A few words about the final exam next **Wednesday, June 11** (2:30-4:20) in this classroom. The exam will cover all materials in the course up to IP, and will contain about six questions, with a mixture of numerical, conceptual, and proof-type questions based on materials from lectures, midterm/quizzes, and homeworks.

- Two **3" by 5"** index cards of handwritten notes, two-sided, will be allowed.

Some additional practice questions are given below. [Disclaimer: These are practice questions only, so they need not bear any resemblance to questions on the exam.]

1. For each positive integers k and l, let $G_{k,l}$ denote the bipartite graph with k vertices on one side, l vertices on the other side, and an arc joining every vertex on one side to every vertex on the other side. Give all values of k and l for which $G_{k,l}$ has a closed eulerian path. Explain your answer.

2. Consider a digraph $G = (V, A)$ with nonnegative arc capacities c_{uv}, $(u, v) \in A$, and a vertex $s \in V$. For each path P in G, define the "capacity" of P to be the *minimum* of the capacity of all the arcs in P. [Thus, if $P = u_1, u_2, ..., u_{p+1}$, then the capacity of P is $\min\{c_{u_1u_2}, ..., c_{u_pu_{p+1}}\}$.] Indicate how you would modify SP-Dijkstra so to find a path from s to each vertex v reachable from s that is of *maximum* capacity. [Hint: Replace "minimum" and "+" in SP-Dijkstra by suitable arithmetic operations.]

3. Consider a bipartite graph $G = (V, A)$ with $V = X \cup Y$. Suppose there exists integer $p \geq 1$ such that $\deg(u) \geq p$ for all $u \in X$ and $\deg(v) \leq p$ for all $v \in Y$. Prove, using Hall's theorem, that the graph has a matching M with $|M| = |X|$.

4.
(a) Give an example of an integer program (IP) that has exactly two feasible solutions. Give an example of an IP that has an infinite number of optimal solutions.
(b) Consider a max IP whose cost function is of the form $c_1x_1 + \cdots + c_nx_n$, where $c_1, ..., c_n$ are integers. Suppose we solve the LP relaxation of this IP and obtain an optimal solution $(x_1^*, ..., x_n^*)$ with non-integer cost, i.e., $c_1x_1^* + \cdots + c_nx_n^*$ is not an integer. Write down a linear inequality that, when added to the LP relaxation, will (i) exclude $(x_1^*, ..., x_n^*)$ from its feasible region and (ii) not exclude any optimal solution of the IP.

5. Consider a digraph $G = (V, A)$ with arc capacities $c_{uv} > 0 \ \forall (u, v) \in A$, and $s \neq t \in V$.

(a) Suppose u_1, u_2, u_3, u_4 is an s-t augmenting path relative to the s-t flow $\{x_{uv}\}_{(u,v)\in A}$. Suppose (u_2, u_3), (u_3, u_4) are forward arcs and (u_2, u_1) is a backward arc. Write down a formula for the augmentation amount Δ.

(b) You notice that there is an s-t cut comprising three forward arcs with capacities of 3, 2, 3. Suppose a psychic tells you that there exists an s-t flow of value 9. Should you believe the psychic? Explain.

Brief Answers to Selected Problems (hopefully correct)

Don't peek at the answers until you have tried doing the problems!

2. The modifications to SP-Dijkstra are as follows:
In Step 0, initialize $d_s \leftarrow \infty$ instead.
In Step 1, pick an $(u, v) \in A$ with $u \in W, y \notin W$ and whose $\min\{d_u, c_{uv}\}$ is maximum among all such arcs; set $d_v \leftarrow \min\{d_u, c_{uv}\}$ instead.

3. For any nonempty $U \subseteq X$, since (i) each vertex in U has degree at least p, (ii) the arcs joined to U form a subset of the arcs joined to $N_A(U)$, (iii) each vertex in $N_A(U)$ has degree at most p, we have the following three inequalities:

$$p|U| \leq (\text{total \#arcs joined to } U) \leq (\text{total \#arcs joined to } N_A(U)) \leq p|N_A(U)|.$$

Dividing both sides by p gives $|U| \leq |N_A(U)|$. Use Hall's theorem, etc.

4. (b) Any IP optimal soln $(x_1, ..., x_n)$ has cost no greater than the cost of the LP relaxation optimal soln $(x_1^*, ..., x_n^*)$, so

$$c_1x_1 + \cdots + c_nx_n \leq c_1x_1^* + \cdots + c_nx_n^*.$$

Since $c_1, ..., c_n$ are integer, the left-hand side is integer, so

$$c_1x_1 + \cdots + c_nx_n \leq \lfloor c_1x_1^* + \cdots + c_nx_n^* \rfloor.$$

Add the above inequality constraint to the IP.

5.

(a) $\Delta = \min\{x_{u_2u_1}, c_{u_2u_3} - x_{u_2u_3}, c_{u_3u_4} - x_{u_3u_4}\}$.

(b) The s-t cut has capacity $3 + 2 + 3 = 8$. By weak duality, every s-t flow has value at most 8. Thus there cannot exist an s-t flow of value 9. So, do NOT believe the psychic!