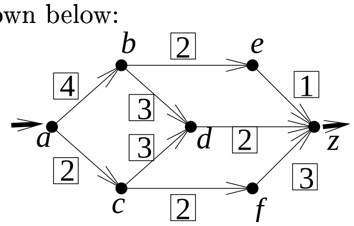
[Homework can be handed in to me or to my mail box in the Math Lounge (opposite the Math main office). Please show your work to receive full credit.]

A. Consider a digraph $G = (V, A)$ and two vertices $s \neq t \in V$.
(a) To hedge against possible arc failures, we wish to find the largest number of paths from s to t that are arc-disjoint (i.e., no two paths share any arc). Formulate this as a maximum flow problem. Explain your answer. [Hint: Use unit capacities.]
(b) If instead of the paths being arc-disjoint, we require that at most k paths (k a positive integer) share any arc. How would your answer to (a) change?

B. Consider the capacitated digraph shown below:



(a) Apply the algorithm MAXFLOW to find a maximum flow and minimum cut from a to z.
(b) For the a-z flow $x_{ab} = x_{be} = x_{ez} = 1$, $x_{ac} = x_{cd} = x_{dz} = 2$ (with zero flow on remaining arcs), find the residual digraph and an a-z augmenting path. Augment flow along the autmenting path to obtain a new a-z flow.

C. Prove that if $\{x_{uv}\}_{(u,v)\in A}$, is an s-t flow of value v and $[S, T]$ is an s-t cut of capacity c (with $T = V \backslash S$), then $v = c$ if and only if $x_{uv} = c_{uv}$ for every $(u, v) \in [S, T]$ and $x_{vu} = 0$ for every $(v, u) \in [T, S]$. [You can use the fact that $v = \sum_{(u,v)\in[S,T]} x_{uv} - \sum_{(v,u)\in[T,S]} x_{vu}$ for any s-t flow $(x_{uv})_{(u,v)\in A}$ of value v.]

D.
(a) Give an example of a digraph $G = (V, A)$ with arc capacities of 1 and two vertices s, t such that $|V| \leq 4$ and the s-t flow of maximum value is not unique.
(b) For your answer to (a). Find an s-t flow of maximum value whose arc flows are *not* all integer.
(c) Give an example of a digraph $G = (V, A)$ with arc capacities of 1 and two vertices s, t such that $|V| = 3$ and the s-t cut of minimum capacity is not unique.
(d) Consider a digraph $G = (V, A)$ with arc capacities c_{uv}, $(u, v) \in A$, and $s \neq t \in V$. Suppose u_1, u_2, u_3, u_4 is an s-t augmenting path relative to an s-t flow $(x_{uv})_{(u,v)\in A}$. Suppose (u_2, u_3) is a forward arc and $(u_2, u_1), (u_4, u_3)$ are backward arcs. Write down a formula for the augmentation amount Δ in terms of the flow and capacity of these arcs.

Bonus. Consider a digraph $G = (V, A)$ with arc capacities c_{uv}, $(u, v) \in A$, and $s \neq t \in V$. Suppose that $[S_1, T_1]$ and $[S_2, T_2]$ are two s-t cuts of minimum capacity. Prove that $[S_1 \cap S_2, T_1 \cup T_2]$ and $[S_1 \cup S_2, T_1 \cap T_2]$ are also s-t cuts of minimum capacity. [Hint: First show that they are s-t cuts. Then show that their capacities equal that of $[S_1, T_1]$ and $[S_2, T_2]$ by considering arcs out of $S_1 \cap S_2$, $S_1 \backslash S_2$, $S_2 \backslash S_1$.]