2, L) \leq 220\varepsilon, \tag{4.29}$$

where L denotes the spine of T.

Proof. We proceed as in Lemma 4.7. Corollary 4.1 says that $c(x, 4r/3) \leq 15\varepsilon$, so we can find a set T of type 3, centered at x, such that $D_{x,4r/3}(E, T) \leq 15\varepsilon$.

Call L the spine of T. Let us first check that

$$\text{dist}\,(y, L) \leq 100\varepsilon r \quad \text{for } y \in E_2 \cap B(x, r). \tag{4.30}$$

Let $y \in E \cap B(x, r)$ be such that $\text{dist}\,(y, L) \geq 100\varepsilon r$, and call $Z = Z(y, 100\varepsilon r)$ the set promised by (1.1). Every point of $Z \cap B(y, 100\varepsilon r)$ lies within $100\varepsilon^2 r$ of E, and hence within $21\varepsilon r$ of T, by definition of T. Since $\text{dist}\,(y, L) \geq 100\varepsilon r$, Lemma 3.4 says that Z coincides with a plane P in $B(y, 60\varepsilon r)$. In addition, Z (and hence also P) goes through y, so we can use P to compute $a(y, 50\varepsilon r)$; we get that $a(y, 50\varepsilon r) \leq 2\varepsilon \leq 10^{-4}$, and Lemma 4.1 says that $y \in E_1$. This proves (4.30).

We still need to show that

$$\text{dist}\,(\xi, E_2) \leq 220\varepsilon r \quad \text{for } \xi \in L \cap B(x, r). \tag{4.31}$$

First consider $\xi \in L \cap B(x, r) \setminus B(x, 220\varepsilon r)$. By definition of T we can find $y \in E \cap \overline{B}(\xi, 20\varepsilon r)$. Then set $Z = Z(y, 220\varepsilon r)$; let us check that

$$D_{\xi,90\varepsilon r}(Z, T) \leq \frac{21}{90} < 1/4. \tag{4.32}$$

Every point of $Z \cap B(\xi, 90\varepsilon r)$ lies within $220\varepsilon^2 r$ of E, hence within $220\varepsilon^2 r + 20\varepsilon r < 21\varepsilon r$ of T (by definition of T, and because $\xi \in B(x, r)$). Similarly, if $t \in T \cap B(\xi, 90\varepsilon r)$, we can find $e \in E$ such that $|e-t| \leq 20\varepsilon r$, then $|e-y| \leq |e-t|+|t-\xi|+|\xi-y| < 130\varepsilon r$, so $e \in E \cap B(y, 130\varepsilon r)$ and we can find $z \in Z$ such that $|z-e| \leq 220\varepsilon^2 r < \varepsilon r$; (4.32) follows.

Recall that T is of type 3 and ξ lies on its spine; then Lemma 3.1 says that $D_{\xi,90\varepsilon r}(T,P) \geq 1/4$ for any plane, so (4.32) forbids Z to coincide with a plane in $B(\xi, 120\varepsilon r)$.

By Lemma 3.4, Z to be of type 2 or 3, with a spine that meets the slightly larger ball $B(\xi, 144\varepsilon r)$. [For sets of type Gi, replace 144 with a geometric constant.] Pick ℓ in the spine of Z, with $|\ell - \xi| \leq 144\varepsilon r$. Obviously $\ell \in B(y, 164\varepsilon r)$ (because $|\xi - y| \leq 20\varepsilon r$), so we can pick $z_1 \in E$ such that $|z_1 - \ell| \leq 220\varepsilon^2 r$. Let us check that

$$\text{(4.33)} \qquad \text{there is a set } Y \text{ of type 2 that coincides with } Z \text{ in } B(z_1, 24\varepsilon r).$$

This is clear if Z is of type 2, so let us assume that it is of type 3. Let z_0 denote its center. By Lemma 3.2, it is enough to show that $|z_0 - z_1| \geq 32\varepsilon r$. Otherwise,

$$\text{(4.34)} \qquad |z_0 - y| \leq |z_0 - z_1| + |z_1 - \ell| + |\ell - y| \leq 32\varepsilon r + 220\varepsilon^2 r + 164\varepsilon r \leq 197\varepsilon r.$$

[For sets of type Gi, we can leave 24 as it is, replace 32 with $24C$, and also replace 197 with a larger geometric constant a_{22}; the next line forces us to replace 220 with a constant $a_{23} > a_{22}$.]

Since $z_0 \in Z \cap B(y, 220\varepsilon r)$, we can find $e \in E$, with $|e - z_0| \leq 220\varepsilon^2 r$. The set $Z + (e - z_0)$ is of type 3, with a center at e, so we can use it to compute $c(e, \varepsilon r)$. Since $D_{y,220\varepsilon r}(Z, E) \leq \varepsilon$, the usual argument shows that $c(e, \varepsilon r) \leq 440\varepsilon$; then Lemma 4.3 gives a point $\eta \in E_3 \cap B(e, \varepsilon r)$. Notice that $|\eta - \xi| \leq |\eta - e| + |e - z_0| + |z_0 - y| + |y - \xi| \leq \varepsilon r + 220\varepsilon^2 r + 197\varepsilon r + 20\varepsilon r \leq 219\varepsilon r$, by (4.34) in particular. Since $|\xi - x| \geq 220\varepsilon r$, we get that $\eta \neq x$. At the same time, $\eta \in B(x, 2r)$ because $\xi \in B(x, r)$, so we have two points of E_3 in $B(x, 2r) \subset B(0, 199/100)$. Lemma 4.4 says that this is impossible, so $|z_0 - z_1| \geq 32\varepsilon r$ and (4.33) holds.

Recall from just above (4.33) that $|z_1 - \ell| \leq 220\varepsilon^2 r$, and ℓ lies on the spine of Z. Hence the spine of $Y' = Y + (z_1 - \ell)$ goes through z_1, and we can use Y' to evaluate $b(z_1, 23\varepsilon r)$. Recall that (4.33) says that Y is $220\varepsilon^2 r$-close to E near $B(z_1, 23\varepsilon r)$; we get that $b(z_1, 23\varepsilon r) \leq 440\varepsilon/23 < 25\varepsilon$. Thus we can apply Lemma 4.5, and there is a point $w \in E_2 \cap B(z_1, \varepsilon r)$. [For sets of type Gi, we then need the constant a_{15} in Lemma 4.5 to be large enough compared to the replacement a_{23} for 220; this is not a serious problem, because a_{23} is a geometric constant coming from Section 3.] Now $|w - \xi| \leq |w - z_1| + |z_1 - \ell| + |\ell - \xi| \leq \varepsilon r + 220\varepsilon^2 r + 144\varepsilon r \leq 146\varepsilon r$.

So we proved (4.31) when $\xi \in L \cap B(x, r) \setminus B(x, 220\varepsilon r)$. We can apply this to a point of $L \cap \partial B(x, 250\varepsilon r)$, and we get the existence of a point in $E_2 \cap B(x, 500\varepsilon r)$. Also, the argument above applies to any ball $B(x, \rho)$ centered at x such that $B(x, 2\rho) \subset B(0, 199/100)$, so by taking small values of ρ, we find a sequence in E_2 that converges to x.

Now x lies in the closure of E_2, so $\operatorname{dist}(\xi, E_2) \leq 220\varepsilon r$ for every $\xi \in B(x, 220\varepsilon r)$. This completes our proof of (4.31), and Lemma 4.8 follows. ■

This ends our general description of E_1 and E_2. As was suggested in Remark 1.1, we shall rapidly restrict to two simple situations. In addition to (1.1), we shall assume that either

$$\text{(4.35)} \qquad Z(0,2) \text{ is a set of type 2, whose spine } L \text{ contains } 0$$

(recall that $Z(x,r)$ is the set given by (1.1)), or else

$$\text{(4.36)} \qquad \text{the origin lies in } E_3 \text{, and } Z(0,2) \text{ is a set of type 3 centered at } 0.$$

In these two cases, we shall establish the slightly more precise (1.7) and (1.8) instead of (1.3) and (1.4).

Let us rapidly say what to do in the other cases. First suppose that we can find $x_0 \in E_3 \cap B(0, 199/100)$. By Lemma 4.4, x_0 is unique.

If $x_0 = 0$, Corollary 4.1 says that $Z(0,2)$ is a set of type 3 centered near the origin, and we may as well suppose that it is centered at 0, as in (4.36) (and replace ε with a slightly larger constant).

If x_0 lies in $B(0, 195/100)$, we can again assume that $Z(0,2)$ is a set of type 3 centered at x_0, and keep the proof below almost exactly as it is (just center the balls B_{i_0} at x_0); we even get (1.7) and (1.8).

If $Z(0,2)$ is a set of type 3 and its center lies in $B(0, 194/100)$, Lemma 4.3 gives a point in $E_3 \cap B(0, 195/100)$, and we are in one of the two previous cases. Otherwise, Lemma 3.2 says that $Z(0,2)$ coincides with a set Z' of type 1 or 2 in $B(0, 4 \cdot \frac{194}{500}) \supset B(0, 155/100)$. If Z' is of type 2, the construction below restricted to $B(0, \frac{3}{2})$ yields the function f promised in Theorem 1.1. This works whether or not 0 lies in the spine of Z'. If Z' is of type 1 for $y \in B(0, \frac{154}{100}) \cap E$, it is easy to see that $a(y, \frac{1}{100}) < 10^{-3}$. In this case, Lemma 4.1 ensures that $a(y,t) \leq 2\varepsilon$, for all $y \in B(0, \frac{154}{100}) \cap E$ and t small. Thus $B(0, \frac{154}{100}) \cap E = \emptyset$, and we are in the standard Reifenberg situation.

Anyway we will find it just as convenient to restrict to the case when $Z(0,2)$ is of type 2 whose spine contains the origin, as in (4.35).

Let us record here that

$$\text{(4.37)} \qquad E_3 \cap B(0, 199/100) \text{ is empty when (4.35) holds.}$$

Indeed, if $x \in E_3 \cap B(0, 199/100)$, Corollary 4.1 says that $c(x, 1/200) \leq 15\varepsilon$, and Lemma 3.1 says that this is incompatible with the fact that $D_{0,2}(Z(0,2), E) \leq \varepsilon$ for a set $Z(0,2)$ of type 2.

Also recall that when (4.36) holds, Lemma 4.4 says that 0 is the only point of $E_3 \cap B(0, 199/100)$.

5 Coverings and partitions of unity

Our general plan is to follow the same scheme as in the standard proof of Reifenberg's theorem (see [To], [DT]) ; thus we shall construct our parameterization by successive deformations of the set $Z(0,2)$. These deformations will occur near $E \cap B(0,2)$, and appropriate partitions of unity play a key role.

We construct one such partition for each scale $2^{-n}, n \geq 0$. We also construct the mapping f in a hierarchical way, which means that we define it first on the spine of $Z(0,2)$ (see §6), then on $Z(0,2)$ itself (see §7), and finally on the rest of $B(0,2)$ (see §9). Our partitions of unity reflect this.

In the more general situation of Theorem 2.2, only one minor modification is needed in this section. Since we have less control on the various constants that arise in the previous section, the small security constant 10^{-20} that is used below needs to be replaced with a smaller constant, that depends on n, d, and τ_0.

For $n \geq 0$ given, we construct a "good" covering of E at scale 2^{-n}. If (4.36) holds, we first cover $E_3 \cap B(0, 199/100) = \{0\}$ with a unique ball $B_{i_0} = B(0, 2^{-n-20})$. For accounting reasons, we set $I_3 = I_3(n) = \{i_0\}$. If (4.35) holds, we simply take $I_3 = \emptyset$ and choose no ball.

Next we want to cover

$$E_2' = E_2 \cap B(0, 198/100) \setminus \frac{7}{4} B_{i_0} \tag{5.1}$$

(or just $E_2' = E_2 \cap B(0, 198/100)$ if (4.35) holds). Here and below, λB is a notation for the ball with the same center as B and λ times the radius. Select a maximal subset of E_2', with the property that different points of E_2' lie at mutual distances at least 2^{-n-40}. Call x_i, $i \in I_2 = I_2(n)$ (with $I_2 \cap I_3 = \emptyset$), the points of this set, and set $r_i = 2^{-n-40}$ and $B_i = B(x_i, r_i)$ for $i \in I_2$. By maximality,

$$\text{the balls } \overline{B}_i,\ i \in I_2, \text{ cover } E_2'. \tag{5.2}$$

Then we take care of E_1. Set

$$V_2 = \frac{15}{8} B_{i_0} \cup \bigcup_{i \in I_2} \frac{7}{4} B_i \quad \text{and} \quad E_1' = E_1 \cap B(0, 197/100) \setminus V_2 \tag{5.3}$$

(forget about B_{i_0} if (4.35) holds), and pick a maximal subset of E_1' with the property that different points of E_1' lie at mutual distances at least 2^{-n-60}. Call x_i, $i \in I_1 = I_1(n)$, with $I_1 \cap (I_2 \cup I_3) = \emptyset$, the points of this set, and set $r_i = 2^{-n-60}$ and $B_i = B(x_i, r_i)$ for $i \in I_1$. Then

$$\text{the balls } \overline{B}_i,\ i \in I_1, \text{ cover } E_1'. \tag{5.4}$$

Let us go one more step, and set

$$V_1 = \frac{31}{16} B_{i_0} \cup \bigcup_{i \in I_2} \frac{15}{8} B_i \cup \bigcup_{i \in I_1} \frac{7}{4} B_i \quad \text{and} \quad E_0' = \mathbb{R}^3 \setminus V_1, \tag{5.5}$$

pick a maximal set in E_0' with points at mutual distances at least 2^{-n-80}, call its points x_i, $i \in I_0 = I_0(n)$, with $I_0 \cap (I_1 \cup I_2 \cup I_3) = \emptyset$ and set $r_i = 2^{-n-80}$ and $B_i = B(x_i, r_i)$ for $i \in I_0$. Again

$$\text{the balls } \overline{B}_i,\ i \in I_0, \text{ cover } E_0'. \tag{5.6}$$

Let us check that

$$E_2 \cap B(0, 198/100) \subset \frac{7}{4} B_{i_0} \cup \Big[\bigcup_{i \in I_2} \overline{B}_i \Big]. \tag{5.7}$$

Indeed, if $x \in E_2 \cap B(0, 198/100)$ does not lie in $\frac{7}{4} B_{i_0}$, then it lies in E_2', and (5.2) gives the result. Similarly,

$$E \cap B(0, 197/100) \subset \frac{15}{8} B_{i_0} \cup \Big[\bigcup_{i \in I_2} \frac{7}{4} B_i \Big] \cup \Big[\bigcup_{i \in I_1} \overline{B}_i \Big] \tag{5.8}$$

because, if $x \in E \cap B(0, 197/100)$ lies out of $\frac{15}{8} B_{i_0} \cup \Big[\bigcup_{i \in I_2} \frac{7}{4} B_i \Big]$, then it lies in E_1' by (4.23) and (5.3); thus (5.8) follows from (5.4). Finally,

$$\frac{31}{16} B_{i_0} \cup \Big[\bigcup_{i \in I_2} \frac{15}{8} B_i \Big] \cup \Big[\bigcup_{i \in I_1} \frac{7}{4} B_i \Big] \cup \Big[\bigcup_{i \in I_0} \overline{B}_i \Big] = \mathbb{R}^3, \tag{5.9}$$

again because the union of the first three pieces is V_1, and by (5.6).

Now we define a partition of unity. Set $I = I(n) = I_3 \cup I_2 \cup I_1 \cup I_0$. For each $i \in I$, pick a smooth function $\widetilde{\theta}_i$ such that

$$\widetilde{\theta}_i(x) = 1 \text{ in } 2B_i, \ \widetilde{\theta}_i(x) = 0 \text{ out of } 3B_i, \text{ and } 0 \leq \widetilde{\theta}_i(x) \leq 1 \text{ everywhere.} \tag{5.10}$$

We may choose the $\widetilde{\theta}_i$ as translations and dilations of a same model. In any case they are chosen so that

$$|\nabla \widetilde{\theta}_i| \leq C 2^n \text{ and } |\nabla^2 \widetilde{\theta}_i| \leq C 2^{2n}. \tag{5.11}$$

Set $\Theta = \sum_{i \in I} \widetilde{\theta}_i$. Since by (5.9), $\{2B_i\}_{i \in I}$ covers $\mathbb{R}^3$, $1 \leq \Theta(x)$. Moreover note that $\Theta(x) \leq C < \infty$ because the choice of balls ensures that they only overlap a bounded number of times. Hence we can set

$$\theta_i(x) = \widetilde{\theta}_i(x) / \Theta(x) \text{ for } x \in \mathbb{R}^3. \tag{5.12}$$

The usual computations yield

$$\sum_{i \in I(n)} \theta_i(x) = 1, \ |\nabla \theta_i| \leq C 2^n, \text{ and } |\nabla^2 \theta_i| \leq C 2^{2n} \text{ for } x \in \mathbb{R}^3. \tag{5.13}$$

The point of our complicated choice of balls is that we shall have a good control on the supports of the θ_i. For instance, if (4.36) holds,

$$\operatorname{dist}\big(\frac{3}{2} B_{i_0}, 100 B_i\big) \geq 2^{-n-30} \text{ when } i \in I_2 \cup I_1 \cup I_0. \tag{5.14}$$

Indeed we chose $x_i \notin \frac{7}{4}B_{i_0}$, then 2^{-n-30} and the radius of $100B_i$ are both much smaller than one fourth of the radius of B_{i_0} (which is 2^{-n-22}). Similarly,

$$\text{dist}\,(\frac{3}{2}B_j, 100B_i) \geq 2^{-n-50} \text{ when } j \in I_2 \text{ and } i \in I_1 \cup I_0 \tag{5.15}$$

and

$$\text{dist}\,(\frac{3}{2}B_j, 100B_i) \geq 2^{-n-70} \text{ when } j \in I_1 \text{ and } i \in I_0. \tag{5.16}$$

We also claim that

$$\text{dist}\,(\frac{7}{4}B_{i_0}, 100B_i) \geq 2^{-n-50} \text{ when } i \in I_1 \cup I_0. \tag{5.17}$$

Indeed, for $i \in I_1 \cup I_0$, we chose $x_i \notin \frac{15}{8}B_{i_0}$. Similarly,

$$\text{dist}\,(\frac{7}{4}B_j, 100B_i) \geq 2^{-n-70} \text{ when } j \in I_2 \text{ and } i \in I_0. \tag{5.18}$$

Finally,

$$\text{dist}\,(\frac{15}{8}B_{i_0}, 100B_i) \geq 2^{-n-30} \text{ when } i \in I_0, \tag{5.19}$$

this time because $x_i \notin \frac{31}{16}B_{i_0}$.

The inequalities (5.14)-(5.19) have implications on the supports of the θ_i. For instance, (5.14) says that on $\frac{3}{2}B_{i_0}$ all the θ_i, $i \neq i_0$, vanish (and so $\theta_{i_0}(x) = 1$). Also,

$$\theta_i(x) = 0 \ \text{ when } \ \text{dist}\Big(x, \frac{7}{4}B_{i_0} \cup \bigcup_{j \in I_2} \frac{3}{2}B_j\Big) \leq 2^{-n-70} \text{ and } i \in I_1 \cup I_0, \tag{5.20}$$

by (5.17) and (5.15). By (5.7), this forces

$$\theta_i(x) = 0 \ \text{ when } \ \text{dist}\,(x, E_2 \cap B(0, 198/100)) \leq 2^{-n-70} \text{ and } i \in I_1 \cup I_0, \tag{5.21}$$

and hence, by (5.13),

$$\sum_{i \in I_3 \cup I_2} \theta_i(x) = 1 \ \text{ when } \ \text{dist}\,(x, E_2 \cap B(0, 198/100)) \leq 2^{-n-70}. \tag{5.22}$$

Finally,

$$\theta_i(x) = 0 \ \text{ when } \ \text{dist}\Big(x, \frac{15}{8}B_{i_0} \cup \bigcup_{j \in I_2} \frac{7}{4}B_j \cup \bigcup_{j \in I_1} \frac{3}{2}B_j\Big) \leq 2^{-n-70} \ \text{ and } i \in I_0, \tag{5.23}$$

by (5.19), (5.18), and (5.16), so (5.8) says that

$$\theta_i(x) = 0 \ \text{ when } \ \text{dist}\,(x, E \cap B(0, 197/100)) \leq 2^{-n-70} \ \text{ and } i \in I_0, \tag{5.24}$$

and then, by (5.13),

$$\sum_{i \in I_3 \cup I_2 \cup I_1} \theta_i(x) = 1 \ \text{ when } \ \text{dist}\,(x, E \cap B(0, 197/100)) \leq 2^{-n-70}. \tag{5.25}$$

6 A first parameterization of E_2

We are now ready to start a construction of the mapping f. To simplify the description, we start with the case when (4.35) holds, i.e., when $Z(0,2)$ is of type 2. As we shall see in Section 8, the modifications that are needed when (4.36) holds are mostly cosmetic.

We still want to proceed by layers, so we shall start with a parameterization of a big part of E_2, which we shall mostly be interested to define on

$$\Gamma = L \cap \overline{B}(0, 197/100)), \tag{6.1}$$

where L denotes the spine of $Z(0,2)$.

Recall from Lemma 4.7, (4.37), and (4.25), that E_2 is a closed locally Reifenberg-flat set of dimension 1 in $B(0,199/100)$. This allows us to reproduce a standard Reifenberg argument and produce a parameterization for E_2. As some details of the construction are needed later, we include it here.

Our parameterization f^* is the limit of a sequence of mappings f_n^* defined on Γ and constructed by induction. We start with $f_0^*(x) = x$, and then set

$$f_{n+1}^* = g_n^* \circ f_n^* \text{ for } n \geq 0. \tag{6.2}$$

In turn the deformation g_n^* is defined by

$$g_n^*(x) = \sum_{i \in I(n)} \theta_i(x) \psi_i^*(x), \tag{6.3}$$

where θ_i is as in the partition of unity defined in the previous section, and ψ_i^* is a suitable deformation, which pushes points closer to E_2. Even though we mostly care about the values of g_n^* and the ψ_i^* in a neighborhood of $\Gamma_n = f_n^*(\Gamma)$, it will be just as easy to define them everywhere. We take $\psi_i^*(x) = x$ everywhere when $i \in I_1(n) \cup I_0(n)$; this does not matter too much, because we shall prove that Γ_n stays very close to E_2 and hence (5.21) ensures that the θ_i's, $i \in I_1(n) \cup I_0(n)$, vanish there. We still need to define ψ_i^* when $i \in I_2$. The following notation will also be used in later sections.

Recall that when $i \in I_2(n)$, x_i lies in $E_2' = E_2 \cap B(0, 198/100) \setminus \frac{7}{4} B_{i_0}$ (but in this section there is no B_{i_0}), so Lemma 4.7 (applied to the pair $(x_i, 10r_i)$, with $r_i = 2^{-n-40}$) tells us that there is a set $Y_i = Y(x_i, 10r_i)$ of type 2, whose spine L_i goes through x_i, and such that

$$D_{x_i,15r_i}(E, Y_i) \leq 24\varepsilon \text{ and } D_{x_i,10r_i}(E_2, L_i) \leq 600\varepsilon. \tag{6.4}$$

For the moment, we just need the second half of (6.4).

When $i \in I_2(n)$, we simply take for ψ_i^* the orthogonal projection onto L_i. Thus we have defined ψ_i^* for all $i \in I(n)$, and $g_n^*(x)$ and $f_n^*(x)$ are defined for all x and n. Several estimates are needed before we can continue our construction. First we claim that

$$D_{x_i,10r_i}(L_i, L_j) \leq C\varepsilon \text{ when } i, j \in I_2(n) \text{ and } 8B_i \text{ meets } 8B_j. \tag{6.5}$$

Indeed suppose that $8B_i$ meets $8B_j$, and pick x in the intersection. Let us first estimate $D_{x,r_i}(L_i, L_j)$. For $y \in L_i \cap B(x, r_i)$, (6.4) says that we can find $z \in E_2$, with $|z - y| \leq 6000\varepsilon r_i$. Then $y \in 10B_j$, and we can find $t \in L_j$ such that $|t - z| \leq 6000\varepsilon r_j = 6000\varepsilon r_i$. So $\operatorname{dist}(y, L_j) \leq 12000\varepsilon r_i$. Similarly, $\operatorname{dist}(y, L_i) \leq 12000\varepsilon r_i$ for $y \in L_j \cap B(x, r_i)$. So $D_{x,r_i}(L_i, L_j) \leq C\varepsilon$. Define L_i by a point $y_i \in L_i \cap B(x, r_i)$ and a unit vector v_i, and proceed similarly with L_j; we can choose y_j such that $|y_j - y_i| \leq C\varepsilon r_i$, and it is easy to see that $|v_j - v_i| \leq C\varepsilon$ or $|v_j + v_i| \leq C\varepsilon$. Then (6.5) follows. Next

$$|\psi_i^*(x) - \psi_j^*(x)| \leq C\varepsilon 2^{-n} \text{ when } i, j \in I_2(n) \text{ and } x \in 8B_i \cap 8B_j. \tag{6.6}$$

Indeed, $8B_i$ meets $8B_j$, so $D_{x_i,10r_i}(L_i, L_j) \leq C\varepsilon$ by (6.5). Again it is a matter of elementary geometry that the orthogonal projections $\psi_i^*(x)$ and $\psi_j^*(x)$ are close to each other (for instance, write $L_i = y_i + \mathbb{R}v_i$ and $L_j = y_j + \mathbb{R}v_j$ as above and compute). For the same reason, the (constant) derivatives of ψ_i^* and ψ_j^* are $C\varepsilon$-close, i.e.,

$$|D\psi_i^* - D\psi_j^*| \leq C\varepsilon \text{ when } i, j \in I_2(n) \text{ and } 8B_i \text{ meets } 8B_j. \tag{6.7}$$

For each x such that

$$\operatorname{dist}(x, E_2 \cap B(0, 198/100)) \leq 2^{-n-70}, \tag{6.8}$$

(5.21) tells us that all the $\theta_i(x)$ with $i \notin I_2(n)$, vanish at x. This allows us to pick $j(x) \in I_2(n)$ such that $\theta_{j(x)}(x) \neq 0$, use (6.6) and (5.10) to show that $|\psi_i^*(x) - \psi_{j(x)}^*(x)| \leq C\varepsilon 2^{-n}$ whenever $\theta_i(x) \neq 0$, and get that

$$|g_n^*(x) - \psi_{j(x)}^*(x)| \leq C_1\varepsilon 2^{-n}, \tag{6.9}$$

because $g_n^*(x)$ is an average of the $\psi_i^*(x)$. Here C_1, just like the next constants C_j, is a simple geometric constant. Since $\psi_{j(x)}^*(x) \in L_{j(x)} \cap 3B_{j(x)}$, (6.4) says that

$$\operatorname{dist}(g_n^*(x), E_2) \leq C_2\varepsilon 2^{-n-1}. \tag{6.10}$$

Let $y \in E_2$ minimize the distance to x; notice that $y \in 6B_{j(x)}$ because $x \in 3B_{j(x)}$ and hence $\operatorname{dist}(x, E_2) \leq |x - x_{j(x)}| \leq 3r_{j(x)}$. Then by (6.4)

$$|\psi_{j(x)}^*(x) - x| = \operatorname{dist}(x, L_{j(x)}) \leq |x - y| + \operatorname{dist}(y, L_{j(x)}) \leq \operatorname{dist}(x, E_2) + 6000\varepsilon r_{j(x)}. \tag{6.11}$$

Hence by (6.9) and (6.11)

$$|g_n^*(x) - x| \leq \operatorname{dist}(x, E_2) + C_3\varepsilon 2^{-n}. \tag{6.12}$$

We shall remember that all this happens for $x \in B(0, 2)$ such that (6.8) holds.

We are ready to show by induction that if $z \in \Gamma = L \cap \overline{B}(0, 197/100)$, then

$$\operatorname{dist}(f_n^*(z), E_2) \leq C_2\varepsilon 2^{-n}, \tag{6.13}$$

and

$$(6.14) \qquad |f_{n+1}^*(z) - f_n^*(z)| \leq (C_2 + C_3)\varepsilon 2^{-n}$$

for every $n \geq 0$.

Indeed we know that (6.13) holds for $n = 0$ (if we did not choose C_2 too small). Also, suppose that (6.13) holds for n and (6.14) holds for all $m < n$ (a vacuous condition when $n = 0$). Then $|f_n^*(z) - z| \leq 4(C_2 + C_3)\varepsilon$ (by repeated use of (6.14)), so $f_n^*(z)$ lies inside or very close to $B(0, 197/100)$. Hence $\mathrm{dist}\,(f_n^*(z), E_2 \cap B(0, 198/100)) \leq C_2\varepsilon 2^{-n} \leq 2^{-n-70}$ (by (6.13) and if ε is small enough; our bound of 10^{-25} should be more than enough). That is,

$$(6.15) \qquad x = f_n^*(z) \text{ satisfies (6.8).}$$

Then $\mathrm{dist}\,(f_{n+1}^*(z), E_2) = \mathrm{dist}\,(g_n^*(f_n^*(z)), E_2) \leq C_2\varepsilon 2^{-n-1}$ by (6.10). That is, (6.13) holds for $n+1$. In addition, $|f_{n+1}^*(z) - f_n^*(z)| = |g_n^*(x) - x| \leq \mathrm{dist}\,(x, E_2) + C_3\varepsilon 2^{-n} \leq (C_2 + C_3)\varepsilon 2^{-n}$ by (6.12) and (6.13). Thus (6.13) and (6.14) hold for every n.

By (6.14), $\{f_n^*\}$ converges uniformly on Γ to some limit f^*, and

$$(6.16) \qquad ||f^* - f_n^*||_\infty \leq 2(C_2 + C_3)\varepsilon 2^{-n}.$$

Next we examine the Lipschitz properties of $\Gamma_n = f_n^*(\Gamma)$.

Lemma 6.1 *The restriction of f_n^* to Γ is of class C^2, with a derivative that does not vanish. For each $i \in I_2(n)$, $\Gamma_n \cap 5B_i$ is contained in a $C_4\varepsilon$-Lipschitz graph $G_{n,i}$ over L_i, that meets $B(x_i, C_5\varepsilon 2^{-n})$.*

Proof. Let us prove all this by induction. The case when $n = 0$ is clear, because $F_0^* = id$, Γ is a straight line, and its closeness to all L_i, $i \in I_2(0)$, is an easy consequence of (4.35) and (6.4). So let us assume that we know the lemma for Γ_n, and prove it for $\Gamma_{n+1} = g_n^*(\Gamma_n)$.

We start with the Lipschitz description. Let $i \in I_2(n+1)$ be given. Recall that $x_i \in E_2' = E_2 \cap B(0, 198/100)$, by (5.1). Then (5.7) ensures that $x_i \in \overline{B}_j$ for some $j \in I_2(n)$. Here the situation is a little simpler because $I_3(n) = \emptyset$, but the argument will be very similar when $I_3(n) \neq \emptyset$. By the induction assumption, there is a $C_4\varepsilon$-Lipschitz graph $G_{n,j}$ over L_j that contains $\Gamma_n \cap 5B_j$. Notice that $5B_i \subset 4B_j$, so $g_n^*(\Gamma_n \setminus 5B_j)$ does not meet $5B_i$, by (6.14). Thus we may restrict our attention to points coming from $\Gamma_n \cap 5B_j \subset G_{n,j} \cap 5B_j$.

Let $z \in G_{n,j} \cap 5B_j$ be given, and denote by τ a unit tangent vector to $G_{n,j}$ at z. Observe that

$$(6.17) \qquad \mathrm{dist}\,(z, E_2) \leq C'\varepsilon 2^{-n},$$

where C' depends on C_4 and C_5. Indeed $\mathrm{dist}\,(z, L_j) \leq C_5\varepsilon 2^{-n} + 11C_4\varepsilon 2^{-n-40}$, because $G_{n,j}$ is a $C_4\varepsilon$-Lipschitz graph that goes through $B(x_j, C_5\varepsilon 2^{-n})$, and $x_j \in L_j$ by definition of Y_j and L_j. Now (6.17) follows from (6.4).

Let $k \in I(n)$ be such that $\theta_k(z) \neq 0$. We know that $k \in I_2(n)$, by (6.17) and (5.21). More precisely, we should make sure that $C'\varepsilon$ in (6.17) is less than 2^{-70}. We claim that this is the case, otherwise we could have chosen a smaller constant than 10^{-25} as an upper bound for ε. Then by (6.6) and (6.7)

$$|D\psi_k^* - D\psi_j^*| \leq C\varepsilon, \quad \text{and} \quad |\psi_k^* - \psi_j^*| \leq C\varepsilon 2^{-n} \text{ in } 5B_j. \tag{6.18}$$

Since $Dg_n^*(z) = \sum_{k\in I(n)} \theta_k(z) D\psi_k^*(z) + \sum_{k\in I(n)} D\theta_k(z)\psi_k^*(z)$ by (6.3),

$$|Dg_n^*(z) - D\psi_j^*| \leq \sum_{k\in I(n)} \theta_k(z)|D\psi_k^* - D\psi_j^*| + \sum_{k\in I(n)} |D\theta_k(x)||\psi_k^*(z) - \psi_j^*(z)| \leq C_6\varepsilon \tag{6.19}$$

because $\sum_k \theta_k = 1$ and $\sum_k D\theta_k = 0$, and by (5.13) and (6.18). Here C_6 is a geometric constant, and in particular does not depend on our future choice of C_5.

Since $G_{n,j}$ is a $C_4\varepsilon$-Lipschitz graph over L_j and ψ_j^* is the orthogonal projection onto L_j, $|\tau - D\psi_j^*(\tau)| \leq C_4\varepsilon$, hence

$$|Dg_n^*(z)(\tau) - \tau| \leq |Dg_n^*(z)(\tau) - D\psi_j^*(\tau)| + C_4\varepsilon \leq (C_4 + C_6)\varepsilon \tag{6.20}$$

by (6.19). In particular, $|Dg_n^*(z)(\tau)| \geq 1 - (C_4 + C_6)\varepsilon \geq 99/100$.

Set $v = Dg_n^*(z)(\tau)$; since $v \neq 0$, it is a tangent vector to $g_n^*(G_{n,j})$ locally . Recall that τ is the unit tangent vector to $G_{n,j}$ at x.

We want to show that it makes a small angle with L_j. Call $\pi_j^\perp$ the projection in the direction orthogonal to L_j; then $\pi_j^\perp(v) = \pi_j^\perp(Dg_n^*(z)(\tau)) = \pi_j^\perp(Dg_n^*(z)(\tau) - D\psi_j^*(\tau))$, hence $|\pi_j^\perp(v)| \leq C_6\varepsilon$ by (6.19). Since $|v| \geq 99/100$, v makes a small angle with L_j. Also,

$$D_{x_j,10r_j}(L_i, L_j) \leq C\varepsilon, \tag{6.21}$$

by the proof of (6.5). [The small difference is that now $j \in I_2(n)$ and $i \in I_2(n+1)$, so B_j is twice as big as B_i ; otherwise, we still have that $x_i \in B_i \cap \overline{B}_j$, which is more than enough for the proof to go through.]

Thus L_i makes a small angle with L_j and v makes a small angle with L_i. We deduce from this (and the fact that g_n^* is smooth on $G_{n,j} \cap 5B_j$, with a non-vanishing derivative) that

$$g_n^*(G_{n,j} \cap 5B_j) \text{ is (contained in) a } C_7\varepsilon\text{-Lipschitz graph } G_{n+1,i} \text{ over } L_i\,, \tag{6.22}$$

with C_7 depending only on C_6 and other geometric constants. Recall that we already checked that every point of $\Gamma_{n+1} \cap 5B_i$ lies in $g_n^*(G_{n,j} \cap 5B_j)$, so we just need to take C_4 larger than C_7 to get the part of the induction that says that $\Gamma_{n+1} \cap 5B_i$ is contained in a $C_4\varepsilon$-Lipschitz graph over L_i.

Next we check that $G_{n+1,i}$ meets $B(x_i, C_5\varepsilon 2^{-n-1})$. By the induction hypothesis, we can find $z \in G_{n,j} \cap B(x_j, C_5\varepsilon 2^{-n})$. Set $z_1 = g_n^*(z)$; thus $z_1 \in g_n^*(G_{n,j} \cap 5B_j) \subset G_{n+1,i}$, by (6.22). Call $z' = \psi_j^*(z)$ the orthogonal projection of z onto L_j. By (6.18), $|\psi_k^*(z) - z'| \leq C\varepsilon 2^{-n}$ when $\theta_k(z) \neq 0$, and hence $|z_1 - z'| = |g_n^*(z) - z'| \leq C\varepsilon 2^{-n}$, with a constant C that does not depend on C_5. This proves that $z_1 \in B(x_j, (C_5 + C)\varepsilon 2^{-n}) \subset B_j$ (because $z' \in B(x_j, C_5\varepsilon 2^{-n})$, just like z as $x_j \in L_j$), and that $\operatorname{dist}(z_1, L_j) \leq C\varepsilon 2^{-n}$. By (6.21), $\operatorname{dist}(z_1, L_i) \leq C\varepsilon 2^{-n}$. By definition of a graph, there is a point $z_2 \in G_{n+1,i}$, with $\psi_j^*(z_2) = x_i$. Then $\operatorname{dist}(z_2, L_i) \leq C\varepsilon 2^{-n}$ (because $\operatorname{dist}(z_1, L_i) \leq C\varepsilon 2^{-n}$ and the slope between z_1 and z_2 is small). Then $z_2 \in B(x_i, C\varepsilon 2^{-n})$. The constant C that we get does not depend on C_5, so we can choose $C_5 \geq C$, and $G_{n+1,i}$ meets $B(x_i, C_5\varepsilon 2^{-n-1})$ as needed.

Finally we need to check that the derivative of the restriction of f_{n+1}^* to Γ does not vanish. Let $t \in \Gamma$ be given, and set $z = f_n^*(t)$. By (6.14), $|f_n^*(t) - t| \leq 2(C_2 + C_3)\varepsilon$, so $f_n^*(t)$ lies well inside $B(0, 198/100)$ (because $z \in \overline{B}(0, 197/100)$ by (6.1). By (5.7), $z \in \overline{B}_j$ for some $j \in I_2(n)$. We can use the graph $G_{n,j}$ provided by our induction assumption. Call τ_0 a unit tangent vector to Γ and set $\tau = Df_n^*(t)(\tau_0)$. By induction assumption, $\tau \neq 0$. Then τ is a tangent vector to $G_{n,j}$. We can follow the argument between (6.17) and (6.20), and we get that $Dg_n^*(f_n^*(t))(\tau) \neq 0$. Thus, $Df_{n+1}^*(t)(\tau_0) \neq 0$, as needed. This completes our proof of Lemma 6.1. ∎

Next we claim that if $n \geq 0$, $i \in I_2(n)$, and $x_i \in B(0, 196/100)$,

$$\Gamma_n \cap 5B_i = G_{n,i} \cap 5B_i, \tag{6.23}$$

where $G_{n,i}$ is still as in Lemma 6.1.

First recall from Lemma 6.1 that Γ_n is a C^1 curve, with just two extremities. If none of the two extremities lies in $5B_i$, then only two things can happen. Either $\Gamma_n \cap 5B_i$ is empty, or else Γ_n enters $5B_i$ at some point. Since $\Gamma_n \cap 5B_i \subset G_{n,i}$ and the derivative of f_n^* on Γ does not vanish, Γ_n has to follow $G_{n,i}$ without turning back, until it eventually leaves $5B_i$. Then it runs along $G_{n,i} \cap 5B_i$ the whole way through, and (6.23) holds.

It will be easier to check by induction that (6.23) holds for $i \in I_2(n)$ such that $\operatorname{dist}(x_i, \mathbb{R}^3 \setminus B(0, 197/100)) \geq \sum_{m=0}^{n} 2^{-m-40}$. The two extremities of Γ_n lie within $2(C_2 + C_3)\varepsilon$ of $\partial B(0, 197/100)$, by (6.1) and (6.16), so they lie out of $5B_i$ and we just need to exclude the case when $\Gamma_n \cap 5B_i$ is empty.

When $n = 0$, $\Gamma_n = \Gamma$ meets B_i. Indeed $x_i \in E_2 \cap B(0, 197/100)$, hence Lemma 4.6 says that $Z(x_i, 10^{-2})$ is of type 2 or 3, with a spine that passes at distance at most $17\varepsilon 10^{-2}$ from x_i; this forces L to pass through $\frac{1}{2}B_i$, because otherwise $Z(0,2)$ coincides with a plane in $\frac{5}{12}B_i$ (by Lemma 3.4), and then $D_{x_i,10^{-3}}(Z(x_i, 10^{-2}), Z(0,2)) \geq 1/5$ (by Lemma 3.1); this is impossible because $Z(x_i, 10^{-2})$ and $Z(0,2)$ are both so close to E near $B(x_i, 10^{-3})$ (by (1.1)). So L meets $\frac{1}{2}B_i$ and Γ meets B_i and (6.23) holds for $n = 0$.

Finally let us assume our claim for some $n \geq 0$ and prove it for $n+1$. Let $i \in I_2(n+1)$ be as above. By (5.1), $x_i \in E_2' = E_2 \cap B(0, 198/100)$; by (5.7), $x_i \in \overline{B}_j$ for some $j \in I_2(n)$. Thus $|x_j - x_i| \leq 2^{-n-40}$ (because $x_i \in \overline{B}_j$), hence

$$\operatorname{dist}(x_j, \mathbb{R}^3 \setminus B(0, 197/100)) \geq \operatorname{dist}(x_i, \mathbb{R}^3 \setminus B(0, 197/100)) - 2^{-n-40} > \sum_{m=0}^{n} 2^{-m-40}. \tag{6.24}$$

By induction assumption, (6.23) holds for n and j, and in particular we can find $\xi \in \Gamma_n \cap B(x_j, C_5\varepsilon 2^{-n})$ (see Lemma 6.1). Now $g_n^*(\xi) \in \Gamma_{n+1} \cap 5B_i$, by (6.14). Thus $\Gamma_{n+1} \cap 5B_i \neq \emptyset$ and (6.23) holds for B_i, as needed. This completes our proof of (6.23) by induction.

Next we want to show that f^* is biHölder. Let us check that

$$(1 - C\varepsilon)\,\mathrm{dist}\,(y,z) \leq \mathrm{dist}\,(g_n^*(y), g_n^*(z)) \leq (1 + C\varepsilon)\,\mathrm{dist}\,(y,z) \tag{6.25}$$

when y, $z \in \Gamma_n$ are such that $|y - z| \leq 2^{-n-40}$. By (6.13), $\mathrm{dist}\,(y, E_2) \leq C_2\varepsilon 2^{-n}$; more trivially, y lies in $B(0, 197/100)$ or very close to it. Then (5.7) says that y lies within $C\varepsilon 2^{-n}$ of B_i for some $i \in I_2(n)$. Let $G_{n,i}$ be the Lipschitz graph of Lemma 6.1; since both y and z lie on $G_{n,i} \cap 3B_i$, we can replace $\mathrm{dist}\,(y,z)$ with the length of the arc of Γ_n between these points, and make a relative error less than $C\varepsilon$. We can proceed similarly with $\mathrm{dist}\,(g_n^*(y), g_n^*(z))$ and the arc-length on $g_n^*(G_{n,j} \cap 5B_j)$ (for the appropriate j), which is also a $C\varepsilon$-Lipschitz graph by (6.22). Finally, the ratio between the arc-lengths can be computed in terms of the derivative $|Dg_n^*(z)(\tau(z))|$ on Γ_n. Thus using (6.20), we get (6.25).

Our biHölder estimate will follow from (6.25) by a rather mechanical argument, which will be repeated later in this paper. Let y, $z \in \Gamma$ be given, and set $y_n = f_n^*(y)$ and $z_n = f_n^*(z)$. Assume that $|z - y| < 1$; we shall take care of the other case later. As long as $|y_n - z_n| \leq 2^{-n-40}$ we can apply (6.25) and get that

$$(1 - C\varepsilon)^{n+1}|y - z| \leq |y_{n+1} - z_{n+1}| \leq (1 + C\varepsilon)^{n+1}|y - z|. \tag{6.26}$$

How long can this last? If (6.26) holds for $n - 1$ and $|y_n - z_n| \leq 2^{-n-40}$, we have that

$$(1 - C\varepsilon)^n|y - z| \leq |y_n - z_n| \leq 2^{-n-40} \tag{6.27}$$

hence $n\log_2(1 - C\varepsilon) + \log_2(|y - z|) \leq -n - 40 \leq -n$, or equivalently

$$\log_2(\frac{1}{|y - z|}) \geq n + n\log_2(1 - C\varepsilon) \geq n(1 - C\varepsilon). \tag{6.28}$$

This cannot happen for n large, thus there is a smallest n_0 such that $|y_{n_0} - z_{n_0}| > 2^{-n_0-40}$, and we even know that (6.28) holds for $n_0 - 1$, so

$$n_0 \leq 1 + (1 - C\varepsilon)^{-1}\log_2(\frac{1}{|y - z|}) \leq 1 + (1 + C'\varepsilon)\log_2(\frac{1}{|y - z|}). \tag{6.29}$$

Now we can use (6.14) to say that

$$\big||y_n - z_n| - |y_{n_0} - z_{n_0}|\big| \leq C\varepsilon 2^{-n_0} \tag{6.30}$$

for $n > n_0$, which leads to

$$\big||f^*(y) - f^*(z)| - |y_{n_0} - z_{n_0}|\big| \leq C\varepsilon 2^{-n_0} \tag{6.31}$$

and then, since $|y_{n_0} - z_{n_0}| > 2^{-n_0-40}$, to

$$(1 - C\varepsilon)|y_{n_0} - z_{n_0}| \leq |f^*(y) - f^*(z)| \leq (1 + C\varepsilon)|y_{n_0} - z_{n_0}| \tag{6.32}$$

and

$$(6.33) \qquad (1-C\varepsilon)^{n_0+1}|y-z| \leq |f^*(y)-f^*(z)| \leq (1+C\varepsilon)^{n_0+1}|y-z|,$$

by (6.26). That is, $\left|\log\frac{|f^*(y)-f^*(z)|}{|y-z|}\right| \leq C\varepsilon(n_0+1) \leq C\varepsilon\left[2+(1+C'\varepsilon)\log_2(\frac{1}{|y-z|})\right] \leq C''\varepsilon\big[(1+\log_2(\frac{1}{|y-z|})\big]$, by (6.29). Then we take exponentials and obtain that

$$(6.34) \qquad (1-C\varepsilon)\,|y-z|^{1+C\varepsilon} \leq |f^*(y)-f^*(z)| \leq (1+C\varepsilon)\,|y-z|^{1-C\varepsilon}.$$

For $y,z\in\Gamma$ with $1<|z-y|<4$ (6.14) or (6.16) imply

$$(6.35) \qquad (1-C\varepsilon)\,|y-z| \leq |f^*(y)-f^*(z)| \leq (1+C\varepsilon)\,|y-z|,$$

which yields (6.34). Thus f^* is bi-Hölder on Γ, with exponents that are as close to 1 as we want.

Observe that it is clear from (6.13) that $f^*(\Gamma)\subset E_2$. Also,

$$(6.36) \qquad f^*(\Gamma) \text{ contains } E_2\cap B(0,196/100).$$

Indeed let $z\in E_2\cap B(0,196/100)$; for every large n, z lies in $\overline{B}_i$ for some $i\in I_2(n)$, $x_i\in B(0,196/100)$ too, and (6.23) says that B_i meets Γ_n (recall from Lemma 6.1 that $G_{n,i}$ meets $B(x_i,C\varepsilon 2^{-n})$). Thus $\operatorname{dist}(z,\Gamma_n)\leq 2^{-n-20}$, and $z\in f^*(\Gamma)$ by compactness.

7 A parameterization of E when $E_3=\emptyset$

We still assume that (4.35) holds, so that in particular $E_3\cap B(0,199/100)=\emptyset$ by (4.37), and we want to define the restriction to a big part of $Z(0,2)$ of the parameterization f. As before, we shall define a mapping everywhere, but we shall only care about $f(z)$ when z lies in the set

$$(7.1) \qquad Y = Z(0,2)\cap B(0,195/100).$$

We still want to get f as the limit of mappings f_n, where $f_0(x)=x$, and

$$(7.2) \qquad f_{n+1} = g_n\circ f_n \text{ for } n\geq 0, \text{ with } g_n(x) = \sum_{i\in I(n)}\theta_i(x)\psi_i(x)$$

and where the ψ_i, $i\in I(n)$, are suitable deformations that will be defined soon. We can again take $\psi_i(x)=x$ for $i\in I_0(n)$ (but $f_n(Y)$ will avoid the support of ψ_i anyway), but this time we need to do something when $i\in I_1(n)$. Let us first define ψ_i in this case.

Recall that when $i\in I_1(n)$, B_i is a ball of radius $r_i=2^{-n-60}$ centered at some $x_i\in E_1'\subset E\cap B(0,197/100)$; see near (5.3). Also recall from (5.15) and (5.17) that

$$(7.3) \qquad \operatorname{dist}\Big(100B_i, \frac{7}{4}B_{i_0}\cup\bigcup_{j\in I_2}\frac{3}{2}B_j\Big) \geq 2^{-n-50},$$

where we only mention i_0 here to convince the reader that things will also work when (4.36) holds. Then (5.7) says that

$$\mathrm{dist}\,(100B_i, E_2) \geq 2^{-n-50}, \tag{7.4}$$

where we also use the fact that $x_i \in B(0, 197/100)$. Let us check that

$$Z(x_i, 100r_i) \text{ coincides with a plane } P_i \text{ in } B(x_i, 60r_i). \tag{7.5}$$

If $Z = Z(x_i, 100r_i)$ is of type 3 and its center lies in $B(x_i, 98r_i)$, we can choose $\xi \in E$ close to the center, apply Lemma 4.3 to $B(\xi, r_i)$, and we get that $B(\xi, r_i)$ meets E_3. This is impossible, by (7.3) (or (7.4) and the fact that points of E_3 are always limits of points of E_2).

Otherwise, and if (7.5) fails, Lemma 3.4 says that the spine of Z meets $\overline{B}(x_i, 80r_i)$. Pick $\xi \in E \cap B(x_i, 81r_i)$, close to the spine of Z. Then Z coincides with a set of type 2 in $B(\xi, 2r_i)$, we can apply Lemma 4.5 to $B(\xi, r_i)$, and we get that $B(\xi, r_i)$ meets E_2. This is impossible too, by (7.4). So (7.5) holds. For the generalization in Theorem 2.2, we would just need to replace 100 with a larger constant and make 2^{-20} and ε somewhat smaller.

By definition, $Z(x_i, 100r_i)$ passes through x_i. We take

$$\psi_i = \pi_i, \text{ the orthogonal projection onto } P_i, \text{ when } i \in I_1. \tag{7.6}$$

Now we need to define ψ_i when $i \in I_2(n)$. There is a constraint, because since we want to have $f = f^*$ on Γ, we shall demand that

$$\psi_i(x) = \psi_i^*(x) \text{ for } x \in \Gamma_n \cap \mathrm{support}(\theta_i). \tag{7.7}$$

This will suffice, because if we have (7.7) for all $i \in I_2(n)$, then $g_n = g_n^*$ on Γ_n, since Γ_n does not meet the support of the θ_i, $i \in I_1(n) \cup I_0(n)$, by (5.21) and (6.13).

We want to use a standard map associated to a propeller in the plane. Let $\mathrm{Prop} \subset \mathbb{R}^2$ denote the same standard propeller centered at the origin as in Definition 2.2. Call D_j, $j = 1, 2, 3$ its three branches. Choose a Lipschitz map h such that $h(x)$ is the orthogonal projection of x onto (the line through) D_j when $\mathrm{dist}\,(x, D_j) \leq |x|/10$, say. We shall see that the values of h in the other regions do not really matter, because images of points of Y will not fall there. To simplify the notation later, choose h smooth away from the origin, such that $h(\lambda x) = \lambda h(x)$ for $\lambda > 0$, and also invariant under symmetries with respect to the D_j and rotations of 120°. It is slightly unpleasant that h is not C^2 at the origin, but the restrictions to the three sectors where $\mathrm{dist}\,(x, D_j) \leq |x|/10$ are (they are even affine).

So far h was defined on $\mathbb{R}^2$; we extend it to a map defined on $\mathbb{R}^3$ by setting $h(x, t) = (h(x), t)$, with the obvious notation. This is the map associated to the set $Y_0 = \mathrm{Prop} \times \mathbb{R}$ of Definition 2.2. If Z is any set of type 2, we define a mapping h_Z by $h_Z = \tau \circ h \circ \tau^{-1}$, where τ is an isometry of $\mathbb{R}^3$ that sends Y_0 to Z.

Return to $i \in I_2(n)$. Let Y_i be the set of type 2 that was introduced in (6.4) and was already used to define ψ_i^*. Recall from Lemma 6.1 that $\Gamma_n \cap 5B_i$ is contained in a $C_4\varepsilon$-Lipschitz graph $G_{n,i}$ over L_i, the spine of Y_i. To illustrate the construction, choose coordinates of $\mathbb{R}^3$ so that x_i is the origin and L_i is the first axis. Parameterize $G_{n,i}$ near B_i by $x \to (x, \zeta(x))$. Let us record for future use that

$$|\zeta(0)| \leq C\varepsilon 2^{-n} \text{ and } |\zeta'(x)| \leq C_4\varepsilon, \tag{7.8}$$

by Lemma 6.1. We already decided that $\psi_i^*(x, \zeta(x)) = (x, 0)$ on Γ_n, and we want to extend this. So we set

$$\eta_i(x, y) = (x, y - \zeta(x)) \text{ for } x \in \mathbb{R} \text{ and } y \in \mathbb{R}^2, \tag{7.9}$$

and then

$$\psi_i(z) = h_i(\eta_i(z)) \text{ for } z \in \mathbb{R}^3, \tag{7.10}$$

where we set $h_i = h_{Y_i}$. Notice that (7.7) holds because in $3B_i$, η_i maps points of Γ_n to L_i just by the same formula as the orthogonal projection ψ_i^*, and then h_i leaves them alone.

This completes our definition of the ψ_i, g_n, and f_n. Now we need to do the same sort of estimates as in the last section. Hopefully, the similarity will help reduce the amount of work, but we may need to be a little more careful with the analogue of Lemma 6.1.

Our set $Y = Z(0,2) \cap B(0, 195/100)$ is composed of three faces. Let F be one of these faces, and set $F_n = f_n(F)$. From time to time, we need to recall which initial face things are coming from, and in this case we add an index l, $1 \leq l \leq 3$, as an exponent. Thus there are three faces F^l, and for each n three sets F_n^l.

First let us check that points do not move too much. Observe that for g_n and in (7.2)

$$|g_n(z) - z| \leq 2^{-n-37} \text{ for } z \in \mathbb{R}^3, \tag{7.11}$$

just because $|\psi_i(x) - x| \leq 6r_i \leq 2^{-n-37}$ in $3B_i$ (the support of θ_i), and then $g_n(z)$ is a convex combination of the $\psi_i(x)$. This will be useful to make sure that our local descriptions of the $F_n \cap 5B_i$ are not upset by points coming from outer space.

Now we want to prove, at the same time and by induction on n, the following collection of estimates. Set

$$\rho_n = \frac{195}{100} - \sum_{k=0}^{n} 2^{-k-30}. \tag{7.12}$$

First we want to prove that

$$\operatorname{dist}(z, E) \leq C_1\varepsilon 2^{-n} \text{ for } z \in F_n \cap B(0, \rho_n). \tag{7.13}$$

Next consider $i \in I_1(n)$ such that $x_i \in B(0, \rho_n)$. We will show that two of the sets $F_n^l \cap 5B_i$, $1 \leq l \leq 3$, are empty, and for the third one there is a $C_2\varepsilon$-Lipschitz graph $A_{n,i}$ over P_i such that $A_{n,i}$ meets $B(x_i, C_3\varepsilon 2^{-n})$ and

$$F_n^l \cap 5B_i = A_{n,i} \cap 5B_i. \tag{7.14}$$

Similarly, if $i \in I_2(n)$, $x_i \in B(0,\rho_n)$, and $1 \leq l \leq 3$, we can find a face V_i^l of Y_i, a (closed) Lipschitz domain S_i^l in the plane P_i^l that contains V_i^l, with

$$D_{x_i,10r_i}(V_i^l, S_i^l) \leq C_5\varepsilon \tag{7.15}$$

(we leave out the name C_4, to avoid confusion with Lemma 6.1), and a $C_6\varepsilon$-Lipschitz graph $T_{n,i}^l$ over S_i^l, such that $T_{n,i}^l$ meets $B(x_i, C_7\varepsilon 2^{-n})$ and

$$F_n^l \cap 5B_i = T_{n,i}^l \cap 5B_i. \tag{7.16}$$

Moreover, the three faces V_i^l, $1 \leq l \leq 3$, are different, and the three $T_{n,i}^l$ are bounded by $\Gamma_n \cap 5B_i$ (which is a Lipschitz graph, by Lemma 6.1 and (6.23)).

Let us first check that all the properties above hold for $n = 0$. Here the F_n^l are the three faces of $Y = Z(0,2) \cap B(0, 195/100)$, so (7.13) follows directly from the fact that $D_{0,2}(Z(0,2), E) \leq \varepsilon$. The Lipschitz description in (7.14) for $i \in I_1(0)$ follows from (7.5), Lemma 3.1, and the same fact. Similarly, for $i \in I_2(0)$, the same fact and Lemma 3.1 force $Z(0,2)$ to coincide with a set of type 2 in $5B_i$, with a spine that gets very close to x_i (see Lemma 4.6), and the description in (7.14)-(7.15) follows. So now we assume that $n \geq 0$, and that the properties hold for $m \leq n$.

Let $z \in F_n^l \cap B(0,\rho_n)$ be given. By (7.13) and (5.8), z lies $C_1\varepsilon 2^{-n}$ close to some B_i, $i \in I_1(n)$ or to some $\frac{7}{4}B_i$, $i \in I_2(n)$. Choose such an index i, and call it $j(z)$. Also call I_z the set of indices $i \in I(n)$ such that $z \in 3B_i$. We already know from (7.13) and (5.24) that $I_z \subset I_2(n) \cup I_1(n)$.

Let $i \in I_z$ be given, and let us assume that $x_i \in B(0,\rho_n)$, so that we can use the descriptions above. Note that in this case $F_n^l \cap 5B_i$ is not empty (because it contains z), so we have (7.14) or (7.16). Since the plane P_i (when (7.14) holds), or $P_i = P_i^l$ (when (7.16) holds) is reasonably well determined by $A_{n,i} \cap 5B_i \cap B(z, 2^{-n-80})$ or $T_{n,i} \cap 5B_i \cap B(z, 2^{-n-80})$, we have that

$$D_{z,2^{-n}}(P_i, P_j) \leq C\varepsilon \text{ for } i,j \in I_z. \tag{7.17}$$

With this proof, C depends on C_3 and C_6, but since we also know that P_i comes from $Z(x_i, 100r_i)$ as in (7.5) or Y_i as in (6.4), and similarly for P_j, we can easily improve this first estimate and get (7.17) with a geometric constant (i.e., that does not depend on our future choices of constants C_k).

Then let us look more carefully at the ψ_j and check that

$$|\psi_i(z) - \psi_j(z)| \leq C\varepsilon 2^{-n} \text{ for } i,j \in I_z. \tag{7.18}$$

The main point will be that for $i \in I_z$,

$$|\psi_i(z) - \pi_i(z)| \leq C\varepsilon 2^{-n}, \tag{7.19}$$

where π_i denotes the orthogonal projection onto P_i. When $i \in I_1(n)$, we even have that $\psi_i(z) = \pi_i(z)$, by (7.6). When $i \in I_2(n)$, and since $z \in 3B_i$, (7.16) says that $z \in T_{n,i}^l \cap 5B_i$.

Also, $T^l_{n,i} \cap 5B_i$ is a piece of Lipschitz graph over P_i, bounded by $\Gamma_n \cap 5B_i$; its image under η_i is still a Lipschitz graph, but it is now bounded by an arc of $\eta_i(\Gamma_n)$, which is an arc over L_i. See (7.9) and the few lines above it. Since the Lipschitz constant for $\eta_i(T^l_{n,i} \cap 5B_i)$ is very small (by (7.8) and (7.9) in particular), and this set is bounded by an arc of L, it is contained in the small sector around V^l_i (the face that is contained in P^l_i), where $h_i = h_{Y_i}$ coincides with the projection π^l_i on P^l_i. Thus (7.10) says that

$$\psi_i(z) = \pi^l_i(\eta_i(z)) \text{ for } z \in T^l_{n,i} \cap 5B_i. \tag{7.20}$$

Now $|\eta_i(z) - z| \leq C\varepsilon 2^{-n}$ on $5B_i$ by the definition (7.9) and the fact that ζ is $C\varepsilon$-Lipschitz and nearly vanishes at the origin (see (7.8)). Then (7.19) follows from (7.20). Once we have (7.19), (7.18) follows from (7.17).

Next we claim that

$$\text{dist}\,(g_n(z), E) \leq C\varepsilon 2^{-n}, \tag{7.21}$$

again with a geometric constant C. When we can find $i \in I_z$ such that $i \in I_1(n)$, this comes directly from (7.18), (7.19), and (7.5) (which says that $\pi_i(z)$ lies very close to E). Otherwise, $i = j(z)$ (say) lies in $I_2(n)$, and (7.21) follows from (7.18), (7.20), and the fact that $\pi^l_i(\eta_i(z))$ lies on the face $V^l_i \subset Y_i$ (by our proof of (7.20)).

Proof of (7.13). We are ready to prove (7.13) for $n+1$, with a geometric constant C_1. Indeed, let $y \in F_{n+1} \cap B(0, \rho_{n+1})$ be given. By construction (i.e., because $F_{n+1} = f_{n+1}(F)$ and by (7.2)), $y = g_n(z)$ for some $z \in F_n$. Since $\rho_{n+1} = \rho_n - 2^{-n-32}$ and by (7.11), z lies in $B(0, \rho_n)$, and so do all the x_i such that $i \in I_z$. Then (7.21) holds, and $\text{dist}\,(y, E) = \text{dist}\,(g_n(z), E) \leq C\varepsilon 2^{-n-1}$, which proves (7.13).

Let us also check that

$$|g_n(z) - z| \leq C\varepsilon 2^{-n} \quad \text{when } z \in F_n \cap B(0, \rho_n) \text{ and } x_{j(z)} \in B(0, \rho_n). \tag{7.22}$$

The constant C depends on C_5 and C_6, but this will be all right. Set $i = j(z)$. First suppose that $i \in I_1(n)$. By (7.13), $\text{dist}\,(z, E) \leq C_1\varepsilon 2^{-n}$; then $\text{dist}\,(z, P_i) \leq C\varepsilon 2^{-n}$ too, by (7.5). [We could also have used the description (7.14) as in our second case.] If $i \in I_2(n)$, we use the induction hypothesis and find that $z \in T^l_{n,i} \cap 5B_i$, as in (7.16), then $\text{dist}\,(z, P^l_i) \leq C\varepsilon 2^{-n}$. In both cases, (7.19) says that $|\psi_i(z) - z| \leq C\varepsilon 2^{-n}$. By (7.18), $|\psi_j(z) - z| \leq C\varepsilon 2^{-n}$ for all other $j \in I_z$; (7.22) follows because $g_n(z)$ is an average of $\psi_j(z)$.

Proof of (7.14). Next we want to prove (7.14) for $n+1$, so let $i \in I_1(n+1)$ be given and assume that $x_i \in B(0, \rho_{n+1})$; we want to look at the sets $F^l_{n+1} \cap 5B_i$. Since $x_i \in E'_1 \subset E \cap B(0, 197/100)$, (5.8) says that x_i lies in $\frac{7}{4}B_j$ for some $j \in I_2(n)$ or in $\overline{B}_j$ for some $j \in I_1(n)$. So let us choose j, so that

$$j \in I_1(n) \text{ and } x_i \in \overline{B}_j \quad \text{or } j \in I_2(n) \text{ and } x_i \in \frac{7}{4}B_j. \tag{7.23}$$

In both cases, we shall apply the induction assumption to get a good description of $F_n \cap 5B_j$, and use it to get information on $F_{n+1} \cap 5B_i$. First notice that

$$|x_j| \leq |x_i| + 2r_j \leq \rho_{n+1} + 2^{-n-39} < \rho_n, \tag{7.24}$$

so we can use the induction assumption to describe $F_n \cap 5B_j$.

Next check that for $l = 1, 2, 3$,

$$F^l_{n+1} \cap 5B_i = g_n(F^l_n \cap 5B_j) \cap 5B_i. \tag{7.25}$$

Clearly $F^l_{n+1} \cap 5B_i$ contains $g_n(F^l_n \cap 5B_j) \cap 5B_i$. Conversely, if $y \in F^l_{n+1} \cap 5B_i$, then $y = g_n(z)$ for some $z \in F^l_n$; we just need to check that $z \in 5B_j$. Let us first check that the condition in (7.22) holds. By (7.11), $|z - y| \leq 2^{-n-37}$; also $|y| \leq |x_i| + 5r_i \leq \rho_{n+1} + 5r_i$ because $y \in 5B_i$, so $|z| \leq \rho_{n+1} + 2^{-n-37} + 5 \cdot 2^{-n-60} < \rho_n$. Since $|x_{j(z)} - z| \leq 3r_{j(z)} \leq 3 \cdot 2^{-n-40}$, we also get that $|x_{j(z)}| < \rho_n$, and (7.22) can be applied. So $|y - z| = |g_n(z) - z| \leq C\varepsilon 2^{-n}$, and z lies very close to $5B_i$. Now $6B_i \subset 5B_j$, either because $x_i \in \overline{B}_j$ and $r_j = 2r_i$ (when $\in I_1(n)$), or because $x_i \in \frac{7}{4}B_j$ and $r_j = 2^{21}r_i$ (when $j \in I_2(n)$). So $z \in 5B_j$, as needed for (7.25).

We continue with the proof of (7.14) for $i \in I_1(n+1)$, and now consider any point z such that

$$z \in F_n \cap 5B_j \text{ and } g_n(z) \in 10B_i. \tag{7.26}$$

In particular, we assume for the moment that such a point exists. Let $l \in \{1, 2, 3\}$ be such that $z \in F^l_n$. Observe that

$$|z| \leq |x_j| + 5r_j \leq |x_i| + 7r_j \leq \rho_{n+1} + 7 \cdot 2^{-n-40} \tag{7.27}$$

by (7.23) and because $x_i \in B(0, \rho_{n+1})$. Then $|x_k| \leq \rho_{n+1} + 10 \cdot 2^{-n-40} < \rho_n$ for every $k \in I_z$. Hence z and the x_k, $k \in I_z$, lie in $B(0, \rho_n)$ and we can use the description near (7.17)–(7.22).

Pick any $k \in I_z$, and let us start with the most delicate case a priori when

$$k \in I_2(n). \tag{7.28}$$

First case. We assume that we can find z satisfying (7.26) and $k \in I_z \cap I_2(n)$.

Recall that B_i is a ball of radius $r_i = 2^{-n-1-60}$ centered at some $x_i \in E'_1 \subset E \cap B(0, 197/100)$; see near (5.3). Also recall from (5.15) and (5.17) that

$$\operatorname{dist}\Big(100B_i, \frac{7}{4}B_{i_0} \cup \bigcup_{j \in I_2(n+1)} \frac{3}{2}B_j\Big) \geq 2^{-n-1-50} > 1000r_i, \tag{7.29}$$

Then (5.7) says that $\operatorname{dist}(z, E_2) \geq 1000r_i$, where we also use the fact that $x_i \in B(0, \rho_{n+1}) \subset B(0, 195/100)$.

By (7.28), $r_k = 2^{21} r_i$, so B_k is much larger than B_i. Also, $z \in 3B_k$ because $k \in I_z$, so $B(z, 1000r_i) \subset 4B_k$. Recall that $Y_k = Y(x_k, 10r_k)$ is a set of type 2 such that (6.4) holds. In particular, its spine L_k is such that $D_{x_k,10r_k}(E_2, L_k) \leq 600\varepsilon$, and hence

$$\text{dist}\,(z, L_k) \geq 400r_i\,. \tag{7.30}$$

We checked just above (7.28) that $|x_k| < \rho_n$, so we can use the description of the three $F_n^m \cap 5B_k = T_{n,k}^m \cap 5B_k$ near (7.16). Notice also that $\text{dist}\,(z, E) \leq C_1\varepsilon 2^{-n}$ by (7.13), applied to $z \in F_n \cap 5B_j$ (we checked in (7.24) that $x_j \in B(0, \rho_n)$). By (6.4), $\text{dist}\,(z, Y_k) \leq C\varepsilon 2^{-n}$. By (7.30) it lies far from L_k, so there is only one face of Y_k that gets close to z. Since $z \in F_n^l \cap 5B_k$ by definition of l above, z belongs to the corresponding $T_{n,k}^l$ alone, and the face of Y_k that is close to z is V_k^l. This also says that l above is unique, i.e., z only lies on one F_n^l.

Notice that $|z - x_i| \leq |z - g_n(z)| + |g_n(z) - x_i| \leq C\varepsilon 2^{-n} + 10r_i \leq 11r_i$ because $g_n(z) \in 10B_i$ and we can use (7.22). In $50B_i$ we know that E is very close to P_i, by (7.5), and at the same time to the face V_k^l of Y_k that gets close to z (by (6.4) and (7.30)). So

$$D_{x_i,40r_i}(P_i, V_k^l) \leq C\varepsilon, \tag{7.31}$$

and also

$$|D\pi_k^l - D\pi_i| \leq C\varepsilon, \tag{7.32}$$

where π_k^l is the orthogonal projection onto the plane P_k^l that contains V_k^l, and π_i is the orthogonal projection onto P_i. In addition, C in (7.31) and (7.32) is a geometric constant (i.e., does not depend on the C_i's of the induction hypothesis).

Recall from (7.16) that $F_n^l \cap 5B_k = T_{n,k}^l \cap 5B_k$, where $T_{n,k}^l$ is a $C_6\varepsilon$-Lipschitz graph over a set $S_k^l \subset P_k^l$ that is very close to V_k^l (see (7.15)). Call $A = A_k^l$ a $C_6\varepsilon$-Lipschitz function from the whole P_k^l to an orthogonal line, whose graph contains $T_{n,k}^l$. Since z is far from the spine L_k by (7.30) and $B(z, 25r_i) \subset 4B_k$ (because $k \in I_z \cap I_2(n)$, hence $z \in 3B_k$),

$$F_n^l \text{ and } T_{n,k}^l \text{ coincide with the graph of } A \text{ in } B(z, 20r_i). \tag{7.33}$$

Recall from (7.20) that $\psi_k(w) = \pi_k^l(\eta_k(w))$ for $w \in T_{n,k}^l \cap 5B_k$ (which includes all points of $F_n^l \cap B(z, 18r_i)$). The proof also says that $\psi_k(w) = \pi_k^l(\eta_k(w))$ near w if w does not lie on the boundary Γ_n. Then

$$|D\psi_k(w) - D\pi_k^l| \leq C\varepsilon, \tag{7.34}$$

where again C is a geometric constant, because (7.8) and (7.9) say that $|D\eta_k - I| \leq \varepsilon$.

We shall also need information on the indices $m \in I(n)$ such that $m \in I_y$ for some $y \in F_n^l \cap B(z, 20r_i)$. When $m \in I_2(n)$, we can repeat the discussion above, replace the information that $z \in 3B_k$ with the slightly weaker fact that $y \in 3B_m$, which still implies that $B(z, 20r_i) \subset 4B_m$, for instance, and get that

$$|D\pi_m^l - D\pi_i| \leq C\varepsilon \tag{7.35}$$

(as in (7.32)), then $\psi_m(w) = \pi^l_m(\eta_m(w))$ for $w \in F^l_n \cap 5B_m$, and

$$|D\psi_m(w) - D\pi^l_m| \leq C\varepsilon. \tag{7.36}$$

The index l stays the same, because it is determined by the fact that $z \in F^l_n$ and y lies close to z, so $y \in F^l_n$. Also, C is a geometric constant.

When $m \in I_1(n)$, we can use the fact that by (7.5), E is $100r_m$-close to P_m in $50B_m$. It is also $10\varepsilon r_k$-close to V^l_k and P^l_k in $B(y, 50r_i)$, because $z \in 3B_k$, $y \in B(z, 20r_i)$, and by (7.30). Now $50B_m$ contains $B(y, 45r_i)$ because $m \in I_y$ and $r_m = r_i$, so we get that $D_{y,40r_i}(P_m, V^l_k) \leq C\varepsilon$. We also get that $|D\pi_m - D\pi^l_k| \leq C\varepsilon$ as in (7.32), hence $|D\pi_m - D\pi_i| \leq C\varepsilon$ by (7.32), as in (7.35). In this case, we have the simpler formula $\psi_m = \pi_m$, and (7.36) holds as well.

We may now collect the estimates (7.35), (7.36), and their analogues for $m \in I_1(n)$, to get that for every $y \in F^l_n \cap B(z, 20r_i)$,

$$\begin{aligned} |Dg_n(y) - D\pi_i| &\leq \sum_{m\in I(n)} \theta_m(y)|D\psi_m(y) - D\pi_i| + \Big|\sum_{m\in I(n)} D\theta_m(y)\psi_m(y)\Big| \\ &\leq C\varepsilon + \sum_{m\in I(n)} |D\theta_m(y)||\psi_m(y) - g_n(y)| \leq C\varepsilon, \end{aligned} \tag{7.37}$$

by (7.2), and where the last inequality comes from (5.13), (7.18) and the fact that $g_n(y)$ is an average of the $\psi_m(y)$, $m \in I_y$.

We may now return to the description of $F^l_n \cap B(z, 20r_i)$ given by (7.33), and see what happens when we apply g_n. If w_1, $w_2 \in F^l_n \cap B(z, 19r_i)$, there is a curve in $F^l_n \cap B(z, 20r_i)$, with length at most $(1 + CC_6\varepsilon)|w_2 - w_1|$, which goes from w_1 to w_2 (just project on P^l_k and join by a line segment). We can integrate Dg_n on this curve to estimate $g_n(w_2) - g_n(w_1)$; this yields

$$\begin{aligned} |\pi_i^\perp(g_n(w_2)) - \pi_i^\perp(g_n(w_1))| &\leq C\varepsilon|w_2 - w_1| \\ \text{and } |\pi_i(g_n(w_2)) - \pi_i(g_n(w_1))| &\geq \frac{99}{100}|w_2 - w_1|, \end{aligned} \tag{7.38}$$

where $\pi_i^\perp = I - \pi_i$, and by (7.37). Again C is a geometric constant (we seem to use C_6 in the definition of the curve, but the main point is that $CC_6 \leq 10^{-3}$, say).

Already (7.38) forces $g_n(F^l_n \cap B(z, 19r_i))$ to be contained in a Lipschitz graph Λ with constant $C\varepsilon$ over π_i. Let us check that there is no hole. For $w \in F^l_n \cap B(z, 19r_i)$, we can still apply (7.22), because $|w| \leq |z| + 19r_i < \rho_{n+1} + 7 \cdot 2^{-n-40} + 19r_i < \rho_n$ by (7.27), and even $|x_{j(w)}| \leq |w| + 3 \cdot 2^{-n-40} < \rho_n$. Then

$$|g_n(w) - w| \leq C\varepsilon 2^{-n} \text{ for } w \in F^l_n \cap B(z, 19r_i). \tag{7.39}$$

We do not care that C here is not a geometric constant; the main point is that $C\varepsilon 2^{-n}$ is much smaller than r_i, so we can use a a little bit of degree theory (or monodromy, or an inversion theorem) to deduce from the description of (7.33) and (7.39) that $\pi_i(g_n(F_n \cap B(z, 19r_i)))$

actually contains $P_i \cap B(\pi_i(g_n(z)), 18r_i)$. So we found a $C\varepsilon$-Lipschitz graph Λ over π_i, such that, if we set $D = B(g_n(z), 18r_i)$,

$$g_n(F_n^l \cap B(z, 19r_i)) \cap D = \Lambda \cap D. \tag{7.40}$$

Recall from (7.26) that $g_n(z) \in 10B_i$; thus D contains $5B_i$ and (7.40) implies that

$$g_n(F_n^l \cap B(z, 19r_i)) \cap 5B_i = \Lambda \cap 5B_i. \tag{7.41}$$

Let us check that

$$g_n(F_n^l \cap 5B_j) \cap 5B_i \subset g_n(F_n^l \cap B(z, 19r_i)). \tag{7.42}$$

Observe that $|g_n(w) - w| \leq C\varepsilon 2^{-n}$ when $w \in F_n^l \cap 5B_j$, by the proof of (7.39) or by (7.22). If in addition $g_n(w) \in 5B_i$, then $|z - w| \leq |g_n(w) - g_n(z)| + C\varepsilon 2^{-n} \leq 15r_i + C\varepsilon 2^{-n} < 16r_i$ (because $g_n(w) \in 5B_i$ and $g_n(z) \in 10B_i$, by (7.40)), so $w \in B(z, 19r_i)$. This proves (7.42).

Now (7.25) and (7.42) say that $F_{n+1}^l \cap 5B_i \subset g_n(F_n^l \cap 5B_j) \cap 5B_i \subset g_n(F_n^l \cap B(z, 19r_i)) \cap 5B_i$. Incidentally, the converse inclusion is trivial. We compare with (7.41) and get that

$$F_{n+1}^l \cap 5B_i = \Lambda \cap 5B_i. \tag{7.43}$$

This is the same thing as (7.14), but we also need to check the small additional properties that go with (7.14).

First we need to check that Λ meets $B(x_i, C_3\varepsilon 2^{-n-1})$. Recall that $g_n(z)$ lies in $\Lambda \cap 10B_i$ by (7.26) and (7.40); since Λ is a $C\varepsilon$-Lipschitz graph over P_i and the center x_i lies in P_i, it is enough to check that $\operatorname{dist}(g_n(z), P_i) \leq C\varepsilon 2^{-n-1}$ for some geometric constant C. By (7.13) (for $n+1$), $\operatorname{dist}(g_n(z), E) \leq C\varepsilon 2^{-n-1}$. (We can safely consider now that the constant in (7.13) is geometric, since we proved (7.13) already.) Then $g_n(z)$ lies very close to P_i, because $g_n(z) \in 10B_i$ and by (7.5). Thus Λ meets $B(x_i, C_3\varepsilon 2^{-n-1})$, as needed.

We also need to check that that two of the $F_{n+1}^l \cap 5B_i$ are empty and the third one is a graph. With our assumptions, we already found l such that $F_{n+1}^l \cap 5B_i$ is a graph, we just need to check that the other two $F_{n+1}^m \cap 5B_i$ are empty.

Let j be as in (7.23) and (7.25); If $j \in I_1(n)$, then by induction assumption (relative to (7.14)), only one $F_n^m \cap 5B_j$ is nonempty, and (7.25) says that at only one $F_{n+1}^l \cap 5B_i$ is nonempty, as needed. If $j \in I_2(n)$, we know that $x_i \in \frac{7}{4}B_j$. If $y \in F_{n+1}^m \cap 5B_i$, (7.25) says that $y = g_n(z)$ for some $z \in F_n^m \cap 5B_j$; then z satisfies (7.26), and so $|g_n(z) - z| \leq C\varepsilon 2^{-n}$ by (7.22)). Then $z \in 11B_i \subset 3B_j$ (recall that $r_j = 2^{21}r_i$ because $j \in I_2(n)$), so j lies in I_z. Then we can do the argument above (from (7.28) down) with $k = j$. We get that $z \in F_n^l$ for the only l such that $|D\pi_j^l - D\pi_i| \leq C\varepsilon$ (as in (7.32)), for instance. In other words, $m = l$. This completes proof of (7.14) and the related facts, but only in first case when we can find z as in (7.26) and $k \in I_z \cap I_2(n)$.

The other case. First we should check that $F_{n+1} \cap B_i$ is not empty. Let j be as in (7.23), and first assume that $j \in I_2(n)$; by induction assumption, we have a description of $F_n \cap 5B_j$ as a union of three pieces of Lipschitz graphs that implies that every point of $Y_j \cap 4B_j$ lies

$C\varepsilon 2^{-n}$-close to some point of F_n (here C depends on C_5 and C_6, but this will not matter). Since $x_i \in E \cap \frac{7}{4}B_j$, it is $10\varepsilon r_j$-close to $Y_j = Y(x_j, 10r_j)$, so it is also $C\varepsilon 2^{-n}$-close to F_n. Let $z \in F_n$ be such that $|z - x_i| \leq C\varepsilon 2^{-n}$; then $|z| \leq |x_i| + C\varepsilon 2^{-n} \leq \rho_{n+1} + C\varepsilon 2^{-n}$, $|x_{j(z)}| \leq |z| + 3 \cdot 2^{-n-40} < \rho_n$, and (7.22) holds. That is, $|g_n(z) - z| \leq C\varepsilon 2^{-n}$, hence $|g_n(z) - x_i| \leq C\varepsilon 2^{-n}$, and even though C depends on the C_i's, we get that $g_n(z) \in B_i$.

If $j \in I_1(n)$, then by (7.14) $F_n \cap 5B_j$ is a Lipschitz graph that stays close to P_j. Since now $x_i \in E \cap \overline{B}_j$, x_i is also very close to P_j, and we can find $z \in F_n$ such that $|z - x_i| \leq C\varepsilon 2^{-n}$; the same argument as before shows that $g_n(z) \in F_{n+1} \cap B_i$.

So we can find $y \in F_{n+1} \cap B_i$. By (7.25), $y = g_n(z)$ for some $z \in F_n \cap 5B_j$. In particular, we can find z such that (7.26) holds.

Now let z be any point such that (7.26) holds. As before, (7.27) shows that (7.22) holds, i.e., $|g_n(z) - z| \leq C\varepsilon 2^{-n}$, and hence $z \in 11B_j$.

Pick $k \in I_z$. Since we are no longer in our first case, $k \in I_1(n)$. Let P_k be as in (7.5) and the definition of ψ_k. Thus $\psi_k = \pi_k$, where π_k is the orthogonal projection onto P_k. Notice that $z \in 11B_i \cap 3B_k$ (because $k \in I_z$), so (7.5) says that E is $C\varepsilon r_k$-close to both P_i and P_k in $B(z, 11r_k)$, and $D_{z,10r_k}(P_i, P_k) \leq C\varepsilon$. Hence $|D\psi_k - D\pi_i| = |D\pi_k - D\pi_i| \leq C\varepsilon$ for $k \in I_z$ and, after averaging,

$$|Dg_n(z) - D\pi_i| \leq C\varepsilon. \tag{7.44}$$

See (7.37) for a little more detail about the proof.

Suppose in addition that $g_n(z) \in 6B_i$ (instead of $10B_i$ in (7.26)). Recall that $|g_n(z) - z| \leq C\varepsilon 2^{-n}$ by (7.22), so z lies very close to $6B_i$ too. Let j be as in (7.23). If $j \in I_2(n)$, then B_j is much larger than B_i, hence $z \in 3B_j$; this is impossible, because we assumed that I_z does not meet $I_2(n)$. So $j \in I_1(n)$, $r_j = 2r_i$, $x_i \in \overline{B}_j$, $6B_i \subset 4B_j$, and z lies in $4B_j$ or very close to it.

A first consequence of this is that z lies in the only F_n^l, $1 \leq l \leq 3$, such that $F_n^l \cap 5B_j$ is not empty. This proves that

$$F_{n+1} \cap 5B_i = g_n(F_n^l \cap 5B_j) \cap 5B_i = F_{n+1}^l \cap 5B_i \tag{7.45}$$

for that l. Indeed, if $y \in F_{n+1} \cap 5B_i$, (7.25) says that $y = g_n(z)$ for some $z \in 5B_j$, and then $z \in F_n^l$, so $y \in F_{n+1}^l$.

Return to $z \in F_n$ such that $g_n(z) \in 6B_i$, and consider $w \in F_n \cap B(z, 3r_i)$. As usual, (7.22) holds for w because $|x_{j(w)}| \leq |w| + 3 \cdot 2^{-n-40} \leq |z| + 3r_i + 3 \cdot 2^{-n-40} \leq \rho_{n+1} + 7 \cdot 2^{-n-40} + 3r_i + 3 \cdot 2^{-n-40} < \rho_n$ by (7.27). Then $g_n(w) \in 10B_i$, and w satisfies (7.26).

For $k \in I_w$, we know that $k \in I_1(n)$ because we are no longer in our first case, and (7.14) gives a Lipschitz graph $A_{n,k}$ such that $F_n \cap 5B_k = A_{n,k} \cap 5B_k$. Then also $F_n \cap B(z, 3r_i) = A_{n,k} \cap B(z, 3r_i)$ (because $z \in 3B_k$, so $B(z, 3r_i) \subset 5B_k$). We may now repeat the argument near (7.38): given w_1, $w_2 \in F_n \cap B(z, 2r_i)$, we can find a short curve in $F_n \cap B(z, 3r_i)$ from w_1 to w_2; then we use (7.44) to prove that (7.38) holds; this proves that $g_n(F_n \cap B(z, 3r_i))$ is contained in the graph of a Lipschitz function with constant $\leq C\varepsilon$. A monodromy or degree argument then shows that $g_n(F_n \cap B(z, 3r_i))$ coincides with the whole graph in $B(g_n(z), r_i)$, as in (7.40).

So we showed that for every $z \in F_n \cap 5B_j$ such that $g_n(z) \in 6B_i$, there is a Lipschitz function defined on P_i, with Lipschitz constant $\leq C\varepsilon$, whose graph Γ_z satisfies

$$g_n(F_n \cap B(z, 3r_i)) \cap B(g_n(z), r_i) = \Gamma_z \cap B(g_n(z), r_i). \tag{7.46}$$

Also observe that $g_n(w)$ lies out of $B(g_n(z), r_i)$ when w lies out of $5B_j$, by the proof of (7.25), and when w lies in $5B_j \setminus B(z, 3r_i)$, this time by (7.22). Hence (7.46) also says that

$$F_{n+1} \cap B(g_n(z), r_i) = \Gamma_z \cap B(g_n(z), r_i). \tag{7.47}$$

So we got a description like the one in (7.14), with with the smaller radius r_i, but for every ball centered on $F_{n+1} \cap 6B_i$ (i.e., at a point $g_n(z)$, z as above). We also know that each Γ_z contains the point $g_n(z)$, and $\mathrm{dist}\,(g_n(z), E) \leq C\varepsilon 2^{-n}$ by (7.21) or (7.13). Then $\mathrm{dist}\,(g_n(z), P_i) \leq C\varepsilon 2^{-n}$, by (7.5), and C is even a geometric constant.

It is now easy to see that F_{n+1} coincides with a Lipschitz graph $A_{n+1,i}$ in $5B_i$, as needed for (7.14). To prove that $A_{n+1,i}$ meets $B(x_i, C_3\varepsilon 2^{-n-1})$, we can use the point $z \in F_n$ such that $|z - x_i| \leq C\varepsilon 2^{-n}$, which we used to prove that $F_{n+1} \cap B_i$ is not empty. This point lies in $F_n \cap B_i$, so $g_n(z)$ lies on the graph, and we just said that $\mathrm{dist}\,(g_n(z), P_i) \leq C\varepsilon 2^{-n}$ for some geometric constant C. We now deduce that $A_{n+1,i}$ meets $B(x_i, C_3\varepsilon 2^{-n-1})$ from the fact that the Lipschitz constant for $A_{n+1,i}$ is small.

We already know from (7.45) that $F_{n+1} \cap 5B_i = F^l_{n+1} \cap 5B_i$ for a single l, so we completed our proof of (7.14) by induction.

Proof of (7.15)–(7.16). Now we prove the description near (7.15)–(7.16). Let $i \in I_2(n+1)$ be given. This time, (5.7) says that $x_i \in \overline{B}_j$ for some $j \in I_2(n)$. We still have that

$$F^l_{n+1} \cap 5B_i = g_n(F^l_n \cap 5B_j) \cap 5B_i, \tag{7.48}$$

with the same proof as for (7.25). Thus we may restrict to points $z \in 5B_j$. By (7.24), we can apply the induction assumption to get a description of $F_n \cap 5B_j$ in terms of three graphs $T^l_{n,j}$ over domains $S^l_j \subset P^l_j$. We want to study g_n on these sets.

Let us check once more that for $z \in 5B_j$ and $k \in I_z$, x_k lies in $B(0, \rho_k)$. Indeed $|x_k| \leq |z| + 3r_k \leq |x_j| + 5r_j + 3r_k \leq |x_i| + 6r_j + 3r_k \leq \rho_{k+1} + 9 \cdot 2^{-n-40} < \rho_n$, by (7.12). This will allow us to use the estimates (7.17)–(7.22) if needed.

Let us pick an index l and restrict to $z \in F^l_n \cap 5B_j = T^l_{n,j} \cap 5B_j$. Call P^l_j the corresponding plane. Recall that $x_i \in \overline{B}_j$ and $r_j = 2r_i$, so $D_{x_i,9r_i}(Y_i, Y_j) \leq C\varepsilon$ because both sets approximate E well near $B(x_i, 9r_i)$. Then there is a (unique) face V^l_i of Y_i such that

$$D_{x_i,8r_i}(V^l_i, V^l_j) \leq C\varepsilon \ \text{ and } \ |D\pi^l_j - D\pi^l_i| \leq C\varepsilon, \tag{7.49}$$

where π^l_i denotes the orthogonal projection onto the plane P^l_i that contains V^l_i.

If $k \in I_z$ lies in $I_2(n)$, we can do the same argument with i replaced with k, and we get that $|D\pi^l_k - D\pi^l_j| \leq C\varepsilon$. There is no ambiguity with the index l: we decided to look at F^l_n, we know that near z, $T^l_{n,j}$ (or equivalently F^l_n) is a small Lipschitz graph over P^l_j, and for each k, P_k^l corresponds to the only face of Y_k that makes a small angle with P_j^l or with the tangent planes to $T_{n,j}^l$ near z.

If $k \in I_z$ lies in $I_1(n)$, the function ψ_k will merely be the projection π_k onto the plane P_k that shows up in (7.5). We shall abuse notation from time to time and set $P_k^l = P_k$, to avoid distinguishing cases. Since E is close to both P_k and Y_j near z, P_k is very close to one of the planes P_j^m. But then $m = l$, because near z, F_n^l is both a piece of small Lipschitz graph over P_j^m and a small Lipschitz graph over P_k (we can apply the induction hypothesis to describe $F_n \cap 5B_k$, because $x_k \in B(0, \rho_n)$). Altogether, we also have that

$$|D\pi_k - D\pi_i^l| \leq C\varepsilon \quad \text{when } k \in I_z \cap I_1(n) \tag{7.50}$$

By the same computation as for (7.37), we get that

$$|Dg_n(w) - D\pi_i^l| \leq C\varepsilon \quad \text{for } z \in F_n^l \cap 5B_j. \tag{7.51}$$

Recall that in $5B_j$, F_n^l coincides with a $C_6\varepsilon$-Lipschitz graph $T_{n,j}^l$ over a Lipschitz domain $S_j^l \subset P_j^l$. Then let w_1, $w_2 \in F_n^l \cap 5B_j$ be given. We can find a path in $F_n^l \cap 5B_j$ that goes from w_1 to w_2, with length $\leq 2|w_1 - w_2|$. Then we can follow this path to estimate $g_n(w_1) - g_n(w_2)$. That is, if $\gamma : [0,1] \to F_n^l$ is this path, we write that $g_n(w_2) - g_n(w_1) = \int_0^1 Dg_n(\gamma(t)).\gamma'(t)dt$, observe that $\int_0^1 D\pi_i^l.\gamma'(t)dt = \pi_i^l(w_2) - \pi_i^l(w_1)$, and then use (7.51) to show that

$$\begin{aligned} |\pi_i^{l\perp}(g_n(w_2)) - \pi_i^{l\perp}(g_n(w_1))| &\leq C\varepsilon|w_2 - w_1|, \\ |\pi_i^l(g_n(w_2)) - \pi_i^l(g_n(w_1))| &\geq \frac{99}{100}|w_2 - w_1|. \end{aligned} \tag{7.52}$$

In other words, $g_n(F_n^l \cap 5B_j)$ is contained in a Lipschitz graph G over P_i (with constant less than $C\varepsilon$).

We still need to show some surjectivity, and for this we plan to use a little bit of degree theory (or at least an index). We shall use the fact that in $5B_j$, F_n^l coincides with the graph $T_{n,j}^l$ over $S_j^l \subset P_j^l$, that S_j^l is a Lipschitz domain with small constant, and that the boundary of $T_{n,j}^l$ (i.e., the image of the boundary of S_j^l by φ_j^l, the standard parameterization of the graph by the base) is $\Gamma_n \cap 5B_j$ (which indeed is the graph of a Lipschitz function with constant $\leq C\varepsilon$, by Section 6).

We want to construct a simple curve in $T_{n,j}^l \cap 5B_j$. Set $D_0 = B(\pi_j^l(x_j), 48r_j/10) \cap P_j^l$, call D the interior of $D_0 \cap S_j^l$, and call γ_0 the boundary of D. This is a nice simple curve, composed of a piece of $\pi_j^l(\Gamma_n)$ (which is small Lipschitz graph, because Γ_n is a small Lipschitz graph over the spine $L_j \subset P_j^l$), completed by an arc of the circle ∂D_0. Then let γ_1 be the image of γ_0 by the standard parameterization φ_j^l. It is easy to see that $\gamma_1 \subset T_{n,j}^l \cap 5B_j$, and it is still a simple Lipschitz curve. Finally set $\gamma_2 = \pi_i^l(g_n(\gamma_1))$. Once again, γ_2 is a simple curve, because (7.52) says that $\pi_i^l \circ g_n$ is injective on $\gamma_1 \subset F_n^l \cap 5B_j$. It is composed of an arc of $\pi_i^l(\Gamma_{n+1})$ (which is still a Lipschitz curve with small constant) and a curve in P_i^l that looks a lot like an arc of circle and that stays out of $B(\pi_i^l(x_j), 47r_j/10) \cap P_i^l$ because $\pi_i^l \circ g_n$ does not send points of γ_0 too far from where they are.

Consider the two domains in P_i^l bounded by the arc of $\pi_i^l(\Gamma_{n+1})$ mentioned above and $\partial B(\pi_i^l(x_j), 47r_j/10)$, and call them D_1 and D_2. Also call $i(\xi)$ the index of γ_2 with respect to a point $\xi \in P_i^l$. It is easy to deduce from the description of γ_2 above that $i(\xi) = 0$ on one of the two domains (call it D_1), because we can deform γ_2 into a point outside of D_1. Then $i(\xi) \neq 0$ on D_2, for instance because the index changes when we cross the arc of $\pi_i^l(\Gamma_{n+1})$ near x_j, and does not change inside D_2 because we never meet γ_2.

Now set $h = \pi_i^l \circ g_n \circ \varphi_j^l$ on D. We can deform γ_0 into a point in $\overline{D}$, and when we take the image by h we get a deformation of γ_2 into a point in $h(\overline{D})$. If $\xi \in P_i^l$ lies out of $h(\overline{D})$, then the deformation of γ_2 never crosses ξ, the index of the curve respect to ξ does not change along the deformation, and so $i(\xi) = 0$. This does not happen when $\xi \in D_2$, so $D_2 \subset h(\overline{D})$.

Now D_1 does not meet $h(D)$. Indeed suppose that $\xi \in D_1$ can be written $\xi = h(x)$ for some $x \in D$, and select any $y \in D$ such that $h(y) \in D_2$. This is easy to do, for instance pick y near the middle of D. Call ζ a path from x to y in D. Observe that $h(\zeta) \subset B(\pi_i^l(x_j), 49r_j/10)$, because $|h(x)-x| \leq C\varepsilon 2^{-n}$ in D (by (7.22), (7.15) for B_j, and because P_i^l is very close to P_j^l). Then $h(\zeta)$ crosses $\pi_i^l(\Gamma_{n+1})$, because it goes from D_1 to D_2 and stays in $B(\pi_i^l(x_j), 49r_j/10)$. The intersection lies in $\pi_i^l \circ g_n(\Gamma_n) \cap B(\pi_i^l(x_j), 49r_j/10) \subset \pi_i^l \circ g_n(\Gamma_n \cap 5B_j)$, because $\pi_i^l \circ g_n$ barely moves points of Γ_n. In other words, we found $w \in \zeta \subset D$ and $s \in \Gamma_n \cap 5B_j$ such that $h(w) = \pi_i^l \circ g_n(s)$. Recall that D is the interior of $D_0 \cap S_j^l$, so $\varphi_j^l(w)$ lies on $T_{n,j}^l \cap 5B_j \setminus \Gamma_n = F_n^l \cap 5B_j \setminus \Gamma_n$ because $w \in D$ and φ_j^l is injective, and we just said that $\pi_i^l \circ g_n(\varphi_j^l(w)) = \pi_i^l \circ g_n(s)$ (by definition of h). This contradicts (7.52), which says that $\pi_i^l \circ g_n$ is injective on $F_n^l \cap 5B_j$.

Return to the graph G and $g_n(T_{n,j}^l)$. Set $B = B(x_j, 46r_j/10)$. The curve Γ_{n+1} crosses $G \cap B$ neatly (because $\Gamma_{n+1} \cap 5B_j$ is a Lipschitz graph with small constant over L_j, and of course $\Gamma_{n+1} \cap B \subset g_n(F_n^l \cap 5B_j) \subset G$). So Γ_{n+1} splits $G \cap B$ into two halves. The image by π_i^l of the first one (call it T_1) is contained in D_1, and the image of the other one (call it T_2) is contained in D_2. Since D_1 does not meet $h(D)$ and $D_2 \subset h(\overline{D})$, this means that T_1 does not meet $g_n \circ \varphi_i^l(D)$ and $T_2 \subset g_n \circ \varphi_i^l(\overline{D})$.

The second part of the last sentence says that $T_2 \subset F_{n+1}^l$. The first part only says that T_1 does not meet $g_n \circ \varphi_j^l(D)$, but if $y \in T_1 \cap 5B_i$ were to belong to F_{n+1}^l, it would have to lie in $g_n(F_n^l \cap 5B_j)$ by (7.48). But if $x \in F_n^l \cap 5B_j$ is such that $g_n(x) = y$, (7.22) says that $|x - y| \leq C\varepsilon 2^{-n}$, and then $\pi_i^l(x) \in D$ and $y \in g_n \circ \varphi_j^l(D)$, a contradiction because then $\pi_i^l(y) \in h(D)$.

Observe that $5B_i \subset B$ because $r_i = r_j/2$. Thus we showed that $F_{n+1}^l \cap 5B_i = T_2 \cap 5B_i$, which is (7.16) with $T_{n+1,i}^l = T_2$. We also know that T_2 is bounded by an arc of Γ_{n+1}, and since the domain D_2 is bounded by $\pi_i^l(\Gamma_{n+1})$, we also get the desired $C_6\varepsilon$-Lipschitz bound on the domain S_i^l. Next, $T_{n+1,i}^l$ meets $B(x_i, C\varepsilon 2^{-n-1})$ trivially, because it contains an arc of Γ_{n+1} that meets $B(x_i, C\varepsilon 2^{-n-1})$ by Lemma 6.1. Similarly, (7.15) holds by Lemma 6.1.

Finally, we should say that the only intersection of the three $T_{n+1,i}^l \cap 5B_i$, $l = 1, 2, 3$, is $\Gamma_{n+1} \cap 5B_i$. This is because the $T_{n+1,i}^l$'s leave from Γ_{n+1} in directions that almost make 120° angles.

This completes our verification of (7.11)–(7.16) by induction.

It is easy to see that the restriction of f_n to each face F^l is C^1 (and even smoother), but this may not be true of the restriction of f_n to the whole Y. Let us rapidly explain why. Since we know that f_n is C^1 on each face (and all the way up to the spine L), we would just need to know that the three derivatives of the three restrictions, at a given point of L, all come from a same *linear* mapping that is defined on $\mathbb{R}^3$. That is, we take a point x in $f_n(L)$ and ask whether the three derivatives of g_n, that are defined on the three tangent planes at x (to the faces F_n^l), come from a same linear mapping defined on $\mathbb{R}^3$. It would even be enough to consider each piece $\theta_i(x)\psi_i(x)$ separately (by (7.2) and linearity). Near x, the ψ_i for which $\theta_i \neq 0$ are given by a formula like (7.10), and since $\eta_i(x)$ is nicely defined and smooth on $\mathbb{R}^3$, we would just need to consider h_i (now defined on three faces of the $\eta_i(F_n^l)$).

Nothing much happens in the direction of L_i (where tangent maps are the identity), but in the transverse directions, each branch is sent to a corresponding P_i^l by orthogonal projection, and so lengths of vectors are multiplied by a cosine. Now even if we started from a Y_n whose faces made 120° angles everywhere, this property would probably be upset slightly by composing with η, and then it could be that three vectors whose sum is zero are sent by the tangent maps to three vectors whose sum is not zero. So we should not expect to get a C^1 mapping directly. We could probably ameliorate this with some extra work, but we don't think that this is worth the trouble: the fact that the f_n are C^1 on each face, and coincide on the spine, is already quite good, and probably enough for most purposes. See the end of Section 10 for additional comments.

Now consider f, the limit of the f_n, and let us prove (1.8) and the biHölder estimates in (1.5). Set $B = B(0, 19/20)$; notice that $B \subset B(0, \rho_n)$ for all n.

The fact that $f(Y \cap B) \subset E$ comes from (7.13). The fact that $E \cap B \subset f(Y)$ comes from (5.8) (which says that for every n, $E \cap B(0, 197/100)$ is covered by the $7B_i/4$, $i \in I(n)$), plus our description of the F_n^l in the $5B_i$ (and in particular the fact that at least one F_n^l meets B_i). Since f does not move points much (by (7.22) or even (7.11)), we get (1.8) easily.

Let us also prove that

$$(1 - C\varepsilon)\, |y - z|^{1+C\varepsilon} \leq |f(y) - f(z)| \leq (1 + C\varepsilon)\, |y - z|^{1-C\varepsilon} \quad \text{for } y, z \in Y \cap B. \tag{7.53}$$

This is only a part of (1.5), we shall take care of $B(0, 18/20) \setminus Y$ in Section 9, after we modify the values of f away from Y.

We shall prove (7.53) as we did for f^*. We start with the analogue of (6.25), i.e., the fact that

$$(1 - C\varepsilon)\,\mathrm{dist}\,(y, z) \leq \mathrm{dist}\,(g_n(y), g_n(z)) \leq (1 + C\varepsilon)\,\mathrm{dist}\,(y, z) \tag{7.54}$$

when y, $z \in f_n(Y \cap B)$ are such that $|y - z| \leq 2^{-n-60}$.

By (7.13), $\mathrm{dist}\,(z, E) \leq C\varepsilon 2^{-n}$; by (5.8), we can find $i \in I(n)$ such that $z \in 2B_i$, and then $y \in 3B_i$. Moreover, we can use the description of the F_n^l in (7.13)–(7.16), because B lies well inside $B(0, \rho_n)$. If $i \in I_1(n)$, y and z lie in $A_{n,i}$, there is a short curve γ in $A_{n,i}$ that goes from y to z, and we can use the differential of g_n on γ to compute $g_n(y) - g_n(z)$. This differential is $C\varepsilon$-close to $D\pi_i$, by (7.44) (or even part of its proof, since here we do not even need to compare with the projection on some other $P_{i'}$, $i' \in I(n+1)$). This gives (7.54) in this first case.

If $i \in I_2(n)$ and y, z lie in the same F_n^l, we can do the same thing with an arc in $T_{n,i}^l$, using (7.37) instead of (7.44). Finally, if $i \in I_2(n)$ and y, z lie in different faces F_n^l, we introduce a third point $x \in \Gamma_n \cap 3B_i$. Let us choose x so that $|x-y|+|x-z|$ is minimal, but the precise choice won't matter much. Let l and m be such that $y \in F_n^l$ and $z \in F_n^m$. We know that $|g_n(x) - g_n(y) - D\pi_i^l \cdot (x-y)| \leq C\varepsilon|x-y|$ by the same trick as before (integrate on a path from x to y). Also, $|\pi_i^l(x) - \pi_i^l(y) - x + y| \leq C\varepsilon|x-y|$ (because both points lie on the small Lipschitz graph $T_{n,i}^l$). Thus $|g_n(x) - g_n(y) - x + y| \leq C\varepsilon|x-y|$, and similarly $|g_n(x) - g_n(z) - x + z| \leq C\varepsilon|x-z|$. Finally, $|g_n(z) - g_n(y) - z + y| \leq C\varepsilon(|x-y|+|x-z|) \leq C\varepsilon(|y-z|)$ (by choice of x), and (7.54) follows.

We may now continue the discussion as we did between (6.25) and (6.34), and we obtain (7.53) just like we proved (6.34).

8 The case of a tetrahedron

In this section we show how to modify the construction above when (4.36) holds (and hence $E_3 \cap B(0, 199/200) = \{0\}$). As was hinted near the end of Section 4, the argument would work the same way if $E_3 \cap B(0, 199/200) = \{x_0\}$ instead, for some $x_0 \in B(0, 195/100)$; we would just need to center the balls B_{i_0} at x_0. We shall use the same sort of formulas as in Sections 6 and 7, except that for each $n \geq 0$, we have to define the ψ_i^* and the ψ_i differently when $i = i_0$ (i.e., when $x_i = 0$).

For ψ_i^*, we want to define approximate projections on spines of sets of type 3, a little like what we did for h near (7.10). Let T_0 denote a minimal set of type 3 centered at the origin; for instance, take T_0 as in Definition 2.3. Denote by p_0 a function on $\mathbb{R}^3$ which is positively homogeneous of degree 1, Lipschitz (or even C^∞ if we want) on the unit sphere, and such that for each of the four branches D_j of the spine of T_0, p_0 coincides with the orthogonal projection on (the line containing) D_j in the region where $\operatorname{dist}(z, D_j) \leq |z|/10$. We can also arrange symmetry with respect to the isometries that fix T_0, even though this is not needed (it just allows us to have a standard map that we use all the time). For a general set T of type 3, we write $T = R(T_0)$ for some composition R of a rotation and a translation, and we set $p_T = R \circ p_0 \circ R^{-1}$.

For each $n \geq 1$, Corollary 4.1 says that $c(0, 2^{-n-10}) \leq 15\varepsilon$, so we can select a set Z_n of type 3, centered at 0, such that $D_{0,2^{-n-10}}(E, Z_n) \leq 15\varepsilon$. For $n = 0$, take $Z_0 = Z(0, 2)$. Then set $\psi_{i_0}^* = p_{Z_n}$ (n will often stay implicit in our notation).

With this choice of $\psi_{i_0}^*$, we can construct a parameterization of (a good piece of) $E_2 \cup \{0\}$ by the spine of $Z_0 = Z(0, 2)$ (intersected with $B(0, 197/100)$, say), as we did in Section 6. The argument is a simpler version of Section 7, and we shall not repeat it all. Notice that we never need to worry about the radii ρ_n when the balls B_{i_0} come into play, because B_{i_0} is far from $\partial B(0, 197/100)$ (one less worry). Also observe that all the θ_i, $i \neq i_0$, vanish near $\frac{3}{2}B_{i_0}$, by (5.10) and (5.14). So

$$g_n^*(x) = \psi_{i_0}^*(x) = p_{Z_n}(x) \quad \text{for } x \in \frac{3}{2}B_{i_0}. \tag{8.1}$$

Finally notice that the f_n^* and g_n^* preserve the origin, by (8.1).

So we get maps f_n^*, g_n^*, and sets Γ_n. Most of Section 6 stays the same, except that now each Γ_n is composed of four branches Γ_n^m which meet at the origin.

Next consider the mapping f of Section 7. Now we need to define the mappings ψ_{i_0} for $n \geq 0$. Again start from a simpler mapping h_n, which is positively homogeneous of degree 1, smooth on the unit sphere, and such that, for each of the six faces V_l that compose Z_n,

$$h_n(x) = \pi_l(x) \text{ for } x \in \mathcal{C}_l = \{x \in \mathbb{R}^3 \,;\, \mathrm{dist}\,(x, V_l) \leq |x|/100\}, \tag{8.2}$$

where π_l denotes the orthogonal projection on the plane that contains V_l. The simplest is to construct such a map for T_0 and get the other ones by conjugation with isometries as before, but this is not needed (provided that we keep some uniform estimates on the derivatives). As before, the values of h_n out of the six blade-like sectors $\mathcal{C}_l$ of (8.2) do not matter too much, because the $F_n^l \cap 5B_{i_0}$ will stay in the $\mathcal{C}_l$.

Denote by L^m, $1 \leq m \leq 4$, the four half lines that compose the spine of Z_n. We want to precompose with a mapping η like the one in (7.9), whose effect near the origin is to map the four branches Γ_n^m to the corresponding half lines L^m.

For each m, we have a Lipschitz parameterization of $\Gamma_n^m \cap 5B_{i_0}$ by a segment of L^m, which we can write $x \to (x, \zeta^m(x))$ (this time, with $x \geq 0$) in appropriate coordinates (depending on m). We want η to be Lipschitz, with

$$\eta(0) = 0 \text{ and } |D\eta - I| \leq C\varepsilon \tag{8.3}$$

and such that

$$\eta(x, y) = (x, y - \zeta^m(x)), \tag{8.4}$$

in the little cone where $\mathrm{dist}\,((x, y), L^m) \leq 10^{-1}|(x, y)|$. Of course we want such a description simultaneously for all m, but there is no problem because the little cones near the L^m are far from each other. We can even require that $\eta(x, y) = (x, y)$ out of the four cones. Then we set $\psi_{i_0} = h_n \circ \eta$, as before.

Let us check that then f_n coincides with f_n^* on Γ. First, $f_0(x) = f_0^*(x) = x$, so $n = 0$ is all right. If we already know that $f_n = f_n^*$ on Γ and want to check that the same thing holds for f_{n+1}, we just need to see that $\psi_i = \psi_i^*$ on $\Gamma_n \cap 5B_i$ for $i \in I(n)$. Indeed, since we use the same partition of unity for g_n and g_n^*, g_n will coincide with g_n^* on Γ_n. See (6.3) and (7.2). We checked this in Section 7 for $i \neq i_0$, so it is enough to see that $\psi_{i_0} = \psi_{i_0}^*$ on Γ_n.

For $z \in \Gamma_n \cap 3B_{i_0}$, (8.4) says that $\eta(z) = \pi_m(z)$, where $1 \leq m \leq 4$ is such that $z \in \Gamma_n^m$ and π_m denotes the projection of z on the half line L^m. So $\eta(z) = p_{Z_n}(z) = \psi_{i_0}^*(z)$, because p_{Z_n} coincides with π_m in a small cone around L^m; see the definitions above (8.1). Then $\psi_{i_0}(z) = h_n(\pi_m(z)) = \psi_{i_0}^*(z)$, as needed, because $h_n(w) = w$ on L^m, by (8.2).

We need to revise slightly our description of the $F_n^l \cap 5B_i$. The first difference is that now Y is composed of 6 faces, so l should vary between 1 and 6. Here is what we need to prove now.

When $i \in I_1$, not much changes; there is still only one l for which F_n^l meets $5B_i$, and for this one we have the same description as in (7.14).

When $i \in I_2$, only three of the F_n^l meets $5B_i$ (and their names depend on which of the four branches of E_2 contains the center x_i). And for these three F_n^l we have the same description with Lipschitz graphs as in (7.15)–(7.16).

Then we need a description of $F_n^l \cap 5B_{i_0}$. This time, the six faces F_n^l meet $5B_{i_0}$, and the intersections are $C\varepsilon$-Lipschitz graphs T^l over sets S^l (as before), where S^l is a Lipschitz domain in the plane P^l that contains the corresponding face V^l of Z_n, and S^l is very close to V^l (as in (7.15)). In addition, the boundary of the graph T^l is the intersection of $5B_{i_0}$ with the two curves of Γ_n that bound the face F^l.

We need a proof by induction of these descriptions. For $i \in I_1 \cup I_2$, not much needs to be changed. Recall from (5.14) that $100B_i$ does not meet $\frac{3}{2}B_{i_0}$ when $i \neq i_0$. We repeat the proof of Section 7, and, whenever B_{i_0} shows up, we observe that we only need the description of the $F_n^l \cap 5B_{i_0}$ (at order n) away from B_{i_0}. We can use this description just as the description of the $F_n^l \cap 5B_j$, $j \in I_2$ (notice that in a small ball away from B_{i_0}, our description of $F_n \cap 5B_{i_0}$ is of the same type as for the $F_n \cap 5B_j$, $j \neq i_0$).

For $i = i_0 \in I(n+1)$, we use the description of the $F_n^l \cap 5B_{i_0}$ at order n, and check that g_n does not destroy things. Away from the origin (say, out of $\frac{1}{2}B_i$), the proof is the same as before, because the maps and the description of F_n are of the same type as in Section 7. Near $\frac{1}{2}B_i$, we can use the fact that by (5.10) and (5.14) only θ_{i_0} in $\frac{3}{2}B_i$, is nonzero, so g_n coincides with ψ_{i_0} near B_{i_0}. We use the fact that by induction assumption, the various faces F_n^l are bounded by the arcs Γ_n^m, their images by η are still reasonably small Lipschitz graphs and lie in the sectors where h_n is a projection, so their images are locally Lipschitz graphs with small constants. Then composing with h_n makes things even better, and in fact $g_n(F_n^l \cap \frac{3}{2}B_{i_0})$ is contained in a face of Z_n.

We also need to make sure that the six $F_n^l \cap 5B_i$, with $i = i_0$, are equal to the full graph T_n^l (the argument for $i \neq i_0$ can stay the same). This is again done with an index argument like the one below (7.52); there are small differences, because S_j^l and D now look like triangular sectors, and the piece of $\pi_j^l(\Gamma_n)$ that bounds D now comes from two arcs Γ_n^m. This does not upset the argument.

Now we need to check the analogue of (7.53), and we start with the analogue of (7.54). We proceed essentially as for (7.54). By (5.8), we can find $i \in I(n)$ such that y and z lie in $3B_i$. We can use the description of the F_n^l, $1 \leq l \leq 6$ (i.e., the analogue of (7.13)–(7.16)), because B lies well inside $B(0, \rho_n)$. The cases where $i \in I_1(n) \cup I_2(n)$ are dealt with exactly as in the proof of (7.54), so we can assume that $i = i_0$.

If y and z lie in the same face F_n^l, we can proceed as in the proof of (7.53). So let us assume that $y \in F_n^l$ and $z \in F_n^m$, with $l \neq m$. We claim that we can find $x \in F_n^l \cap F_n^m \cap 4B_{i_0}$, with $|x - y| + |x - z| \leq 100|y - z|$ (notice that the first condition forces $x \in \Gamma_n$, and even $x = 0$ if the faces F_n^l and F_n^m are not adjacent).

We can always try $x = 0$, which works when $|y| + |z| \leq 100|y - z|$. So let us assume that $|y - z| < (|y| + |z|)/100$. Set $y' = \pi^l(y)$, where π^l denotes the projection onto the plane P_l that contains V^l. Notice that $|y' - y| \leq C\varepsilon|y|$ because $F_n^l \cap 5B_{i_0} \subset T^l$, T^l is a contained in a $C\varepsilon$-Lipschitz graph the plane P^l, and T^l contains the origin. By construction, y' lies is the Lipschitz domain $S^l \subset P^l$. This domain is the projection of $F_n^l \cap 5B_{i_0}$, and it is bounded by two arcs $\pi^l(\Gamma_n^s)$, where the two values of s correspond to the two half lines L^s that bound V^l. These two arcs in P^l are $C\varepsilon$-Lipschitz over L^s, because the Γ_n^s (in space) are $C\varepsilon$-Lipschitz over L^s. Then $\operatorname{dist}(y', V^l) \leq C\varepsilon|y'|$. Pick $y'' \in V^l$, with $|y'' - y'| \leq C\varepsilon|y'|$. Then $|y'' - y| \leq C\varepsilon|y|$ (because $|y' - y| \leq C\varepsilon|y|$). Similarly, we can find $z'' \in V^m$ such that $|z'' - z| \leq C\varepsilon|z|$. Next

$$\begin{aligned} |y'' - z''| &\leq |y - z| + C\varepsilon(|y| + |z|) \leq (10^{-2} + C\varepsilon)(|y| + |z|) \\ &\leq (|y| + |z|)/95 \leq (|y''| + |z''|)/90. \end{aligned} \tag{8.5}$$

This is impossible if y'' and z'' lie in faces of Z_n that are not adjacent, so V^l and V^m are adjacent. Denote by L^s the half line $V^l \cap V^m$. Notice that $|y'' - z''| \geq \operatorname{dist}(y'', P^m) \geq \operatorname{dist}(y'', L^s)/2$ because V^l and V^m make $120°$ angles. That is, $\operatorname{dist}(y'', L^s) \leq 2|y'' - z''| \leq (|y| + |z|)/45$, by (8.5). Then $\operatorname{dist}(y, L^s) \leq (|y| + |z|)/44$, and $\operatorname{dist}(y, \Gamma_n^s) \leq (|y| + |z|)/43$ because Γ_n^s is a $C\varepsilon$-Lipschitz graph over L^s that goes through the origin (recall that 0 is the only point of type 3). Notice also that $|y|$ and $|z|$ are both pretty close to $(|y| + |z|)/2$, because $|y - z| < (|y| + |z|)/100$. Then we can use the description of $F_n^l = T^l$ and $F_n^m = T^m$ near y and z, as Lipschitz graphs over P^l and P^m with a common Lipschitz boundary Γ_n^s, to find a point $x \in \Gamma_n^s$ such that $|x - y| + |x - z| \leq 3|y - z|$. This proves the claim.

Once we have x, we can use it as in Section 7. We use a short path γ in T^l from x to y, obtained for instance by lifting a line segment from $\pi^l(x)$ to $\pi^l(y)$, to show that

$$|y - x - (\pi^l(y) - \pi^l(x))| \leq C\varepsilon|y - x|, \tag{8.6}$$

just because x and y lie on the small Lipschitz graph T^l, and

$$|g_n(y) - g_n(x) - D\pi^l(y) \cdot (y - x)| = \Big| \int_\gamma [Dg_n - D\pi^l] \cdot \gamma' \Big| \leq C\varepsilon|y - x|, \tag{8.7}$$

because $|Dg_n - D\pi^l| \leq C\varepsilon$ on $4B_{i_0} \cap F_n^l$. Thus $|g_n(y) - g_n(x) - (y - x)| \leq C\varepsilon|y - x|$. Similarly, $|g_n(z) - g_n(x) - (z - x)| \leq C\varepsilon|z - x|$, and

$$|g_n(z) - g_n(y) - (z - y)| \leq C\varepsilon(|y - x| + |z - x|) \leq 100C\varepsilon(|y - x|), \tag{8.8}$$

by definition of x. This proves (7.54); (7.53) follows as before.

9 Extension to $\mathbb{R}^3$

The mapping f that was constructed in the previous sections is not the final one for Theorem 1.1 and Remark 1.1. So far, we focused on the restriction of f to $Z(0, 2)$, but in this section we need to modify f away from $Z(0, 2)$, in particular to make sure that the result is injective. Notationally it will be more convenient to leave f and the f_n as they are, and denote by $\widetilde{f}$ and $\widetilde{f}_n$ the required modifications. Thus the true mapping for Theorem 1.1 and Remark 1.1 is the mapping $\widetilde{f}$ constructed in this section.

We want to define $\widetilde{f}_n$ and $\widetilde{f}$ near $B(0, 18/10) \subset \mathbb{R}^3$; we shall use the same formula as before, i.e., we start from $\widetilde{f}_0(x) = x$ and set

$$\widetilde{f}_{n+1} = \widetilde{g}_n \circ \widetilde{f}_n \text{ for } n \geq 0, \text{ with } \widetilde{g}_n(x) = \sum_{i \in I(n)} \theta_i(x) \widetilde{\psi}_i(x), \tag{9.1}$$

with the same functions θ_i as before, but slightly different $\widetilde{\psi}_i$. This time we shall need all the θ_i, even with $i \in I_0$.

Since we want $\widetilde{f}_n$ to coincide with f_n on Y, we require that

$$\widetilde{\psi}_i(x) = \psi_i \text{ on } 3B_i \cap F_n, \tag{9.2}$$

where F_n is the union of the three or six faces $F_n^l = f_n(F^l)$, and the F^l are the faces of $Z(0, 2) \cap B(0, 195/100)$.

When $i \in I_0(n)$, the simplest is to set $\psi_i(x) = x$; the requirement (9.2) is empty because $3B_i$ does not meet F_n, by (5.24) and since the F_n^l stay close to E (by (7.13)).

Next take $i \in I_1(n)$. Recall from (7.14), or its analogue when $Z(0, 2)$ is of type 3, that only one of the F_n^l meets $5B_i$, and that for this one $F_n^l \cap 5B_i = A_{n,i} \cap 5B_i$ for some $C_2\varepsilon$-Lipschitz graph $A_{n,i}$ over P_i. Call $a_{n,i} : P_i \to P_i^{\perp}$ the corresponding Lipschitz function, and set $\widetilde{\psi}_i(x, y) = (x, y - a_{n,i}(x))$ for $(x, y) \in P_i \times P_i^{\perp}$ (we identify $\mathbb{R}^3$ with $P_i \times P_i^{\perp}$). Here the requirement (9.2) holds by definition of $a_{n,i}$ and because ψ_i is the orthogonal projection onto P_i.

Now consider $i \in I_2(n)$. This time the $F_n^l \cap 5B_i$ coincide with three half-Lipschitz graphs $T_{n,i}^l$ that meet along Γ_n (see near (7.16)), and on these half-Lipschitz graphs we need to take $\widetilde{\psi}_i = \psi_i = h_i \circ \eta_i$, as in (7.10). Let us check that we can extend this mapping to $4B_i$, in such a way that $\widetilde{\psi}_i$ is $(1 + C\varepsilon)$-bilipschitz on $4B_i$, and

$$|\widetilde{\psi}_i(x) - x| \leq C\varepsilon 2^{-n} \text{ on } 4B_i \text{ and } |D\widetilde{\psi}_i(x) - I| \leq C\varepsilon \text{ on } 4B_i \setminus \Gamma_n. \tag{9.3}$$

Recall from (7.9) that η_i, whose role is to move Γ_n to the spine L_i of Y_i, is a $(1 + C\varepsilon)$-bilipschitz mapping that satisfies (9.3). It is also smooth, because Γ_n is smooth. Thus it is enough to find an extension $\widetilde{h}_i$ to $\frac{9}{2}B_i$, of the restriction of h_i to $\eta_i(F_n \cap 5B_i)$, with the required properties. Recall that near each $\eta_i(F_n^l \cap 5B_i)$, h_i coincides with the orthogonal projection onto the plane P^l that contains the face V^l of Y_i that lies close to $\eta_i(F_n^l \cap 5B_i)$. Set $\mathcal{C}^l = \{z \in \mathbb{R}^3 ; \operatorname{dist}(z, V^l) \leq \frac{1}{10}|z|\}$; we claim that we can even choose $\widetilde{h}_i$ so that $\widetilde{h}_i(w) = w$ when w lies out of the three $\mathcal{C}^l$. To define $\widetilde{h}_i$ on $\mathcal{C}^l$, pick coordinates such that $V^l = \{(x, y, z) ; x \geq 0 \text{ and } y = 0\}$; notice that $\eta_i(F_n^l \cap 5B_i)$ is contained in the graph of some $C\varepsilon$-Lipschitz function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F(0, y) = 0$ for $y \in \mathbb{R}$ (because η_i sends Γ_n to L_i), and $h_i(x, y, z) = (x, y)$ on that set. We pick a smooth function φ on $\mathbb{R}$ such that $\varphi(t) = 1$ for $|t| \leq 1/20$, $\varphi(t) = 0$ for $|t| \geq 1/10$, and $|\varphi'(t)| \leq 30$ everywhere, and set

$$\widetilde{h}_i(x, y, z) = (x, y, z) - \varphi\big(y/\sqrt{x^2 + y^2}\big) F(x, y) \text{ on } \mathcal{C}^l. \tag{9.4}$$

It is easy to see that $\widetilde{h}_i(x,y,z)$ has all the required properties and that $\widetilde{\psi}_i = \widetilde{h}_i \circ \eta_i$ satisfies (9.3). [Notice that $|F(x,y)| \leq C\varepsilon\sqrt{x^2+y^2}$ because $F(0,y)=0$, that $|y/\sqrt{x^2+y^2}| \leq C\varepsilon < 1/20$ on $\eta_i(F_n^l \cap 5B_i)$, and that the $(1+C\varepsilon)$-bilipschitzness follows from the second part of (9.3).]

The definition of $\widetilde{\psi}_i$ for $i = i_0$ when (4.36) holds and $Z(0,2)$ is of type 3 is similar. We have a good description of $F_n \cap 5B_i$, this time as a union of six Lipschitz faces T_n^l bounded by four arcs of Γ_n, which themselves meet nicely at the origin, as described in the last section. Again, we know $\widetilde{\psi}_i$ on these six faces, and we claim that there is a $(1+C\varepsilon)$-bilipschitz extension to $4B_i$ such that (9.3) holds. Call Γ_n^s, $1 \leq s \leq 4$, the four arcs of Γ_n, and denote by L^s the corresponding branch of the spine of Z_n; thus the spine of Z_n is the union of the four L^s. Let η be as in (8.3) and (8.4); thus η is $(1+C\varepsilon)$-bilipschitz and satisfies the analogue of (9.3). Notice that η sends $\Gamma_n^s \cap 5B_i$ to L^s, by (8.4). By (8.3), it sends T_n^l to a subset of a $C\varepsilon$-Lipschitz graph over the plane P^l that contains the corresponding face V^l of Z^n, and this graph goes through the two L^s that bound V^l (because η sends the $\Gamma_n^s \cap 5B_i$ to the L^s). That is, we have the same sort of description of $\eta(F_n \cap 5B_i)$ as for $F_n \cap 5B_i$ itself, except that now Γ_n is replaced with the spine of Z_n.

We shall take $\widetilde{\psi}_i = \psi_i^\sharp \circ \eta$ for some $\psi_i^\sharp$. Since $\psi_i = h_n \circ \eta$ (see below (8.4)), (9.2) will hold if $\psi_i^\sharp = h_n$ on $\eta(3B_i \cap F_n)$. By (8.2) and our description of $\eta(F_n \cap 5B_i)$, h_n coincides with the orthogonal projection π^l onto P^l, on the set $\eta(F_n^l \cap 5B_i)$. So we just have to find an extension $\psi_i^\sharp$ to $\frac{9}{2}B_i$, say, of the map defined on the union of the six Lipschitz graphs (that contain the) $\eta(F_n^l \cap 5B_i)$, that is equal to π^l on $\eta(F_n^l \cap 5B_i)$. We can even require that $\psi_i^\sharp(z) = z$ out of the sectors $\mathcal{C}_l$ of (8.2), while in $\mathcal{C}_l$ we use a formula like (9.4). This gives a bilipschitz $\psi_i^\sharp$ that satisfies (9.3), and then $\widetilde{\psi}_i = \psi_i^\sharp \circ \eta$ also satisfies (9.3) because η does.

This completes our definition of $\widetilde{\psi}_i$ in all cases. So we also have a full definition of $\widetilde{f}_n$ and $\widetilde{f}$. Let us check that $\widetilde{f}$ satisfies the conclusions of Remark 1.1.

First we have that $|\widetilde{g}_n(x) - x| \leq C\varepsilon$ everywhere, by (9.1) and (9.3). Then $|\widetilde{f}(x) - x| \leq C\varepsilon$, again by (9.1). This proves (1.6). The second half of (1.7) follows immediately; the first part of (1.7) follows from (1.6) and a little bit of degree theory. That is, for $x \in B(0, 17/10)$, the mapping $F_1 : \partial B(0,1) \to \partial B(0,1)$ defined by $F_1(\xi) = \dfrac{\widetilde{f}(18\xi/10) - x}{|\widetilde{f}(18\xi/10) - x|}$, is homotopic to the identity (by (1.6)), so it has degree 1, and if x did not lie in $\widetilde{f}(B(18/10))$, the mappings F_t, $0 \leq t \leq 1$, defined by $F_t(\xi) = \dfrac{\widetilde{f}(18t\xi/10) - x}{|\widetilde{f}(18t\xi/10) - x|}$, would provide a homotopy from F_1 to a constant.

We already proved (1.8) a little above (7.53), so we just need to check (1.5), i.e., that

$$(1 - C\varepsilon)\,|y - z|^{1+C\varepsilon} \leq |\widetilde{f}(y) - \widetilde{f}(z)| \leq (1 + C\varepsilon)\,|y - z|^{1-C\varepsilon} \tag{9.5}$$

for $y, z \in B(0, 19/10)$. As before, it is enough to prove the analogue of (6.25) or (7.54), i.e., that

$$(1 - C\varepsilon)\,\mathrm{dist}\,(y,z) \leq \mathrm{dist}\,(\widetilde{g}_n(y), \widetilde{g}_n(z)) \leq (1 + C\varepsilon)\,\mathrm{dist}\,(y,z) \tag{9.6}$$

when y, $z \in B(0, 185/100)$ are such that $|y - z| \leq 2^{-n-80}$. By (9.1), it is enough to check that for $j \in I(n)$,

$$|\widetilde{\psi}_j(y) - \widetilde{\psi}_j(z) - (y - z)| \leq C\varepsilon|z - y| \text{ for } y, z \in 4B_j. \tag{9.7}$$

This last follows from (9.3).

We finally completed our verification of Theorem 1.1.

10 C^1 estimates when E is asymptotically flatter

All the estimates above were done with ε fixed, but we can do better if the distance in our basic assumption (1.1) improves when r tends to 0. In this section, we assume that there is a sufficiently small nondecreasing positive function $\varepsilon(r)$ such that for each $x \in E \cap B(0, 2)$ and $r > 0$ such that $B(x, r) \subset B(0, 2)$, there is a minimal cone $Z(x, r)$ that contains x such that

$$D_{x,r}(E, Z(x, r)) \leq \varepsilon(r). \tag{10.1}$$

In particular, we still have (1.1) with $\varepsilon = \varepsilon(2)$, and we assume that $\varepsilon(2)$ is so small that we can apply the construction above. We want to say that when $\varepsilon(r)$ tends to 0, the construction above comes with slightly better estimates. We shall assume that

$$\sum_{k \geq 0} \varepsilon(2^{-k}) < +\infty, \tag{10.2}$$

and then show that the restriction of our mapping f to each of the three or six faces of $Z(0, 2) \cap B(0, 17/10)$, is of class C^1.

Set $\varepsilon_k = \varepsilon(2^{-k})$. For technical reasons, it will be more convenient to assume that ε_k never drops too brutally, i.e., that

$$\varepsilon_k \leq C_0 \varepsilon_{k+1} \text{ for } k \geq 0 \tag{10.3}$$

and some constant $C_0 > 0$. This is not really an additional assumption, because we can always replace $\{\varepsilon_k\}$ above with a larger sequence that satisfies (10.3) and such that $\sum_{k \geq 0} \varepsilon_k$ still converges.

The point of assuming (10.3) is that when we apply the construction and arguments above, we can replace ε with $C\varepsilon_n$ in our various intermediate statement (such as Lemmas 6.1 and the Lipschitz descriptions in (7.13)-(7.16)). If we did not assume (10.3), our proofs by induction would force us to carry additional terms that control the larger scales.

In particular, the proof above shows that

$$(1 - C\varepsilon_n)\,\mathrm{dist}\,(y, z) \leq \mathrm{dist}\,(\widetilde{g}_n(y), \widetilde{g}_n(z)) \leq (1 + C\varepsilon_n)\,\mathrm{dist}\,(y, z) \tag{10.4}$$

for y, $z \in B(0, 185/100)$ such that $|y - z| \leq 2^{-n-80}$, instead of (9.6).

We need estimates on the first and second derivatives of f near the faces. We start with the mappings g_n^* and f_n^* from Section 6. We see them as mappings defined on $\mathbb{R}^3$, because this will make it easier to differentiate, even though we shall later restrict to a small neighborhood of the spine of $Z(0,2)$. When we differentiate (6.3), we get that

$$Dg_n^* = \sum_{i\in I(n)} D\theta_i \psi_i^* + \sum_{i\in I(n)} \theta_i D\psi_i^* = \sum_{i\in I(n)} D\theta_i \,[\psi_i^* - \psi_j^*] + \sum_{i\in I(n)} \theta_i \, D\psi_i^*, \tag{10.5}$$

where $j \in I(n)$ is any fixed index and we used the fact that $\sum_{i\in I(n)} \theta_i = 1$. Here we just need to differentiate near $x \in \Gamma_n = f_n^*(\Gamma)$, so we can restrict to $i \in I_2(n)$ because the other θ_i vanish near x, and then ψ_i^* is the orthogonal projection onto the line L_i. We select $j \in I_2(n)$ such that $\theta_j(x) \neq 0$, and then

$$|\psi_i^* - \psi_j^*| \leq C\varepsilon_n 2^{-n} \;\text{ and }\; |D\psi_i^* - D\psi_j^*| \leq C\varepsilon_n \;\text{ near } x, \tag{10.6}$$

by (6.6) and (6.7). Thus (10.5) yields

$$|Dg_n^* - D\psi_j^*| \leq C\varepsilon_n \;\text{ near } x, \tag{10.7}$$

where we used the fact that $D\psi_j^* = \sum_{i\in I(n)} \theta_i D\psi_j^*$, and (5.13) to control the $|D\theta_i|$. In particular,

$$|Dg_n^*| \leq 1 + C\varepsilon_n \;\text{ near } x, \tag{10.8}$$

When we differentiate a second time in (10.5), we get that

$$D^2 g_n^* = \sum_{i\in I(n)} D^2\theta_i \,[\psi_i^* - \psi_j^*] + 2 \sum_{i\in I(n)} D\theta_i \,[D\psi_i^* - D\psi_j^*], \tag{10.9}$$

because $\sum_{i\in I(n)} D\theta_i = 0$ and the $D^2\psi_i^*$ vanish (each ψ_i^* is a linear projection). Thus

$$|D^2 g_n^*| \leq C\varepsilon_n 2^n \;\text{ near } x, \tag{10.10}$$

by (10.6) and (5.13). Next f_n^* is the composition of the g_k^*, $k < n$ (see (6.2)), so

$$Df_n^* = Dg_{n-1}^* \circ Dg_{n-2}^* \circ \cdots \circ Dg_0^*, \tag{10.11}$$

where we do not write the arguments to save space. When we differentiate once more, we get a sum of terms with one second derivative of some g_k^*, composed with a collection of first-order differentials; thus

$$|D^2 f_n^*| \leq C\varepsilon_n 2^n \prod_{0\leq k<n} (1 + C\varepsilon_n) \leq C\varepsilon_n 2^n, \tag{10.12}$$

by (10.8), (10.10), and (10.2).

Let us now consider the restriction of f_n^* to Γ, the intersection of the spine of $Z(0,2)$ with $\overline{B}(0,197/100)$. When $Z(0,2)$ is of type 3 we can still use the discussion above, we just need restrict our attention to $\Gamma\setminus\{0\}$. The derivative of this restriction is (locally, and modulo identification of Γ with an interval in $\mathbb{R}$) a vector-valued function $f_n^{*\prime}$, and $|f_n^{*\prime}(x)| \geq \prod_{0\leq k<n}(1-C\varepsilon_n) \geq C^{-1}$ by multiple applications of (6.20), or equivalently of (10.7) and the fact that Γ_n is a small Lipschitz graph over L_i.

We need these estimates to control the parameterization $x \to (x,\zeta(x))$ of the graph $G_{n,i}$ that we use in (7.8)-(7.10). Observe that $|\zeta''(x)| \leq C\varepsilon_n 2^n$ because $|f_n^{*\prime}(x)| \geq C^{-1}$ and $|f_n^{*\prime\prime}(x)| \leq |D^2 f_n^*| \leq C\varepsilon_n 2^n$ (by (10.12)). Then

$$|D\eta_i - I| \leq C\varepsilon_n \ \text{ and } \ |D^2\eta_i| \leq C\varepsilon_n 2^n, \tag{10.13}$$

by (7.8) and (7.9).

Next fix a face F_n^l and a point z in the interior of F_n^l. Assume that $z\in B(0,195/100)\subset B(0,\rho_n)$, so that we can use the estimates below (7.16). Let $i\in I(n)$ be such that $z\in 3B_i$ (i.e., with the notation of Section 7, $i\in I_z$). If $i\in I_1(n)$, (7.6) says that $\psi_i=\pi_i$, the orthogonal projection onto a plane P_i. If $i\in I_2(n)$, $\psi_i = h_i\circ\eta_i$ by (7.10), and (7.20) says that

$$\psi_i = \pi_i^l\circ\eta_i \ \text{ near } z, \tag{10.14}$$

because z lies in the interior of F_n^l. Here π_i^l is the orthogonal projection onto the plane that contains the appropriate face of Y_i. If $Z(0,2)$ is of type 3 and $i\in I_3(n)=\{i_0\}$, the same argument shows that $\psi_i=\pi_i^l\circ\eta_i$ near z, where π_i^l is the orthogonal projection onto the plane that contains some face of Z_n. The case when $i\in I_0(n)$ does not happen, as noted below (7.16), by (7.13) and (5.24). Set $\pi_i^l=\pi_i$ when $i\in I_1(n)$, to uniformize the notation.

If $j\in I(n)$ is some other index such that $z\in 3B_j$, then $|D\pi_j^l - D\pi_i^l|\leq C\varepsilon_n$, by the proof of (7.19) (recall, we used the fact that near z, F_n^l is at the same time a small Lipschitz graph over P_i^l and P_i^l). We easily deduce from this, (10.13), (10.14) and its analogue for $i\in I_3(n)$ (both applied to j), that

$$|D\psi_j - D\pi_i^l| \leq C\varepsilon_n \text{ and } |D^2\psi_j|\leq C\varepsilon_n 2^n \ \text{ near } z, \tag{10.15}$$

We also know from (7.18) and (7.19) that $|\psi_j(z)-\pi_i^l(z)|\leq C\varepsilon_n 2^{-n}$. The same computation as for (10.5) and (10.7) yields

$$Dg_n = \sum_{j\in I(n)} D\theta_j\psi_j + \sum_{j\in I(n)}\theta_j D\psi_j = \sum_{j\in I(n)} D\theta_j\,[\psi_j-\pi_i^l] + \sum_{i\in I(n)}\theta_j\, D\psi_j, \tag{10.16}$$

by (7.2), and then

$$|Dg_n - D\pi_i^l| \leq C\varepsilon_n + \Big|\sum_{i\in I(n)}\theta_j\,[D\psi_j - D\pi_i^l]\Big| \leq C\varepsilon_n \tag{10.17}$$

near z. Next

$$D^2 g_n = \sum_{j\in I(n)} D^2\theta_j\,[\psi_j - \pi_i^l] + 2\sum_{j\in I(n)} D\theta_j\,[D\psi_j - D\pi_j^l] + \sum_{j\in I(n)} \theta_j D^2\psi_j\,, \tag{10.18}$$

so $|D^2 g_n| \leq C\varepsilon_n 2^n$ near z, by (10.15) and the line below it. The same proof as for (10.12) now shows that

$$|D^2 f_n| \leq C\varepsilon_n 2^n \ \text{ on the interior of } F^l \cap B(0, 194/100). \tag{10.19}$$

(we only consider $x \in B(0, 194/100)$ to make sure that all the $f_n(x)$ lie in $B(0, 195/100)$.)

Let x in the interior of $F^l \cap B(0, 194/100)$ be given, and let u be a unit tangent vector to F^l at x. Set $z_n = f_n(x)$ and $u_n = Df_n(x)(u)$. Then u_n is a tangent vector to F_n^l at z_n, and

$$\begin{aligned} |u_{n+1} - u_n| &= |Dg_n(z_n)(u_n) - u_n| \leq |Dg_n(z_n)(u_n) - \pi_i^l(u_n)| + |\pi_i^l(u_n) - u_n| \\ &\leq C\varepsilon_n |u_n| + |\pi_i^l(u_n) - u_n| \leq C\varepsilon_n |u_n| \end{aligned} \tag{10.20}$$

where π_i^l is associated to z_n as above, by (10.17), and because F_n^l is a small Lipschitz graph over P_i^l near z_n. Multiple applications of this yield $C^{-1} \leq |u_n| \leq C$, by (10.2), and $|u_{n+1} - u_n| \leq C\varepsilon_n$.

We just proved that the derivatives $Df_n(x)$ converge, uniformly in x in the interior of $F^l \cap B(0, 194/100)$, to some limit G. Since we already know that the f_n converge to f, we get that f is continuously differentiable on the interior of $F^l \cap B(0, 194/100)$, with $Df = G$. In addition, $|Df(u) - Df_n(u)| \leq \sum_{k\geq n} |u_{k+1} - u_k| \leq C\widehat{\varepsilon}_n$, where we set $\widehat{\varepsilon}_n = \sum_{k\geq n} \varepsilon_k$. Then, if x and y both lie in the interior of $F^l \cap B(0, 194/100)$,

$$\begin{aligned} |Df(x) - Df(y)| &\leq |Df(x) - Df_n(x)| + |Df_n(x) - Df_n(y)| + |Df_n(y) - Df(y)| \\ &\leq C\widehat{\varepsilon}_n + |Df_n(x) - Df_n(y)| \leq C\widehat{\varepsilon}_n + C\varepsilon_n 2^n |x - y|, \end{aligned} \tag{10.21}$$

by (10.19). We choose n such that $2^{-n} \leq |x - y| < 2^{-n+1}$, and we get that

$$|Df(x) - Df(y)| \leq C\widehat{\varepsilon}_n \leq C \sum_{k\geq 0\,;\,|x-y|\leq 2^{-k}} \varepsilon(2^{-k}). \tag{10.22}$$

Of course we can let x or y tend to the boundary of F^l. So we proved that the restriction of f to each face $F^l \cap B(0, 194/100)$ is of class C^1, with a modulus of continuity for Df that depends on the decay of the function $\varepsilon(r)$. For instance, if $\varepsilon(r) \leq Cr^\alpha$ for some $\alpha \in (0,1)$, we get corresponding $C^{1,\alpha}$ estimates for f on F^l.

Now the reader should not pay too much attention to this more precise result; we proved it because it is easy (once we have the other estimates), but it is also easy to prove directly that on each face, there is a tangent plane at x that depends nicely on x. Indeed, we can use the good approximation of E near x by planes $P(x, 2^{-k})$ (or sets of type 2 or 3 for the scales that are larger than $\operatorname{dist}(x, E_2)$) to show that the $P(x, 2^{-k})$ vary slowly with k, then tend to a plane $P(x)$. Then $P(x)$ is tangent to E at x, and we can use the fact that $P(x)$ is close to $P(x, 2^{-k})$ to estimate the distance from $P(x)$ to $P(y)$ (compare $P(x, 2^{-k})$ to $P(y, 2^{-k})$, with 2^{-k} a little larger than $|x - y|$).

Our construction does not give a mapping f that is C^1 on $Z(0, 2)$ near its spine, but we claim that in the main case when the sets $Z(x, r)$ are minimal cones, it would not be hard to construct a C^1 mapping.

For the discussion, let us assume that $Z = Z(0, 2)$ is a set of type $G3$; then E_3 is not empty, and without real loss of generality we may assume that its only point is the origin. We know from the discussion above that near $B(0, 1)$, E is composed of C^1 faces, and in particular the direction of the tangent plane to E is continuous on each face. Then E has a tangent cone Z at the origin, whose faces are obtained by taking the tangent planes to the faces of E there. Also, it is rather easy to see that Z is the limit, when r tends to 0, of the $Z(0, r)$, so it satisfies the same constraints as the sets of type $GT3$. If all the sets $Z(x, r)$ are minimal cones, their faces always make 120° angles along their edges (see [D3]), hence this is also the case for Z. We just need to find a C^1 diffeomorphism φ, defined near $B(0, 1)$, and that sends $Z \cap B(0, 1)$ to E. To simplify the notation, we shall not keep track of the precise balls where our mappings are defined, and pretend that we work on the whole space.

Denote by Z^i, $1 \leq i \leq m$, the faces of Z, and by F^i the corresponding faces of E (observe that Z is bilipschitz-equivalent to the set $Z(0, 2)$ used to construct f above, because they are close to each other and have a simple structure). The spine of Z is composed of half lines L^j, $1 \leq j \leq k$, and we denote by γ^j the corresponding curves that compose E_2. By the discussion above, each γ^j is a C^1 curve that leaves from the origin, where it is tangent to L^j.

As a first step, we construct a C^1 diffeomorphism φ_1 that sends each L^j to γ^j, and the three tangent planes to the Z^i along L^j to the tangent planes to the F^i. Let us first parameterize γ^j with L^j. Denote by π^j the orthogonal projection on the line that contains L^j, and for $x \in L^j$ denote by $x + h(x)$ the point of γ_j with the same projection, i.e., such that $\pi^j(h(x)) = 0$. Since γ^j is C^1 and is tangent to L^j at the origin, $|x|^{-1}|h(x)|$ and $|h'(x)|$ are small and tend to 0 at 0.

We want to use Whitney's theorem to extend h from $L = \cup_j L^j$ (where we just defined it) to the whole space, and for this we need to define a jet Dh on L. If e_j denotes the unit vector in the direction of L^j, we have to take $Dh(x) \cdot e_j = h'(x)$ for $x \in L^j$ (where h' denotes the derivative of h along L^j, with a slight abuse of notation).

Fix j, and denote by F^1, F^2, and F^3 the three faces of E that contain γ^j. For $x \in L^j$ and $1 \leq i \leq 3$, denote by $e_i(x)$ the unit vector that is tangent to E at $x + h(x)$, is orthogonal to $h'(x)$, and points in the direction of F^i. Note that $e_i(x)$ is a continuous function of x, because F^i is C^1. Also observe that the three faces F^i, $1 \leq i \leq 3$, make 120° angles at $x + h(x)$, because the sets $Z(x + h(x), r)$, for r small, are sets of type $G2$ whose faces make 120° angles. Then the three $v_i(x)$ make 120° angles, hence they lie in a 2-plane $H^j(x)$. In addition, $H^j(x)$ is a continuous function of x (because it is spanned by $e_1(x)$ and $e_2(x)$). There is a unique isometry $R^j(x)$ from $H^j(0)$ to $H^j(x)$ that maps each $v_i(0)$ to $v_i(x)$; we set $Dh(x) \cdot v = R^j(x)(v) - v$ for $v \in H^j(0)$. Finally, we set $Dh(x) \cdot v = 0$ when v is orthogonal to e_j and to $H^j(0)$. This defines a continuous function Dh on L^j, with $Dh(0) = 0$. Then Dh is also defined and continuous on L.

We also need to check that $|h(y)-h(x)-Dh(x)\cdot(y-x)| \leq |x-y|o(|y-x|)$ for $x,y\in L$ (and some function $o(\cdot)$ that tends to 0 at 0). When x and y lie in the same branch L^j, $Dh(x)\cdot(y-x)$ comes from $h'(x)$, and the estimate just holds because h is C^1 on L^j. When they lie in different branches, $|x|+|y| \leq C|y-x|$, hence $|Dh(x)\cdot(y-x)| \leq |Dh(x)||y-x| \leq o(|x|)|x-y| \leq |x-y|o(|y-x|)$ because Dh is continuous on L^j and $Dh(0)=0$. Since h and h' vanish at 0, then $|h(x)|+|h(y)| \leq o(|x|)|x|+o(|y|)|y| \leq |x-y|o(|y-x|)$ and we get the desired estimate. Thus the Whitney extension theorem alows us to extend h to a C^1 mapping defined on $\mathbb{R}^n$, and even with $||Dh||_\infty$ small (see [EG] §6.5).

Now set $\varphi_1(x)=x+h(x)$; thus φ_1 is a C^1 diffeomorphism. In addition, φ_1 maps L^j to γ^j, and the tangent plane to Z^i at $x\in L$ to the tangent plane to Z^i at $\varphi_1(x)=x+h(x)$. Thus $E'=\varphi_1^{-1}(E)$ is a set composed of C^1 faces like E, but in addition the set $E_2'=\varphi_1^{-1}(E_2)$ is equal to L, and the faces of $\varphi_1^{-1}(E')$ are tangent to the Z^i there. If we ever construct a diffeomorphism φ_2 that sends Z to E', we can take $\varphi=\varphi_1\circ\varphi_2$ and we will be done.

First fix $i\leq m$. Denote by π_i the orthogonal projection onto the plane P^i that contains Z^i. For $y\in Z^i$, denote by $h_i(y)$ the point of $(P^i)^\perp$ such that $y+h_i(y)\in(F')^i$ (where $(F')^i$ is the face of E' that lies close to Z^i). Then h_i is C^1 on Z^i (by the implicit function theorem), and h_i and its derivative both vanish on ∂Z^i (which is a union of two half lines L^j).

We choose a function θ_i such that $0\leq\theta_i\leq 1$, $\theta_i=1$ on Z^i, θ_i is smooth and homogeneous on $\mathbb{R}^n\setminus\partial Z^i$ and supported in $W_i=\{x\,;\ \mathrm{dist}\,(x,Z^i)\leq\alpha|x|\}$ for some small constant $\alpha>0$ (so that the W_i are disjoint), and such that $|\nabla\theta_i(x)|\leq C\,\mathrm{dist}\,(x,\partial Z^i)^{-1}$. Then we set

$$\varphi_2(x)=x+\sum_{1\leq i\leq m}\theta_i(x)\,h_i(\pi_i(x)) \tag{10.23}$$

for $x\in\mathbb{R}^n$. Notice that $\varphi_2(x)\in(F')^i$ on Z^i, by definition of h_i, so we just need to check that φ_2 is C^1, with a differential close to the identity. On $W_i\setminus\partial F^i$,

$$|D\varphi_2(x)-I|\leq|Dh_i(\pi_i(x))|+C\,\mathrm{dist}\,(x,\partial Z^i)^{-1}|h_i(\pi_i(x))| \tag{10.24}$$

because $|\nabla\theta_i(x)|\leq C\,\mathrm{dist}\,(x,\partial Z^i)^{-1}$; it is bounded by a small constant and tends to 0 on ∂F^i because h_i is C^1 and vanishes on ∂Z^i as well as its derivative. Then the gluing along the L^j is C^1, $||D\varphi_2||$ is small, and φ_2 is a C^1 diffeomorphism, as needed. This completes our sketch for the construction of a local C^1 parameterization of E when we have (10.1) and (10.2), with sets $Z(x,r)$ whose faces always make 120° angles along the spine.

Our proof seems to go through in ambient dimension 3, without the 120° angles constraint (use the fact that the $e_i(x)$ span a two-dimensional space $H^j(x)$ and modify the definition of $R_j(x)$), but not in higher dimensions when $e_i(x)$ span a three-dimensional space; to be fair, we did not try to exclude that case.

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