Lifting K3 Surfaces to Characteristic 0

Notes on Deligne's Article by Matt Ward

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1 Preliminaries

Warning: These are just notes I typed while reading the paper which can be found in *Surfaces Algébriques* edited by Giraud, Illusie, and Raynaud. This is not well-referenced, but all the appropriate references can be found at the end of original paper. For the most part, I have tried to add the information from referenced facts so that one doesn't need to continually look random facts up. I then just wrote the place I found it (not properly cited).

Again, as just noted this is merely a translation of the paper written by Illusie plus a few added remarks found in other papers and none of this should be attributed to me. Since these are only notes there are probably many typos and several places can certainly be expanded to be made clearer. I'd love to know about these!! Please email me suggestions and typos.

Let k be an algebraically closed field of characteristic $p > 0$, and denote $W = W(k)$ the ring of Witt vectors on k. It is well-known (e.g. in Serre's *Local Fields*) that for any perfect field, there is a unique absolutely unramified DVR with residue field k and fraction field of characteristic 0. This is our W. Another way to think of W is as infinite sequences of elements from k, e.g. $(a_0, a_1, a_2, \dots)$ endowed with the unique multiplication and addition that make W a ring with the properties that the sum and product are given by polynomials with integral coefficients, and every Witt polynomial is a homomorphism $W \to W$.

Note that localization at the generic point $W_{(0)}$ gives a characteristic 0 field, which we will try to lift to. Given a flat map $X \to \mathrm{Spec}(W)$, the closed fiber X_0 is a scheme defined over k and the generic fiber X_η is a scheme defined over a characteristic 0 field. So X_η is a deformation of X_0, i.e. a lift of X_0 to characteristic 0.

Let X_0 be a K3 surface over k. Our definition of a K3 surface is a projective surface over k such that $\omega_{X_0} = \Omega^2_{X_0} \simeq \mathcal{O}_{X_0}$, and $H^1(X, \mathcal{O}_X) = 0$.

Proposition 1.1 *1. The Hodge-de Rham Spectral Sequence* $E_1^{ij} = H^j(X_0, \Omega^i_{X_0/k}) \Rightarrow H^*_{DR}(X_0/k)$ *degenerates at the* E_1*-page, and the Hodge numbers* $h^{ij} = \dim_k H^j(X_0, \Omega^1_{X_0/k})$ *are*

$$\begin{matrix} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{matrix}$$

2. $H^i(X_0, T) = 0$ *for* $i = 0$ *or* 2*, and* $\dim_k H^1(X_0, T) = 20$ *where* $T \simeq \Omega^\vee_{X_0}$ *is the tangent sheaf.*

3. The W*-modules of the crystalline cohomology* $H^i_{crys}(X_0/W)$ *are free of rank* $1, 0, 22, 0, 1$ *for* $i = 0, 1, 2, 3, 4$.

Proof In the paper that is impossible to find *Inseparable Morphisms of Algebraic Surfaces* by Rudakov and Shafarevich it is proved that K3 surfaces cannot admit a global vector field, i.e. $H^0(X_0, T) = 0$. By triviality of $\Omega^2_{X_0} \simeq \mathcal{O}_{X_0}$ we get that $T \simeq \Omega^1_{X_0}$. The previous vector field comment immediately shows that $h^{1,0} = 0$ and by Serre duality $H^2(X_0, \Omega^1_{X_0}) \simeq H^0(X_0, T)^\vee = 0$ so $h^{1,2} = 0$. Now all of the other Hodge numbers except the middle one follow immediately from the K3 property or the fact that X_0 is a projective surface over an algebraically closed field.

The E_1 page of the Hodge-de Rham Spectral Sequence (H-dR SS) only has a possiblity of 9 non-zero terms. The shown table of Hodge numbers gives the dimension of the k-vector space in each of these places (except the middle which is still unkown). The zeros are the important part since every differential at a non-zero place comes from a 0 and goes to a 0. Thus the E_2 page is exactly the same as the E_1. Checking the E_2 page gives that the same thing happens, so $E_2^{ij} = E_3^{ij}$. Once we hit E_3 all differentials are long enough that they are in the zero range and hence $E_3^{ij} = E_\infty^{ij}$. Without even knowing the middle spot in the Hodge diamond we can conclude that the H-dR SS degenerates at E_1.

Let's now assume that $h^{1,1} = s$. From the H-dR SS we know that $\dim_k H^i_{dR}(X_0/k) = 1, 0, s+2, 0, 1$ for $i = 0, 1, 2, 3, 4$. These are sometimes called the de-Rham Betti numbers which are known to satisfy, even in positive characteristic, $c_2(X_0) = 1 - 0 + (s+2) - 0 + 1 = s + 4$. Thus we only need to calculate $c_2(X_0)$. Using Serre duality and the above we have $\chi(\mathcal{O}_{X_0}) = 2$, and since K_{X_0} is trivial, $c_1^2(K) = K \cdot K = 0$. These are all the pieces of the Noether formula: $c_1^2 + c_2 = 12\chi(X_0)$, which gives $c_2 = 24$, i.e. $s = 20$ as we wanted to show.

We already cited a reference for $H^0(X_0, T) = 0$. The rest of Part 2 follows from part 1 by noting that by triviality of $\Omega^2_{X_0}$, we get $\Omega^1_{X_0} \simeq T$, so $0 = h^{1,2} = \dim H^2(X_0, \Omega^1_{X_0}) = \dim H^2(X_0, T)$ The last part follows similarly from knowing $h^{1,1} = 20$.

Part 3 follows from part 1 by noting that the vector spaces $H^i_{DR}(X_0/k)$ have dimensions $1, 0, 22, 0, 1$, so by the Universal Coefficients Theorem we get the exact sequence $0 \to H^i_{crys}(X_0/W) \to H^i_{DR}(X_0/k) \to \mathrm{Tor}_1^W(H^{i+1}(X_0/W), k) \to 0$, and hence these are the ranks of the crystalline cohomology as free W-modules. ■

Define S to be the formal versal space of deformations X_0 (i.e. the prorepresenting object of Def_{X_0}) on the category of local Artin W-algebras with residue field k.

Corollary 1.2 *S is universal and formally smooth of dimension* 20*. I.e. there is a W-isomorphism to* $\mathrm{Spf}(W[[t_1, \ldots, t_{20}]])$.

Proof The deformations are parametrized by $H^1(X_0, T)$, and we saw in the second part of the theorem that $\dim_k H^1(X_0, T) = 20$. ■

For the purposes of this paper we'll denote the category of local Artin W-algebras with residue field k by Art (suppressing the W and k). Let $\mathcal{L}_0$ be an invertible sheaf on X_0. Define the deformation functor $\mathrm{Def}(X_0, \mathcal{L}_0)$: Art $\to$ Set that sends A to the isomorphism class of couples $(X, \mathcal{L})$ where X is a flat deformation of X_0 over A and $\mathcal{L}$ is an extension of $\mathcal{L}_0$.

We've already noted that Def_{X_0} is prorepsentable. We also have a forgetful natural transformation $\mathrm{Def}(X_0, \mathcal{L}_0) \to \mathrm{Def}_{X_0}$. Define the universal deformation of X_0 over S by $\mathfrak{X}$.

Proposition 1.3 *The functor* $\mathrm{Def}(X_0, \mathcal{L}_0)$ *is prorepresentable and the forgetful natural transformation is a closed immersion defined by a single equation.*

Proof Note that the proposition means that there is a closed formal subscheme $\Sigma(\mathcal{L}_0) \subset S$ such that $\mathcal{L}_0$ extends to an invertible sheaf on $\mathfrak{X} \times_S \Sigma(\mathcal{L}_0)$, and the extension is unique.

First, assuming we have prorepresentability, we'll show $\eta : \mathrm{Def}(X_0, \mathcal{L}_0) \to \mathrm{Def}_{X_0}$ is a closed immersion. Let $R' \in \mathrm{Art}$ be a complete ring and $S' = \mathrm{Spf}(R')$ and let $(X', \mathcal{L}')$ be a deformation over S' such that for all $A \in \mathrm{Art}$ the associated map $\mathrm{Hom}(R', A) \to \mathrm{Def}(X_0, \mathcal{L}_0)(A)$ is surjective and if $A = k[\varepsilon]$ we have a bijection.

Define R to be the ring of S. Now consider the homomorphism $u : R \to R'$ (from prorepresentability of Def_{X_0}). This induces the following $\mathrm{Hom}(R', A) \to \mathrm{Def}(X_0, \mathcal{L}_0)(A) \to \mathrm{Def}_{X_0}(A) = \mathrm{Hom}(R, A)$ and the composition is $\phi = \mathrm{Hom}(u, A)$. This gives what we intended to show.

We'll show prorepresentability by verifying Schlessinger's criterion. It suffices to check that if u is surjective, then ϕ is injective, so η is bijective. By a lemma in Schlessinger's paper, it is equivalent to prove that if m (respectively m') is the maximal ideal of R (respectively R'), the map $m/(pR + m^2) \to m'/(pR' + m'^2)$ induced by u is surjective, or that the map on tangent spaces at the origen is injective:

$$\mathrm{Hom}(u, k[\varepsilon]) : \mathrm{Def}(X_0, \mathcal{L}_0)(k[\varepsilon]) \to \mathrm{Def}_{X_0}(k[\varepsilon])$$

Let T' be the sheaf on X_0 of automorphisms of the trivial deformation of $(X_0, \mathcal{L}_0)$ over $k[\varepsilon]$. We get a short exact sequence $0 \to \mathcal{O}_{X_0} \to T' \to T \to 0$, where the map $T' \to T$ is defined by forgetting $\mathcal{L}_0$. Note T is considered the sheaf of automorphisms of the trivial deformation of X_0 over $k[\varepsilon]$.

It is standard to identify (canonically) the map on tangent spaces with $H^1(X_0, T') \to H^1(X_0, T)$ coming from $T' \to T$. Since $H^1(X_0, \mathcal{O}_{X_0}) = 0$ the long exact sequence on cohomology shows that this map is injective. Thus Schlessinger's criterion is satisfied and we have prorepresentability of $\mathrm{Def}(X_0, \mathcal{L}_0)$.

Recall that we have a closed immersion of formal schemes $S' \to S$ and on rings this is $u : R \to R' = R/I$. To show that the immersion is defined by a single equation, we must show $I = \ker(u)$ is principal. Consider $S'' = \mathrm{Spf}(R/mI)$. This is a thickening of S' over S and has characteristic 0 ideal I/mI.

The obstruction to extending the sheaf $\mathcal{L}'$ defined above on $\mathfrak{X} \times_S S''$ is an element $a \in H^2(X_0, I/mI) = H^2(X_0, \mathcal{O}_{X_0}) \otimes I/mI$, which can be regarded as an element of I/mI by the choice of a basis of $H^2(X_0, \mathcal{O}_{X_0})$.

Let $\Sigma = \mathrm{Spf}\,(R/(mI + (f)))$ where f is a lift of a to I. We thus get that $S' \subset \Sigma \subset S'' \subset S$, and by construction (and functoriality of the obstruction), $\mathcal{L}'$ extends to $\mathfrak{X} \times_S \Sigma$.

But now by the universal property of S', we see that $S' = \Sigma$, i.e. $mI + (f) = I$. Thus by Nakayama's Lemma $I = (f)$, and hence the immersion is defined by a single equation. ∎

2 Main Results

Now we'll state the main theorem of the paper:

Theorem 2.1 *Let $\mathcal{L}_0$ be a non-trivial invertible sheaf on X_0. Then, with the earlier notation, the formal scheme $\Sigma(\mathcal{L}_0)$ prorepresents* $\mathrm{Def}(X_0, \mathcal{L}_0)$ *and is flat over W of relative dimension* 19*.*

Before embarking on a proof of this theorem, we should examine some of the diverse consequences it has. Another way to think about the theorem is that if f is a defining equation for $\Sigma(\mathcal{L}_0)$ on S, then p does not divide f. It also means that $\Sigma(\mathcal{L}_0)$ does not contain the reduction S_0 of S mod p, i.e. $\mathcal{L}_0$ extends to $\mathfrak{X} \times_S S_0$.

Corollary 2.2 *Let $\mathcal{L}_0$ be a non-trivial invertible sheaf on X_0. There exists a curve T, finite over W, a deformation of X_0 to a formal scheme X flat over T, and an extension of $\mathcal{L}_0$ to an invertible sheaf on X.*

Proof We must prove that there is a W-map $T \to \Sigma(\mathcal{L}_0)$, where T is a curve finite over W. As p is not a zero divisor in R' (the ring of $\Sigma(\mathcal{L}_0)$), there exists a system of parameters $x_1, \ldots, x_n$ of the maximal ideal of R'. The quotient $B = R'/(x_1, \ldots, x_n)$ is quasi-finite over W, and hence finite. Thus there is a W local homomorphism of B to a complete DVR C finite over W and the composition of the homomorphisms $R' \to B \to C$ is what we sought. ∎

We get yet another corollary to which we can apply Grothendieck's existence theorem to get the theorem alluded to in the paper:

Corollary 2.3 *Let $\mathcal{L}_0$ be an ample invertible sheaf on X_0. There exists a curve T finite over S, a deformation X_0 to a proper smooth X over T, and an extension of $\mathcal{L}_0$ to an ample invertible sheaf $\mathcal{L}$ on X.*

This corollary is exactly the hypothesis of Grothendieck's theorem for algebraization, and hence we get a lifting to characteristic 0 via W. We'll disregard whether or not this lifting is proper.

Ogus showed that

1. all K3 surfaces over k lift over W, except perhaps the case of supersingular, non-elliptic, which may not even exist due to a conjecture of Artin.

2. If $p > 2$, all K3 surfaces over k lift over $W[\sqrt{p}]$. So for instance, in the single case where $p = 2$ and X_0 is supersingular is not a part of his results.

Corollary 2.4 *If k is the algebraic closure of a finite field over which the surface X_0 is defined, then the Frobenius morphism acts semi-simply on $H^2(X_0, \mathbb{Q}_\ell)$ (where $\ell \neq p$).*

By using the Weil conjectures for K3 surfaces, the generic fiber of X is a K3 surface. This is because being K3 is stable under generization since the K3 property can be expressed by requiring $\chi(\mathcal{O}) = 2$ in characteristic 0, so X_0 satisfies the hypothesis of the Weil conjectures. The conclusion follows from Serre's paper and from the action of Frobenius on ℓ-adic H^1 of an abelian variety over a finite field is semi-simple.

This last corollary did not need that $H^0(X_0, T_{X_0}) = 0$ because if there exists on X a non-zero vector field, so X_0 will be unirational and the conclusion will be still be true by a trace argument.

3 De Rham Cohomology and a Proof of the Main Theorem

We'll continue with the same notation of the previous parts, namely that X_0 is a K3 surface over k, and $\mathfrak{X}/S$ is the universal W-deformation. The first part of this section will review the Gauss-Manin connection, the Hodge filtration, the Frobenius action, and Chern classes of invertible sheaves.

Let $\Omega^\bullet$ be the de Rham complex for the formal scheme $\mathfrak{X}$ (relative to S). So Ω^i is the projective limit of modules of differentials $\Omega^i_{X\times_S S'/S'}$ where S' runs through the infinitesimal neighborhoods of Spec(k) on S.

Define $f : \mathfrak{X} \to S$ to be the projection. By definition, de Rham cohomology of $\mathcal{X}/S$ is formed as the $\mathcal{O}_S$-modules

$$H^i_{DR}(\mathfrak{X}/S) := R^i f_*(\Omega^\bullet)$$

whereas the Hodge cohomology is the $\mathcal{O}_S$-module

$$H^i(\mathfrak{X}, \Omega^j) := R^i f_*(\Omega^j)$$

Since we are working with regular schemes the $\mathcal{O}_{\mathfrak{X}}$-modules $\Omega^i_{\mathfrak{X}/S}$ are locally free of finite type. Also, Grothendieck's finiteness theorem tells us that both Hodge and de Rham cohomology yield coherent sheaves (coherence is preserved under higher direct images of coherent sheaves of a proper map of noetherian schemes).

Recall the following criterion of Grothendieck which we will not prove here (EGA III 7.5.4): If (A, m, k) is a local ring, and $T_\bullet$ a homological functor $\mathrm{Ab}_A \to \mathrm{Ab}$, commuting with inductive limits, and for all i and all finite type A-modules, M, we have that $T_i(M)$ is of finite type and the canonical map $(T_i(M))^\wedge \to \lim_n T_i(M_n)$ is bijective, then the following are equivalent

1. T_p is exact.
2. T_p is right exact and $T_p(M)$ is a free A-module.
3. The canonical maps $T_{p+1}(A) \to T_{p+1}(k)$ and $T_p(A) \to T_p(k)$ are surjective
4. For all n, the maps $T_{p+1}(A_n) \to T_{p+1}(k)$ and $T_p(A_n) \to T_p(k)$ are surjective.
5. For all n, the functor $N \mapsto T_p(N)$ is exact on the category of A_n-modules of finite type.

There is an E_1 spectral sequence relating these two cohomologies called the Hodge-de Rham spectral sequence:
$E_1^{ij} = H^j(\mathfrak{X}, \Omega^i) \Rightarrow H^*_{DR}(\mathfrak{X}/S)$.

Proposition 3.1 *1. The Hodge spectral sequence degenerates at E_1: The $\mathcal{O}_S$-modules $H^j(\mathfrak{X}, \Omega^i)$ are free of finite type, and the canonical arrows $H^j(\mathfrak{X}, \Omega^i) \otimes k \to H^j(X_0, \Omega^i_{X_0/k})$ are isomorphisms.*

2. The $\mathcal{O}_S$ modules $H^i_{DR}(\mathfrak{X}/S)$ are free of finite type, and the canonical arrows $H^i_{DR}(\mathfrak{X}/S) \otimes k \to H^i_{DR}(X_0/k)$ are isomorphisms.

3. The cup product $\langle , \rangle : H^2_{DR}(\mathfrak{X}/S) \otimes H^2_{DR}(\mathfrak{X}/S) \to H^4_{DR}(\mathfrak{X}/S)$ is a perfect pairing.

Proof The first statement follows by applying Grothendieck's criterion to the functor $M \mapsto H^\bullet(\mathfrak{X}, \Omega^i \otimes f^*M)$. By examining the table of Hodge numbers in the first section and using that the Hodge cohomology is a composition of derived functors, we can apply the Grothendieck spectral sequence to get the first statement.

The second statement follows similarly from the first and in turn implies the third by Poincaré duality. ■

Denote by $\Omega^\cdot_{S/W}$ the "formal" de Rham complex of differentials of S/W: i.e. $\Omega^i_{S/W} = \bigwedge^i \Omega^1_{S/W}$, and $\Omega^1_{S/W}$ is the projective limit of complete modules of differentials $\Omega^1_{S_n/W_n}$ where S_n/W_n is the reduction mod p^{n+1} of S/W. It is free over $\mathcal{O}_S$ with basis $dt_1, \dots, dt_{20}$ if $S \simeq \mathrm{Spf}(W[[t_1, \dots, t_{20}]])$.

The $H^i_{DR}(\mathfrak{X}/S)$ come from the canonically integrable Gauss-Manin connection: $\nabla : H^i_{DR}(\mathfrak{X}/S) \to \Omega^i_{S/W} \otimes H^i_{DR}(\mathfrak{X}/S)$. We'll take the approach of Katz-Oda to construct this connection based on the extension $0 \to f^*\Omega^1_{S/W} \to \Omega^1_{\mathfrak{X}/W} \to \Omega^1_{\mathfrak{X}/S} \to 0$.

3.1 Construction of the Gauss-Manin Connection

For this construction we'll always be working with a smooth scheme Z over a field k (no assumptions here). Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_Z$-modules. We'll write Ω for $\Omega^1_{Z/k}$ and unless otherwise noted, all tensor products will be over $\mathcal{O}_Z$. We say that ∇ is a connection on $\mathcal{E}$ if it is a homomorphism $\nabla : \mathcal{E} \to \Omega \otimes \mathcal{E}$ that satisfies the "Leibniz rule".

In other words, $\nabla(fg) = f\nabla(g) + df \otimes g$. This is the standard shorthand meaning $\nabla(U) : \mathcal{E}(U) \to \Omega(U) \otimes \mathcal{E}(U)$ satisfies the rule where $f \in \mathcal{O}_Z(U)$, $g \in \mathcal{E}(U)$ and df is the image of f under the universal map $\mathcal{O}_Z \to \Omega$.

Given a connection ρ, we get homomorphisms for all i, $\rho_i : \Omega^i \otimes \mathcal{E} \to \Omega^{i+1} \otimes \mathcal{E}$. These are given by $\rho_i(\omega \otimes e) = d\omega \otimes e + (-1)^i \omega \wedge \rho(e)$.

The notation is just the one that makes sense: $\rho(e) \in \Omega \otimes \mathcal{E}$, so it looks like $\tau \otimes \epsilon$. So we define $\omega \wedge \rho(e) = \omega \wedge (\tau \otimes \epsilon)$ to be $(\omega \wedge \tau) \otimes \epsilon \in \Omega^{i+1} \otimes \mathcal{E}$.

Now we define the curvature of the connection $K : \mathcal{E} \to \Omega^2 \otimes \mathcal{E}$ to be the map $\rho_1 \circ \rho$. The curvature is related to the other ρ_i by an easy check $\rho_{i+1} \circ \rho_i(\omega \otimes e) = \omega \wedge K(e)$.

This gives some sort of meaning to the curvature now. If the curvature is 0, then the natural de Rham-like sequence we get from a connection by stringing together the ρ_i as follows $0 \to \mathcal{E} \overset{\rho}{\to} \Omega \otimes \mathcal{E} \overset{\rho_1}{\to} \Omega^2 \otimes \mathcal{E} \to \cdots$

is an honest complex that we can take cohomology with respect to, since $\rho_{i+1} \circ \rho_i = 0$.

When this happens we call the connection ρ integrable. Now let $\mathcal{D}er_k(\mathcal{O}_Z)$ be the sheaf of germs of k-derivations of $\mathcal{O}_Z$ into itself. From the fact that the module of differentials is a representing object, we get that as a sheaf of $\mathcal{O}_Z$-modules, $\mathcal{D}er_k(\mathcal{O}_Z) \simeq \mathcal{H}om_{\mathcal{O}_Z}(\Omega, \mathcal{O}_Z)$.

Let $\mathcal{E}nd_k(\mathcal{E})$ be the sheaf of germs of k-linear endomorphisms of $\mathcal{E}$. Given any connection ρ on $\mathcal{E}$ we get an induced $\mathcal{O}_Z$-linear map $\mathcal{D}er_k(\mathcal{O}_Z) \to \mathcal{E}nd_k(\mathcal{E})$ as follows. Let δ be a derivation, then it corresponds to a map $D : \Omega \to \mathcal{O}_Z$.

So consider the composition $\overline{D} : \mathcal{E} \to \Omega \otimes \mathcal{E} \to \mathcal{O}_Z \otimes \mathcal{E} \simeq \mathcal{E}$, where the first is the connection and the second is $D \otimes Id$. This gives the map $\mathcal{D}er_k(\mathcal{O}_Z) \to \mathcal{E}nd_k(\mathcal{E})$ as $\delta \mapsto \overline{D}$.

Note that we get a nice relation between D and $\overline{D}$ as follows $\overline{D}(fe) = D(f)e + f\overline{D}(e)$ and that any map $\mathcal{D}er_k(\mathcal{O}_S) \to \mathcal{E}nd_k(\mathcal{E})$ satisfying this relation comes from a unique connection on $\mathcal{E}$.

Let $\pi : Y \to Z$ be a smooth map of (smooth) k-schemes. We define the relative de Rham cohomology sheaf to be the quasi-coherent sheaf $\mathcal{H}^q_{dR}(Y/Z) := \mathbf{R}^q\pi_*(\Omega^\bullet_{Y/Z})$. Note that the R is bolded to mean we take hypercohomology, so that the $\mathbf{R}^q\pi_*$ is the hyperderived functor of π_*. Note that $\mathcal{H}^q_{dR}(Y/Z)$ are sheaves of graded anticommutative $\mathcal{O}_Z$-algebras.

We now will describe a canonical integrable connection on these sheaves called the Gauss-Manin Connection denoted $\nabla := \nabla(Y/Z, q)$. First notice that whenever we have a smooth map the fundamental sequence $0 \to \pi^*(\Omega_{Z/k}) \to \Omega_{Y/k} \to \Omega_{Y/Z} \to 0$ is exact. We now consider the filtration on the complex $\Omega^\bullet_{Y/k}$.

$$\Omega^\bullet_{Y/k} = F^0(\Omega^\bullet_{Y/k}) \supset F^1(\Omega^\bullet_{Y/k}) \supset F^2(\Omega^\bullet_{Y/k}) \supset \cdots$$

where $F^i(\Omega^\bullet_{Y/k}) = \mathrm{im}\left(\Omega^\bullet_{Y/k}[-i] \otimes_{\mathcal{O}_Y} \pi^*\Omega^i_{Z/k} \to \Omega^\bullet_{Y/k}\right)$

The smoothness tells us that $\Omega^i_{Y/k}$ and $\Omega^i_{Z/k}$ are locally free, so the exactness of the fundamental sequence allows us to compute the associated graded with respect to this filtration as $\mathrm{gr}^i := \mathrm{gr}^i(\Omega^\bullet_{Y/k}) = F^i/F^{i+1} = \pi^*(\Omega^i_{Z/k}) \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/Z}[-i]$.

Now the functor $\mathbf{R}^0\pi_*$ takes complexes of (abelian) sheaves on Y to complexes of sheaves on Z, and the derived functors are just $\mathbf{R}^q\pi_*$. Now we can take the spectral sequence associated to a filtration. It takes the form $E_1^{p,q} = \mathbf{R}^{p+q}\pi_*(\mathrm{gr}^p) \Rightarrow \mathbf{R}^q\pi_*(\Omega^\bullet_{Y/k})$.

We can just work out that E_1 term more explictly using our previous calculation.

Substitution gives $\mathbf{R}^{p+q}\pi_*(\mathrm{gr}^p) = \mathbf{R}^{p+q}\pi_*(\pi^*(\Omega^p_{Z/k}) \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/Z}[-p])$, then just by shifting for the first equality, and then projection formula for next isomorophism that whole thing is $\mathbf{R}^q\pi_*(\pi^*(\Omega^p_{Z/k}) \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/Z}) \simeq \Omega^p_{Z/k} \otimes_{\mathcal{O}_Z} \mathbf{R}^q\pi_*(\Omega^\bullet_{Y/Z}) = \Omega^p_{Z/k} \otimes_{\mathcal{O}_Z} \mathcal{H}^q_{dR}(Y/Z)$.

Let's consider the E_1 terms. The degree of d_1 is $(1,0)$ so we get maps (for every q) of the following form $0 \to \mathcal{H}^q_{dR}(Y/Z) \to \Omega^1_{Z/k} \otimes \mathcal{H}^q_{dR}(Y/Z) \to \Omega^2 \otimes \mathcal{H}^q_{dR}(Y/Z) \to \cdots$. The maps shown here are $d_1^{0,q}$ and $d_1^{1,q}$.

It turns out that $\nabla = d_1^{0,q}$ is actually a flat connection. We'll check this next, but taking this to be true, we get the Gauss-Manin connection on $\mathcal{H}^q_{dR}(Y/Z)$. It is just one of the maps we get from the spectral sequence associated to the filtration of the complex $\Omega^{\cdot}_{Y/k}$. Now the filtration is compatible with taking wedge products ($F^i \wedge F^j \subset F^{i+j}$) and the functors $\mathbf{R}^q\pi_*$ are multiplicative, so we have a product structure on the terms of the spectral sequence as follows.

If we take sections of the sheaves over an open, then $E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'}$ by $(e, e') \mapsto e \cdot e'$. If you want the actual construction see Godement. The product satisfies a few important properties. We have a type of anti-commutativity $e \cdot e' = (-1)^{(p+q)(p'+q')} e' \cdot e$. Also we know how it behaves under the differential map: $d_r(e \cdot e') = d_r(e) \cdot e' + (-1)^{p+q} e \cdot d_r(e')$.

In particular, let's look at what this product rule for the differential is for the Gauss-Manin map. For $\nabla = d_1^{0,q} : E_1^{0,q} \to E_1^{1,q}$ which is really mapping $\mathcal{H}^q_{dR}(Y/Z) \to \Omega_{Z/k} \otimes \mathcal{H}^q_{dR}(Y/Z)$, the differential is really just $d_{Z/k} \otimes Id$. Thus that rule says that $\nabla(\omega \cdot e) = d\omega \cdot e + (-1)^0 \omega \cdot \nabla(e)$. This shows that it is a connection.

The curvature is easily seen to be $d_1^{1,q} \circ d_1^{0,q}$ and since the d_1's are maps of a complex we get that it is 0, and hence ∇ is flat and hence the Gauss-Manin connection is integrable. We've now proved the theorem that any smooth k-morphism of smooth k-schemes gives rise to a canonical integrable connection on the relative de Rham cohomology sheaves that is compatible with the cup product.

There is a more explicit way to see what the map is by just a tedious calculuation of how it appears in the spectral sequence. We quickly find that it is the connecting homomorphism when taking the long exact sequence after applying the functor $\mathbf{R}^q\pi_*$ to the exact sequence $0 \to \mathrm{gr}^{p+1} \to F^p/F^{p+2} \to \mathrm{gr}^p \to 0$.

One would hope (via a conjecture of Grothendieck) that there is some sort of relative de Rham-Leray Spectral Sequence: $E_2^{p,q} = \mathbf{H}^p(S, \Omega^\bullet_{Z/k} \otimes_{\mathcal{O}_Z} \mathcal{H}^q(Y/Z)) \Rightarrow H^{p+q}_{dR}(Y/k)$. For the E_2-term to make any sort of sense we needed $\Omega^\bullet_{Z/k} \otimes_{\mathcal{O}_Z} \mathcal{H}^q(Y/Z)$ to be a complex, and since the Gauss-Manin connection is integrable it is a complex. Also, $H^{p+q}_{dR}(Y/k)$ is defined to be $\mathbf{H}^{p+q}(Y, \Omega^\bullet_{Y/k})$.

It turns out that when Z is affine such a spectral sequence exists. Affineness is needed for a nice proof of this because it makes certain cohomologies vanish. Deligne has proved it in a more complicated way when Z is not affine (but still with our standing assumptions).

3.2 Application of Gauss-Manin

Let's now return to our previous notation. In our situation we have that

$$H^i_{DR}(\mathfrak{X}/S) = R^i(f_0)_{crys*}(\mathcal{O}_{\mathfrak{X}_0/W})(S) \quad (*),$$

since $H^i_{DR}(\mathfrak{X}/S)$ is the value on S of a crystal of $\mathcal{O}$-modules on the crystalline site of S_0/W.

Let's unravel this notation a little. First $f_0 : \mathfrak{X}_0 \to S_0$ is just the map induced by pullback $f \times_S S_0$ where $S_0 = S \otimes_W k \simeq \mathrm{Spf}(k[[t_1, \ldots, t_{20}]])$ and $(f_0)_{crys} : (\mathfrak{X}_0/W)_{crys} \to (S_0/W)_{crys}$ is the morphism on the corresponding crystalline sites. It is just deduced by applying morphisms induced by f_0 on the infinitesimal opens of $\mathrm{Spec}(k)$ on S_0.

Now if we use the way ∇ is compatible with the cup product we see that $\langle \nabla x, y\rangle + \langle x, \nabla y\rangle = \nabla\langle x, y\rangle$ where $x, y \in H^2_{dR}(\mathfrak{X}/S)$.

Define $F^i_{Hdg}H^2_{dR}(\mathfrak{X}/S)$ to be the Hodge filtration given by the cohomology of the truncated complexes $\Omega^{\geq i}_{\mathfrak{X}/S} = [0 \to \Omega^i \to \Omega^{i+1} \to \cdots] \hookrightarrow \Omega^\bullet_{\mathfrak{X}/S}$. The induced map on cohomology is an isomorphism $H^2(\mathfrak{X}, \Omega^{\geq i}) \xrightarrow{\sim} F^i_{Hdg}H^2_{dR}(\mathfrak{X}/S)$.

More explicitly, since we have degeneration of the H-dR SS the Hodge filtration is just given by the decomposition of the de Rham cohomology in terms of the Hodge cohomology. Thus $F^i_{Hdg} = \bigoplus_{j=i}^{2} H^{2-j}(\mathfrak{X}, \Omega^j_{\mathfrak{X}/S})$. This shows us that $H^2_{dR}(\mathfrak{X}/S) = F^0_{Hdg} \supset F^1_{Hdg} \supset F^2_{Hdg} \supset F^3_{Hdg} = 0$.

The F^i_{Hdg} are all free $\mathcal{O}_S$-modules and we have the following natural isomorphisms of the associated graded objects $\mathrm{gr}^i = F^i/F^{i+1}$ (we assume $F = F^\bullet_{Hdg}$): $H^2(\mathfrak{X}, \Omega^i) \xrightarrow{\sim} \mathrm{gr}^i_F\, H^2_{dR}(\mathfrak{X}/S)$.

The cup product decomposes compatibly with the filtration by noting $F^1_{Hdg} = (F^2_{Hdg})^{\perp}$. This does not require anything about the K3 property or being a surface. This can be checked after base-changing to $\mathbb{C}$ where we have the Dolbeaut resolution. This in general gives $H^q(X, \Omega^p) \simeq H^{p,q}$ the subspace of $H^n_{dR}(X)$ generated by $dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z_{k_1}} \wedge \cdots \wedge d\overline{z_{k_q}}$.

The cup product $H^n_{dR}(X) \otimes H^n_{dR}(X) \to H^{2n}_{dR}(X) \to \mathbb{C}$ in these terms is simply $\langle \omega, \eta \rangle = \int_X \omega \wedge \eta$. In great generality, whenever we have such a Hodge decomposition, by a local computation we see the wedge itself gives a zero form that we integrate to get the perpendicular relations on the filtration. For example, if X is a surface we get the relation we wanted $(F^2)^{\perp} = F^1$ and if X is a threefold $(F^3)^{\perp} = F^1$ and $(F^2)^{\perp} = F^2$.

Going back to our actual situation, the Gauss-Manin connection satisfies Griffiths transversality which says that $\nabla F^i_{Hdg} H^2_{dR}(\mathfrak{X}/S) \subset \Omega^1_{S/W} \otimes F^{i-1}_{Hdg} H^2_{dR}(\mathfrak{X}/S)$. One should think of this as saying the Guass-Manin connection respects the Hodge filtration "up to a shift of 1". This allows us to pass to the quotient and get a map on associated graded objects: $\mathrm{gr}^i \nabla : \mathrm{gr}^i_F H^2_{dR}(\mathfrak{X}/S) \to \Omega^1_{S/W} \otimes \mathrm{gr}^{i-1}_F H^2_{dR}(\mathfrak{X}/S)$. Using the Leibniz rule we see that this is $\mathcal{O}_S$-linear. By dualizing and using locally freeness we see that such a map is equivalent to a map $\nabla_i : T_{S/W} \to \mathcal{H}om(\mathrm{gr}^i_F H^2_{dR}(\mathfrak{X}/S), \mathrm{gr}^{i-1}_F H^2_{dR}(\mathfrak{X}/S))$ where $T_{S/W}$ is just the usual tangent bundle.

Using our canonical isomorphism of associated graded objects earlier we get the Hom is isomorphic to $\mathcal{H}om(H^{2-i}(\mathfrak{X}, \Omega^i_{\mathfrak{X}/S}), H^{3-i}(\mathfrak{X}, \Omega^{i-1}_{\mathfrak{X}/S}))$. By a technical computation (all of the details of which are given in the article by Katz) this map is related to the Kodaira-Spencer map $\mathrm{Kod}(\mathfrak{X}/S) : T_{S/W} \to H^1(\mathfrak{X}, T_{\mathfrak{X}/S})$. These fit into a triangle:

$$\begin{array}{ccc} T_{S/W} & \longrightarrow & H^1(\mathfrak{X}, T_{\mathfrak{X}/S}) \\ & \searrow & \downarrow \\ & & \mathcal{H}om(H^{2-i}(\mathfrak{X}, \Omega^i_{\mathfrak{X}/S}), H^{3-i}(\mathfrak{X}, \Omega^{i-1}_{\mathfrak{X}/S})) \end{array}$$

where the vertical arrow is defined by the interior product $V \otimes \omega \mapsto V \lrcorner \omega$ a mapping $T_{\mathfrak{X}/S} \otimes \Omega^i_{\mathfrak{X}/S} \to \Omega^{i-1}_{\mathfrak{X}/S}$. I.e. the Gauss-Manin map is just a twisted form of the standard Kodaira-Spencer map. The idea behind this is very general: If $0 \to A \to B \to C \to 0$ is a short exact sequence of $\mathcal{O}_X$ modules (for say a smooth projetive X), then the coboundary $H^q(X, C) \to H^{q+1}(X, A)$ is just the cup product with the extension class in $Ext^1(C, A)$. Taking this fact the above proof is nothing more than noting that the extension class associated to $0 \to f^*\Omega^1_{S/W} \to \Omega^1_{\mathfrak{X}/W} \to \Omega^1_{\mathfrak{X}/S} \to 0$ is the definition of the Kodaira spencer class.

Now if we take the de Rham cup product from earlier and pass to associated graded objects we get a perfect pairing $\mathrm{gr}^i_F H^2_{dR}(\mathfrak{X}/S) \otimes \mathrm{gr}^{2-i}_F H^2_{dR}(\mathfrak{X}/S) \to H^4(\mathfrak{X}/S)$. Using the compatibility between the G-M connection and the pairing we see that $\nabla_1(D) = -\nabla_2(D)^\vee$ for all $D \in T_{S/W}$. I.e. $\langle \nabla_2(D)x, y\rangle + \langle x, \nabla_1(D)y\rangle = 0$ for all $x \in \mathrm{gr}^2_F$ and $y \in \mathrm{gr}^1_F$.

Proposition 3.2 *The maps ∇_i and* $\mathrm{Kod}(\mathfrak{X}/S)$ *are both isomorphisms and hence all maps in the triangle are isomorphisms.*

Examining the long exact sequence $\cdots \to H^1(X_0, \mathcal{O}_{X_0}) \to H^1(X_n, \mathcal{O}^*_{X_n}) \to H^1(X_{n-1}, \mathcal{O}^*_{X_{n-1}}) \to H^2(X_0, \mathcal{O}_{X_0}) \to \cdots$ we see that $Pic(X_n) \to Pic(X_{n-1})$ which is the natural restriction map is injective. Since $\omega_{X_{n-1}} \simeq \omega_{X_n}|_{X_{n-1}}$ we see by induction that the $\Omega^2_{\mathfrak{X}/s}$ is trivial. Thus even on this formal lift $T_{\mathfrak{X}/S} \simeq \Omega^1_{\mathfrak{X}/S}$. This means we can run the arguments from the beginning again to get that $H^1(\mathfrak{X}, T_{\mathfrak{X}/S})$ is free of rank 20 and the canonical map $H^1(\mathfrak{X}, T_{\mathfrak{X}/S}) \otimes k \to H^1(X_0, T_{X_0/k})$ is an isomorphism.

Using this, one can quickly check that $\mathrm{Kod}(\mathfrak{X}/S) \otimes_W k : T_{S/W} \otimes k \to H^1(X_0, T_{X_0/k})$ is an isomorphism. This is because the map is just a linear map from the tangent space at the identity to the canonical identification $S \xrightarrow{\sim} Def_{X_0}$, i.e. a tangent vector corresponds to deforming in that direction.

Now that $\mathrm{Kod}(\mathfrak{X}/S)$ is an isomorphism, since we know the triangle above is commutative we can show the other isomorphism by checking that the vertical arrow is an isomorphism. Recall that the vertical arrow is merely cupping with a certain extension class. Thus it suffices to check that more generally cupping provides a nondegenerate pairing $H^1(X_0, T_{X_0/k}) \otimes H^1(X_0, \Omega^1_{X_0/k}) \to H^2(X_0, \mathcal{O}_{X_0})$.

By definition, choosing a base for $H^0(X_0, \Omega^2_{X_0/k})$ is a choice of trivializing section $\Omega^2_{X_0/k} \simeq \mathcal{O}_{X_0}$. Thus this choice gives us an isomorphism $T_{X_0/k} \to \Omega^1_{X_0/k}$. The pairing can now seen to be nondegenerate because after this choice and the canonical identifications the pairing is exactly the pairing from Serre duality.

Corollary 3.3 *The map* $\mathrm{gr}^1 \nabla$ *is an isomorphism and* $\mathrm{gr}^2 \nabla$ *is injective with free image.*

This tells us that we get the same properties for $\mathrm{gr}^1(\nabla|_{S_0})$ and $\mathrm{gr}^2(\nabla|_{S_0})$ where $\nabla|_{S_0}$ is the Gauss-Manin connection restricted to $H^2_{dR}(\mathfrak{X}_0/S_0)$ and

$S_0 = k[[t_1, \ldots, t_{20}]]$. It follows now that ∇_i is an isomorphism and hence the proposition is proved. $\blacksquare$

4 Compatibility of Frobenius Action

Define $F_{\mathfrak{X}/S} : \mathfrak{X}_0 \to \mathfrak{X}_0^{(p)}$ to be the relative Frobenius map. Since $H^i(\mathfrak{X}/S) = R^i(f_0)_{crys*}(\mathcal{O}_{\mathfrak{X}/W})(S)$ and crystalline cohomology is functorial we see that $F_{\mathfrak{X}_0/S_0}$ induces for any lift of Frobenius $\phi : S \to S$ from S_0 a horizontal $\mathcal{O}_S$-linear $F(\phi) : \phi^* H^i_{dR}(\mathfrak{X}/S) \to H^i_{dR}(\mathfrak{X}/S)$. This homomorphism is an isogeny and the dependence on choice of lift ϕ is controlled by ∇. From now on we will fix a lift ϕ (for example you can think $\phi(t_i) = t_i^p$) and we'll notate $\overline{F} := F(\phi)$. Lastly define $F : H^i_{dR}(\mathfrak{X}/S) \to H^i_{dR}(\mathfrak{X}/S)$ the ϕ-semilinear map which is the composition of $\overline{F}$ with the adjunction map $H^i_{dR}(\mathfrak{X}/S) \to \phi^* H^i_{dR}(\mathfrak{X}/S)$ by $x \mapsto 1 \otimes x$.

Since all maps involved are compatible with the cup product we see that $\langle Fx, Fy\rangle = F\langle x, y\rangle$. As a special case of a theorem of Mazur and Ogus we get the following proposition:

Proposition 4.1 *With the above notation* $F^1_{Hdg} H^2_{dR}(\mathfrak{X}/S) \subset \{x \in H^2_{dR}(\mathfrak{X}/S) : Fx \in pH^2_{dR}(\mathfrak{X}/S)\}$ *and this is an equality when reduced to* $H^2_{dR}(\mathfrak{X}_0/S_0)$.

In other words, if we consider the reduction to S_0 map $F : H^2_{dR}(\mathfrak{X}_0/S_0) \to H^2_{dR}(\mathfrak{X}_0/S_0)$ and the corresponding Hodge filtration we get $F^1 H^2_{dR}(\mathfrak{X}_0/S_0) = \ker F$. We will now check this. By definition, F is the composition of the canonical injection $H^2_{dR}(\mathfrak{X}_0/S_0) \hookrightarrow H^2_{dR}(\mathfrak{X}_0^{(p)}/S_0) = F^*_{S_0} H^2_{dR}(\mathfrak{X}_0/S_0)$ and the $\mathcal{O}_{S_0}$-linear $\overline{F} : H^2_{dR}(\mathfrak{X}_0^{(p)}/S_0) \to H^2_{dR}(\mathfrak{X}_0/S_0)$ defined by the relative Frobenius.

This shows us that $F^1_{Hdg} H^2_{dR}(\mathfrak{X}_0/S_0) = H^2_{dR}(\mathfrak{X}_0/S_0) \cap F^1_{Hdg} H^2_{dR}(\mathfrak{X}_0^{(p)}/S_0)$. Thus it suffices to prove that $F^1_{Hdg} H^2_{dR}(\mathfrak{X}_0^{(p)}/S_0) = \ker \overline{F}$. Recall that the H-dR SS degenerates at E_1, so this gives us an exact sequence

$$0 \to F^1_{Hdg} H^2_{dR}(\mathfrak{X}_0^{(p)}/S_0) \to H^2_{dR}(\mathfrak{X}_0^{(p)}/S_0) \xrightarrow{\pi} H^2(\mathfrak{X}_0^{(p)}, \mathcal{O}) \to 0$$

where π is the canonical projection.

For the other part, note that $F_{\mathfrak{X}_0/S_0}$ acts trivially on $\Omega^i_{\mathfrak{X}_0^{(p)}/S_0}$ for $i \geq 1$. This gives us a commutative square:

$$\begin{array}{ccc} H^2_{dR}(\mathfrak{X}_0^{(p)}/S_0) & \longrightarrow & H^2(\mathfrak{X}_0^{(p)}, \mathcal{O}) \\ \downarrow & & \downarrow \\ H^2_{dR}(\mathfrak{X}_0/S_0) & \longleftarrow & H^2(\mathfrak{X}_0, \mathcal{H}^0(\Omega^\bullet_{\mathfrak{X}_0/S_0})) \end{array}$$

By the Cartier operator and degeneration of H-dR at E_1 we get that the conjugate spectral sequence degenerates at $E_2^{ij} = H^i(\mathfrak{X}_0, \mathcal{H}^j(\Omega^\bullet_{frx_0/S_0})) \Rightarrow H^*_{dR}(\mathfrak{X}_0/S_0)$. This gives that the bottom arrow is of the square is injective. The right vertical arrow is exactly the Cartier isomorphism. Thus the left vertical arrow and π have the same kernel which proves the proposition. ■

5 Crystalline Chern Classes

A more thorough reference is given in Berthelot and Illusie's "Classes de Chern en Cohomologie Cristalline". Denote $Fil^1\Omega^\bullet_{\mathfrak{X}/S}$ to be the subcomplex of $\Omega^\bullet_{\mathfrak{X}/S}$ given by the kernel of projeting $\Omega^\bullet_{\mathfrak{X}/S} \to \mathcal{O}_{\mathfrak{X}_0}$, i.e. $Fil^1\Omega^\bullet_{\mathfrak{X}/S} = (p\mathcal{O}_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}/S} \to \Omega^2_{\mathfrak{X}/S})$.

Since we defined this as a kernel we have the exact sequence $0 \to Fil^1\Omega^\bullet_{\mathfrak{X}/S} \to \Omega^\bullet_{\mathfrak{X}/S} \to \mathcal{O}_{\mathfrak{X}_0} \to 0$. Define $\left(\Omega^\bullet_{\mathfrak{X}/S}\right)^*$ to be the multiplicative de Rham complex: $\mathcal{O}^*_{\mathfrak{X}} \xrightarrow{dlog} \Omega^1_{\mathfrak{X}/S} \xrightarrow{d} \Omega^2_{\mathfrak{X}/S}$. We get the induced exact sequence $0 \to 1 + Fil^1\Omega^\bullet_{\mathfrak{X}/S} \to \left(\Omega^\bullet_{\mathfrak{X}/S}\right)^* \to \mathcal{O}^*_{\mathfrak{X}_0} \to 0$.

The chern class of a line bundle is now defined to be the image of the line bundle under the coboundary map in the long exact sequence in cohomology (followed by the map induced by log) $c_1 : Pic(\mathfrak{X}_0) = H^1(\mathfrak{X}_0, \mathcal{O}^*_{\mathfrak{X}_0}) \to H^2(\mathfrak{X}, Fil^1\Omega^\bullet_{\mathfrak{X}/S})$. The log map is $1 + Fil^1\Omega^\bullet_{\mathfrak{X}/S} \to Fil^1\Omega^\bullet_{\mathfrak{X}/S}$ which is the identity in positive degrees and $1 + x \mapsto \sum(-1)^i \frac{x^{i+1}}{(i+1)}$ in degree 0. We should check to make sure this map makes sense. First note that since $Fil^1\Omega^\bullet_{\mathfrak{X}/S}$ is defined as the kernel of $\Omega^\bullet_{\mathfrak{X}/S} \to \mathcal{O}_{\mathfrak{X}_0}$ and by being a K3 we have $H^1(\mathfrak{X}_0, \mathcal{O}) = 0$ this tells us that $H^2(\mathfrak{X}, Fil^1\Omega^\bullet_{\mathfrak{X}/S}) \to H^2_{dR}(\mathfrak{X}/S)$ is injective. Thus we can consider c_1 to take values in $H^2_{dR}(\mathfrak{X}/S)$.

Proposition 5.1 *If $\mathfrak{L}_0 \in Pic(\mathfrak{X}_0)$ and $x = c_1(\mathcal{L}_0)$, then the following two formulas hold: $Fx = px$ and $\nabla x = 0$.*

To prove this we will need a crystalline interpretation of c_1. Consider the canonical exact sequence $0 \to J_{\mathfrak{X}_0/W} \to \mathcal{O}_{\mathfrak{X}_0/W} \to \mathcal{O}_{\mathfrak{X}_0} \to 0$. We can define $c_{1/W} : Pic(\mathfrak{X}_0) \to H^2_{crys}(\mathfrak{X}_0, J_{\mathfrak{X}_0/W})$. By a similar arguement to (*) it can be verified that $H^2(\mathfrak{X}, Fil^1\Omega^\bullet_{\mathfrak{X}/S}) = R^2(f_0)_{crys*}(J_{\mathfrak{X}_0/W})(S)$ where we get $H^0_{crys}(S_0, R^2(f_0)_{crys*}J_{\mathfrak{X}_0/W}) = \{x \in H^2(\mathfrak{X}, Fil^1\Omega^\bullet_{\mathfrak{X}/S}) : \nabla x = 0\}$.

Now we see that c_1 is the composition of $c_{1/W}$ with $H^2_{crys}(\mathfrak{X}_0, J_{\mathfrak{X}_0/W}) \to H^0_{crys}(S_0, R^2(f_0)_{crys*}J_{\mathfrak{X}_0/W})$ and then the identification given above. This yields the second formula of the proposition. The first follows from functoriality of crystalline cohomology by considering the absolute Frobenius $F_{\mathfrak{X}_0}$. This gives us $Fx = c_1(F^*_{\mathfrak{X}_0}\mathfrak{L}_0) = c_1(\mathfrak{L}_0^p) = px$. This proves the proposition. ■

Again, by functoriality we get for any W-valued point e on S a commutative diagram.

$$\begin{array}{ccc} Pic(\mathfrak{X}) & \stackrel{c_1}{\to} & H^2_{dR}(\mathfrak{X}/S) = H^2_{crys}(\mathfrak{X}_0/S) \\ \downarrow & & \downarrow \\ Pic(X_0) & \stackrel{c_1}{\to} & H^2_{dR}(X/W) = H^2_{crys}(X_0/W) \end{array}$$

where the vertical arrows are the canonical restrictions and the X/W is basechange induced by e.

The last note of this section is that since X_0 is a K3 surface we know that $Pic(X_0)$ coincides with $NS(X_0)$ and is torsion-free. Thus the bottom arrow is injective.

6 Proof of the Main Theorem

Now we'll use the contents of all of these notes in order to prove the original theorem. Recall that we started with X_0/k and produced a formal lift $\mathfrak{X}/S$. The question is whether or not this algebraizes. Suppose $\mathcal{L}_0$ is an ample invertible sheaf on X_0 that extends to $\mathfrak{L}_0$ on $\mathfrak{X}_0$. Define $x = c_1(\mathfrak{L}_0) \in H^2_{dR}(\mathfrak{X}/S)$.

By the commutative square in the last section $e^*x = c_1(\mathcal{L}_0)$. Now by the non-triviality of $\mathcal{L}_0$ and injectivity of the bottom arrow we get $c_1(\mathcal{L}_0) \neq 0$ and hence $x \neq 0$. Recall the two formulas $Fx = px$ and $\nabla x = 0$. From section 2, we noted that the theorem would be proved if we could show that p does not divide the defining equation which is now equivalent to showing p does not divide x. Suppose for contradiction that it does.

Let p^n be the largest power of p dividing x and set $y = p^{-n}x$. This means that the reduction y_0 of y in $H^2_{dR}(\mathfrak{X}_0/S_0)$ is not 0, but we do get that $Fy = py$ and $\nabla y = 0$. Thus $Fy_0 = 0$ which by the alternate description of the Hodge filtration on H^1_{dR} tells us that $y_0 \in F^1_{Hdg}H^2_{dR}(\mathfrak{X}_0/S_0)$. Now since $y_0 \neq 0$ and $\nabla y_0 = 0$ we cannot have that $y_0 \in F^2_{Hdg}H^2_{dR}(\mathfrak{X}_0/S_0)$ otherwise we would contradict the injectivity of $\operatorname{gr}^2 \nabla$.

But now since y_0 is not 0 in $\operatorname{gr}^1_F H^2_{dR}(\mathfrak{X}_0/S_0)$ and $\nabla y_0 = 0$ we contradict the the fact that $\operatorname{gr}^1 \nabla$ is an isomorphism. Thus no power of p can divide x which means p does not divide the defining equation for $\Sigma(\mathcal{L}_0)$ which proves the theorem. ∎