Elementary Aspects of the Theory of Hecke Operators

by

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§1. Introduction

For some time now, I have felt that it would be a good idea to assemble in one spot the various elementary formalities governing the theory of Hecke operators. This, therefore, is the primary purpose of the present note, the thrust of which is to prepare the way for later work.

Let G be a reductive Lie group, Γ a lattice in G, both subject to the usual conditions. Fix an element ς in the commensurator of Γ in G – then one may associate with ς a bounded linear transformation

$$H(\Gamma\varsigma\Gamma) : L^2(G/\Gamma) \to L^2(G/\Gamma),$$

known as a Hecke operator. $H(\Gamma\varsigma\Gamma)$ intertwines the left G-action and the fundamental problem of the theory is to compute, in explicit terms,

$$\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha)),$$

where, say, α is a K-finite function in $C_c^\infty(G)$. Note that when $\varsigma \in \Gamma$, the fundamental problem reduces to that of the Selberg trace formula itself.

In fact, I believe it was Selberg [16-(a), pp. 68-70] who, following up on ideas of Hecke [4] and others, was the first to pose the problem in just this way. Its importance was stressed once again by him in [16-(b), pp. 188-189].

My original impression was that the calculation of

$$\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha))$$

was somehow a more involved undertaking than the calculation of

$$\mathrm{tr}(L_{G/\Gamma}^{\mathrm{dis}}(\alpha)).$$

However, I have since changed my mind and no longer think this to be the case. Of course, various technical complications do crop up but they seem to be tractable and offer no real additional difficulties. To support this contention, I have included an example in the Appendix (viz., rank(Γ) = 1). While no attempt has been made at an exhaustive discussion, nevertheless what is said there does serve to illustrate the kind of changes that actually do occur.

Regarding the organization, in §§2–3, the various definitions and some of their simple consequences are collected. The fundamental problem in addressed in §4. In §5, the uniform case is considered, there being in principle at least, a positive solution in this situation, while, in §6, the nonuniform case is considered, it being a question here of merely setting the stage, so to speak, for a more serious investigation to be conducted elsewhere.

As a convenient general reference, I shall use the monograph

The Theory of Eisenstein Systems, Academic Press, N.Y., 1981,

henceforth abbreviated to TES.

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§2. The Hecke Algebra

Let G be a reductive Lie group, Γ a lattice in G, both subject to the usual conditions (cf. TES, p. 62) – then one may attach to the pair (G,Γ) its Hecke algebra $\mathcal{H}(G,\Gamma)$, the definition and properties of which will be reviewed below.

We shall agree that the volume of the fundamental domain of any lattice in G is to be calculated with respect to some fixed determination of the Haar measure on G via compatible normalization of the quotient measure and the counting measure.

Recall that two subgroups H_1 and H_2 are said to be commensurable if their intersection $H_1 \cap H_2$ is of finite index in each: Symbolically, $H_1 \sim H_2$.

Put now

$$\mathrm{Com}(\Gamma) = \{\varsigma \in G : \varsigma\Gamma\varsigma^{-1} \sim \Gamma\},$$

the commensurator of Γ in G. $\mathrm{Com}(\Gamma)$ is a subgroup of G containing the normalizer of Γ in G. Depending on the circumstances, it can happen that $\mathrm{Com}(\Gamma) = \Gamma$ or that $G \neq \mathrm{Com}(\Gamma) \neq \Gamma$ or that $\mathrm{Com}(\Gamma) = G$. $\mathrm{Com}(\Gamma)$ may or may not be discrete and can be dense. Obviously,

$$\varsigma \in \mathrm{Com}(\Gamma) \quad \text{iff} \quad \mathrm{vol}(G/\Gamma \cap \varsigma\Gamma\varsigma^{-1}) < +\infty.$$

This said, the Hecke ring $\mathcal{H}_{\mathbf{Z}}(G,\Gamma)$ attached to the pair (G,Γ) is the free abelian group on the double cosets $\Gamma\varsigma\Gamma(\varsigma \in \mathrm{Com}(\Gamma))$ equipped with the following law of multiplication (cf. Shimura [18-(a), pp. 304-305]): If

$$\begin{cases} \Gamma\varsigma'\Gamma &= \coprod\limits_{\iota'} \varsigma'_{\iota'}\Gamma \\ \Gamma\varsigma''\Gamma &= \coprod\limits_{\iota''} \varsigma''_{\iota''}\Gamma, \end{cases}$$

then

$$\Gamma\varsigma'\Gamma * \Gamma\varsigma''\Gamma = \sum_{\varsigma} m(\Gamma\varsigma'\Gamma, \Gamma\varsigma''\Gamma; \Gamma\varsigma\Gamma)\Gamma\varsigma\Gamma,$$

where the sum is taken over all those double cosets $\Gamma\varsigma\Gamma$ contained in $\Gamma\varsigma'\Gamma\varsigma''\Gamma$ and

$$m(\Gamma\varsigma'\Gamma, \Gamma\varsigma''\Gamma; \Gamma\varsigma\Gamma)$$

is the number of pairs (ι', ι'') such that

$$\varsigma'_{\iota'}\varsigma''_{\iota''}\Gamma = \varsigma\Gamma.$$

The multiplication is independent of the choice of representatives. It is associative but need not be commutative; also, $\Gamma 1 \Gamma$ serves as the identity element. That being, let

$$\mathcal{H}(G,\Gamma) = \mathcal{H}_{\mathbf{Z}}(G,\Gamma) \underset{\mathbf{Z}}{\otimes} \mathbf{C}.$$

Then by definition, $\mathcal{H}(G,\Gamma)$ is the Hecke algebra attached to the pair (G,Γ).

One may also look at $\mathcal{H}(G,\Gamma)$ as a convolution algebra (cf. Iwahori [7, p. 218]). For this purpose, let $\mathfrak{A}$ be the set of all Γ-right invariant subsets of $\mathrm{Com}(\Gamma)$ – then $\mathfrak{A}$ is a σ-algebra. Given $A \in \mathfrak{A}$, put

$$\mu(A) = \#(A/\Gamma).$$

Then μ is a measure on $\mathfrak{A}$. As such, it is clearly left invariant under $\mathrm{Com}(\Gamma)$. Consider the set

$$C_c(\Gamma\backslash\mathrm{Com}(\Gamma)/\Gamma)$$

of all complex valued functions on $\mathrm{Com}(\Gamma)$ that are biinvariant under Γ and supported by a finite number of double cosets mod Γ. The elements of

$$C_c(\Gamma\backslash\mathrm{Com}(\Gamma)/\Gamma)$$

are $\mathfrak{A}$-measurable and μ-integrable. In addition, if

$$\phi, \psi \in C_c(\Gamma\backslash\mathrm{Com}(\Gamma)/\Gamma),$$

then their convolution

$$\phi * \psi(?) = \int_{\mathrm{Com}(\Gamma)} \phi(?\varsigma)\psi(\varsigma^{-1})d\mu(\varsigma)$$

is again in

$$C_c(\Gamma\backslash\mathrm{Com}(\Gamma)/\Gamma).$$

The characteristic functions $\chi_{\Gamma\varsigma\Gamma}$ of the double cosets $\Gamma\varsigma\Gamma$ form a basis for

$$C_c(\Gamma\backslash\mathrm{Com}(\Gamma)/\Gamma)$$

over $\mathbf{C}$. They can be used to implement an obvious identification

$$\mathcal{H}(G,\Gamma) \simeq C_c(\Gamma\backslash\mathrm{Com}(\Gamma)/\Gamma)$$

which preserves the multiplication. To check this point, write

$$\chi_{\Gamma\varsigma'\Gamma} * \chi_{\Gamma\varsigma''\Gamma} = \sum_{\varsigma} \mu(\Gamma\varsigma'\Gamma, \Gamma\varsigma''\Gamma; \Gamma\varsigma\Gamma)\chi_{\Gamma\varsigma\Gamma}.$$

Then

$$\begin{aligned} \chi_{\Gamma\varsigma'\Gamma} * \chi_{\Gamma\varsigma''\Gamma}(\varsigma) &= \mu(\Gamma\varsigma'\Gamma, \Gamma\varsigma''\Gamma; \Gamma\varsigma\Gamma) \\ &= \mu(\Gamma\varsigma'\Gamma \cap \varsigma(\Gamma\varsigma''\Gamma)^{-1}), \end{aligned}$$

an integer, nonzero (hence positive) iff $\Gamma\varsigma\Gamma$ is a subset of $\Gamma\varsigma'\Gamma\varsigma''\Gamma$. There is a disjoint decomposition

$$\Gamma\varsigma'\Gamma \cap \varsigma(\Gamma\varsigma''\Gamma)^{-1} = \coprod_{\iota'} \varsigma'_{\iota'}\Gamma \cap \varsigma(\Gamma\varsigma''\Gamma)^{-1},$$

so that

$$\begin{aligned}\mu(\Gamma\varsigma'\Gamma, \Gamma\varsigma''\Gamma; \Gamma\varsigma\Gamma) &= \mu(\Gamma\varsigma'\Gamma \cap \varsigma(\Gamma\varsigma''\Gamma)^{-1}) \\ &= \sum_{\iota'} \mu(\varsigma'_{\iota'}\Gamma \cap \varsigma(\Gamma\varsigma''\Gamma)^{-1}).\end{aligned}$$

However, $\varsigma'_{\iota'}\Gamma$ is contained in $\varsigma(\Gamma\varsigma''\Gamma)^{-1}$ iff there exists an ι'' such that

$$\varsigma'_{\iota'}\varsigma''_{\iota''}\Gamma = \varsigma\Gamma.$$

Accordingly,

$$\begin{aligned}\mu(\Gamma\varsigma'\Gamma, \Gamma\varsigma''\Gamma; \Gamma\varsigma\Gamma) &= \sum_{\iota', \iota''} \mu(\varsigma'_{\iota'}\varsigma''_{\iota''}\Gamma \cap \varsigma\Gamma) \\ &= m(\Gamma\varsigma'\Gamma, \Gamma\varsigma''\Gamma; \Gamma\varsigma\Gamma),\end{aligned}$$

as claimed.

For $\varsigma \in \mathrm{Com}(\Gamma)$, write

$$\begin{cases} \mathrm{ind}_R(\varsigma) & = \#\text{ of right cosets of } \Gamma \text{ in } \Gamma\varsigma\Gamma \\ \mathrm{ind}_L(\varsigma) & = \#\text{ of left cosets of } \Gamma \text{ in } \Gamma\varsigma\Gamma. \end{cases}$$

Then, in our situation,

$$\mathrm{ind}_R(\varsigma) = \mathrm{ind}_L(\varsigma).$$

This is a standard remark. Its verification simply depends on the fact that $\mathrm{vol}(G/\Gamma) < +\infty$. Thus, we have

$$\begin{cases} \displaystyle\int_{G/\Gamma\cap\varsigma\Gamma\varsigma^{-1}} = \int_{G/\Gamma} \int_{\Gamma/\Gamma\cap\varsigma\Gamma\varsigma^{-1}} \\ \displaystyle\int_{G/\Gamma\cap\varsigma^{-1}\Gamma\varsigma} = \int_{G/\Gamma} \int_{\Gamma/\Gamma\cap\varsigma^{-1}\Gamma\varsigma}, \end{cases}$$

from which

$$\begin{cases} \mathrm{vol}(G/\Gamma \cap \varsigma\Gamma\varsigma^{-1}) = \mathrm{vol}(G/\Gamma) \bullet [\Gamma : \Gamma \cap \varsigma\Gamma\varsigma^{-1}] \\ \mathrm{vol}(G/\Gamma \cap \varsigma^{-1}\Gamma\varsigma) = \mathrm{vol}(G/\Gamma) \bullet [\Gamma : \Gamma \cap \varsigma^{-1}\Gamma\varsigma] \end{cases}$$

or still

$$\begin{cases} \mathrm{vol}(G/\Gamma \cap \varsigma\Gamma\varsigma^{-1}) & = \mathrm{vol}(G/\Gamma) \bullet \mathrm{ind}_R(\varsigma) \\ \mathrm{vol}(G/\Gamma \cap \varsigma^{-1}\Gamma\varsigma) & = \mathrm{vol}(G/\Gamma) \bullet \mathrm{ind}_L(\varsigma). \end{cases}$$

Since the volumes on the left hand side are equal, it follows that

$$\mathrm{ind}_R(\varsigma) = \mathrm{ind}_L(\varsigma).$$

Calling their common value $\mathrm{ind}(\varsigma)$ then determines a Γ-biinvariant function

$$\mathrm{ind} : \mathrm{Com}(\Gamma) \rightarrow \mathbf{N}$$

satisfying

$$\mathrm{ind}(\varsigma) = \mathrm{ind}(\varsigma^{-1}).$$

Plainly,

$$\begin{aligned}\mathrm{ind}(\varsigma) &= \mu(\Gamma\varsigma\Gamma) \\ &= \int_{\mathrm{Com}(\Gamma)} \chi_{\Gamma\varsigma\Gamma}\, d\mu.\end{aligned}$$

It should also be noted that the equality of ind_R and ind_L implies that it is always possible to choose a common set of representatives ς_ι such that

$$\begin{cases} \Gamma\varsigma\Gamma &= \coprod\limits_{\iota} \varsigma_\iota\Gamma \\ \Gamma\varsigma\Gamma &= \coprod\limits_{\iota} \Gamma\varsigma_\iota, \end{cases}$$

an observation sometimes useful in calculations.

The Hecke algebra $\mathcal{H}(G,\Gamma)$ admits an adjoint operation $\phi \rightarrow \phi^\star$:

$$\phi^\star(\varsigma) = \overline{\phi(\varsigma^{-1})}$$

such that

(i) $(\phi + \psi)^\star = \phi^\star + \psi^\star$;

(ii) $(c\phi)^\star = \bar{c}\phi^\star$;

(iii) $(\phi * \psi)^\star = \psi^\star * \phi^\star$;

(iv) $(\phi^\star)^\star = \phi$.

Otherwise said, $\mathcal{H}(G,\Gamma)$ is a *-algebra. Norming $\mathcal{H}(G,\Gamma)$ by the prescription

$$||\phi|| = \int_{\mathrm{Com}(\Gamma)} |\phi(\varsigma)|\, d\mu(\varsigma),$$

the equality of ind at ς and ς^{-1} gives

$$||\phi|| = ||\phi^\star||.$$

Finally,

$$||\phi * \psi|| \leq ||\phi|| \bullet ||\psi||.$$

The construction of the Hecke algebra $\mathcal{H}(G,\Gamma)$ is perfectly general and can be conveniently carried out in other contexts as well. For a case in point, let P be a Γ-cuspidal split parabolic subgroup of G – then one may form the Hecke algebra $\mathcal{H}(P,\Gamma\cap P)$ even though $\Gamma\cap P$ is not a lattice in P (if $P\neq G$). Of course,

$$P/(\Gamma\cap P)\bullet A\bullet N \leftrightarrow M/\Gamma_M$$

and Γ_M is a lattice in M. Suppose that

$$\varsigma\in\mathrm{Com}(\Gamma)\cap P.$$

Then

$$\varsigma\Gamma\varsigma^{-1}\cap\Gamma$$

is a lattice in G, so

$$\varsigma\Gamma\varsigma^{-1}\cap\Gamma\cap P=\varsigma(\Gamma\cap P)\varsigma^{-1}\cap(\Gamma\cap P)$$

is a lattice in S. From this, we conclude that

$$\varsigma\in\mathrm{Com}(\Gamma\cap P),$$

the commensurator of $\Gamma\cap P$ in P. There is therefore an injection

$$\Gamma\cap P\backslash\mathrm{Com}(\Gamma)\cap P/\Gamma\cap P\hookrightarrow\mathcal{H}(P,\Gamma\cap P).$$

Because

$$\varsigma=m_\varsigma a_\varsigma n_\varsigma\Rightarrow m_\varsigma\in\mathrm{Com}(\Gamma_M),$$

one can also define a map

$$\Gamma\cap P\backslash\mathrm{Com}(\Gamma)\cap P/\Gamma\cap P\to\mathcal{H}(M,\Gamma_M)$$

that in general, however, fails to be injective. It is for this reason preference is given to P rather than to M.

§3. The Hecke Operators

Let $\mathrm{Fnc}(G/\Gamma)$ be the space of complex valued functions f on G/Γ – then one may associate with each $\phi \in \mathcal{H}(G,\Gamma)$ an endomorphism

$$H(\phi) : \mathrm{Fnc}(G/\Gamma) \to \mathrm{Fnc}(G/\Gamma),$$

commonly referred to as a Hecke operator. Historically, Hecke ([4]) took

$$G = \mathbf{SL}(2,\mathbf{R}), \quad \Gamma = \mathbf{SL}(2,\mathbf{Z})$$

and considered the special elements

$$\varsigma_n = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & 1/\sqrt{n} \end{pmatrix} \qquad (n = 1, 2, \ldots)$$

in the commensurator.

Thus write

$$\phi = \sum_{\varsigma} C_\varsigma \chi_{\Gamma\varsigma\Gamma}.$$

Then, to define $H(\phi)$, we need only specify the effect of $\Gamma\varsigma\Gamma$, which we do by letting

$$H(\Gamma\varsigma\Gamma)f = \sum_{\gamma \in \Gamma/\Gamma \cap \varsigma\Gamma\varsigma^{-1}} f \circ R_{\gamma\varsigma},$$

R the right translation operator. It is clear that if

$$\Gamma\varsigma\Gamma = \coprod_{\iota} \varsigma_\iota \Gamma,$$

then

$$H(\Gamma\varsigma\Gamma)f = \sum_{\iota} f \circ R_{\varsigma_\iota}.$$

Indeed,

$$\Gamma = \coprod_{\iota} \gamma_\iota (\Gamma \cap \varsigma\Gamma\varsigma^{-1})$$

$$\Rightarrow$$

$$\Gamma\varsigma\Gamma = \coprod_{\iota} \gamma_\iota \varsigma \Gamma \quad (\varsigma_\iota = \gamma_\iota \varsigma).$$

$\mathrm{Fnc}(G/\Gamma)$ is actually an $\mathcal{H}(G,\Gamma)$-module, i.e., $\forall \phi, \psi \in \mathcal{H}(G,\Gamma)$,

$$H(\phi * \psi) = H(\phi)H(\psi).$$

To see this, it will be enough to show that

$$H(\Gamma\varsigma'\Gamma * \Gamma\varsigma''\Gamma) = H(\Gamma\varsigma'\Gamma)H(\Gamma\varsigma''\Gamma).$$

But, in the notation of §2,

$$\begin{aligned} H(\Gamma\varsigma'\Gamma)H(\Gamma\varsigma''\Gamma)f &= \sum_{\iota'}\Big(\sum_{\iota''} f\circ R_{\varsigma''_{\iota''}}\Big)\circ R_{\varsigma'_{\iota'}} \\ &= \sum_{\iota',\iota''} f\circ R_{\varsigma'_{\iota'}\varsigma''_{\iota''}} \\ &= \sum_{\varsigma} m(\Gamma\varsigma'\Gamma,\Gamma\varsigma''\Gamma;\Gamma\varsigma\Gamma)\sum_{\iota} f\circ R_{\varsigma_\iota} \\ &= H(\Gamma\varsigma'\Gamma * \Gamma\varsigma''\Gamma)f, \end{aligned}$$

the contention.

Let us examine the terms figuring in the definition

$$H(\Gamma\varsigma\Gamma)f = \sum_{\iota} f\circ R_{\varsigma_\iota}.$$

Obviously,

$$f\circ R_{\varsigma_\iota} \in \mathrm{Fnc}(G/\varsigma_\iota\Gamma\varsigma_\iota^{-1}).$$

Moreover,

$$[\Gamma : \Gamma\cap\varsigma_\iota\Gamma\varsigma_\iota^{-1}] < +\infty.$$

Put

$$\Gamma(\varsigma) = \bigcap_{\iota}(\Gamma\cap\varsigma_\iota\Gamma\varsigma_\iota^{-1}).$$

Then still

$$[\Gamma : \Gamma(\varsigma)] < +\infty$$

and

$$\mathrm{ind}(\varsigma) = \frac{\sum_{\iota}[\varsigma_\iota\Gamma\varsigma_\iota^{-1} : \Gamma(\varsigma)]}{[\Gamma : \Gamma(\varsigma)]}.$$

Therefore the set of cuspidal (percuspidal) split parabolic subgroups of G is the same for both Γ and $\Gamma(\varsigma)$ (cf. TES, p. 37), although, of course, the number of cusps may very well be different. In particular: Γ and $\Gamma(\varsigma)$ share the same Siegel domains.

Here are some examples of $\mathcal{H}(G,\Gamma)$-submodules of $\mathrm{Fnc}(G/\Gamma)$.

(1) Let

$$\begin{cases} S(G/\Gamma) \\ R(G/\Gamma) \end{cases}$$

be the space of

$$\begin{cases} \text{slowly increasing functions} \\ \text{rapidly decreasing functions} \end{cases}$$

on G/Γ (cf. TES, p. 77) – then each of these spaces is $\mathcal{H}(G,\Gamma)$-stable. In fact,

$$\begin{cases} f \in S(G/\Gamma) \Rightarrow f \circ R_{\varsigma_\iota} \in S(G/\varsigma_\iota\Gamma\varsigma_\iota^{-1}) \\ f \in R(G/\Gamma) \Rightarrow f \circ R_{\varsigma_\iota} \in R(G/\varsigma_\iota\Gamma\varsigma_\iota^{-1}), \end{cases}$$

so

$$\begin{cases} f \in S(G/\Gamma) \Rightarrow H(\Gamma\varsigma\Gamma)f \in S(G/\Gamma(\varsigma)) \\ f \in R(G/\Gamma) \Rightarrow H(\Gamma\varsigma\Gamma)f \in R(G/\Gamma(\varsigma)). \end{cases}$$

Since $H(\Gamma\varsigma\Gamma)f$ is Γ-invariant, it follows that

$$\begin{cases} H(\Gamma\varsigma\Gamma)f \in S(G/\Gamma(\varsigma)) \Rightarrow H(\Gamma\varsigma\Gamma)f \in S(G/\Gamma) \\ H(\Gamma\varsigma\Gamma)f \in R(G/\Gamma(\varsigma)) \Rightarrow H(\Gamma\varsigma\Gamma)f \in R(G/\Gamma). \end{cases}$$

(2) Let $\mathfrak{A}(G/\Gamma)$ be the space of automorphic forms on G/Γ – then

$$\mathfrak{A}(G/\Gamma) = \bigcup_{F,I} \mathfrak{A}(G/\Gamma, F, I),$$

where, in the notation of TES (pp. 77-78), $\mathfrak{A}(G/\Gamma, F, I)$ is the finite dimensional subspace of $\mathfrak{A}(G/\Gamma)$ comprised of those f such that

$$\begin{cases} \overline{\chi}_F * f &= f \qquad (F \subset \widehat{K}) \\ I * f &= 0 \qquad (I \subset \mathfrak{Z}). \end{cases}$$

We claim that $\mathfrak{A}(G/\Gamma, F, I)$ is $\mathcal{H}(G/\Gamma)$-stable, hence that $\mathfrak{A}(G/\Gamma)$ is too. In fact, if

$$f \in \mathfrak{A}(G/\Gamma, F, I),$$

then

$$\begin{cases} \overline{\chi}_F * H(\Gamma\varsigma\Gamma)f &= H(\Gamma\varsigma\Gamma)(\overline{\chi}_F * f) = H(\Gamma\varsigma\Gamma)f \\ I * H(\Gamma\varsigma\Gamma)f &= H(\Gamma\varsigma\Gamma)(I * f) = 0. \end{cases}$$

That the growth condition obtains is a consequence of what was said in the first example.

The Hecke algebra $\mathcal{H}(G,\Gamma)$ also operates on $L^2(G/\Gamma)$. It is in fact easy to check that

$$f \in L^2(G/\Gamma) \Rightarrow H(\Gamma\varsigma\Gamma)f \in L^2(G/\Gamma),$$

the assignment

$$H(\Gamma\varsigma\Gamma) : L^2(G/\Gamma) \to L^2(G/\Gamma)$$

being actually a bounded linear transformation. Because

$$\begin{aligned}\int_{G/\Gamma} H(\Gamma\varsigma\Gamma)f\bullet\bar{g} &= \int_{G/\Gamma\cap\varsigma\Gamma\varsigma^{-1}} f\circ R_\varsigma\bullet\bar{g}\\ &= \int_{G/\varsigma\Gamma\varsigma^{-1}} f\circ R_\varsigma\bullet\int_{\varsigma\Gamma\varsigma^{-1}/\Gamma\cap\varsigma\Gamma\varsigma^{-1}}\bar{g}\\ &= \int_{G/\Gamma} f\bullet H(\Gamma\varsigma^{-1}\Gamma)\bar{g},\end{aligned}$$

the adjoint of $H(\Gamma\varsigma\Gamma)$ is $H(\Gamma\varsigma^{-1}\Gamma)$, that is,

$$H(\Gamma\varsigma\Gamma)^* = H((\Gamma\varsigma\Gamma)^*).$$

In other words: The action of $\mathcal{H}(G,\Gamma)$ on $L^2(G/\Gamma)$ gives rise to a *-representation of $\mathcal{H}(G,\Gamma)$. Since

$$H(\phi)\bullet L_{G/\Gamma}(x) = L_{G/\Gamma}(x)\bullet H(\phi) \quad (x\in G),$$

the $H(\phi)$ are intertwining operators. Consequently,

$$H(\phi)f = f\underset{G/\Gamma}{*}D_\phi,$$

D_ϕ a distribution on G/Γ, left invariant under Γ.

The Hecke operators associated with the ς in the normalizer of Γ are unitary ($\varsigma\Gamma = \Gamma\varsigma$). E.g.: If Γ_0 is a normal subgroup of finite index in Γ, then $\forall\varsigma\in\Gamma$,

$$H(\Gamma_0\varsigma\Gamma_0): L^2(G/\Gamma_0)\to L^2(G/\Gamma_0)$$

is unitary.

As is well-known (cf. TES, p. 23), there is an orthogonal decomposition

$$L^2(G/\Gamma) = \sum_{C}\oplus L^2(G/\Gamma;C),$$

parameterized by the association classes C. Since Hecke operators respect Eisenstein series (cf. §6), each of the

$$L^2(G/\Gamma;C)$$

is invariant under $\mathcal{H}(G,\Gamma)$. The same is thus true of

$$L^2_{\text{dis}}(G/\Gamma)(= L^2(G/\Gamma;\{G\}): \quad \text{the discrete spectrum}$$

and

$$L^2_{\text{con}}(G/\Gamma)(=\sum_{C\neq\{G\}}\oplus L^2(G/\Gamma;C)): \quad \text{the continuous spectrum.}$$

We can, moreover, split $L^2_{\text{dis}}(G/\Gamma)$ into an orthogonal direct sum of

$$L^2_{\text{cus}}(G/\Gamma) \text{ (the space of cusp forms on } G/\Gamma)$$

and

$$L^2_{\text{res}}(G/\Gamma) \text{ (the space of residual forms on } G/\Gamma),$$

both of which are again invariant under $\mathcal{H}(G,\Gamma)$.

The above space are modules for $C_c^\infty(G)$ or $\mathcal{C}^1(G)$, qua $L_{G/\Gamma}$. Either action commutes with that of $\mathcal{H}(G,\Gamma)$ and our chief concern in the sequel is with the interplay between them; cf. infra.

It is sometimes necessary to relativize the considerations supra from G to P. No problems arise in so doing since the general theory provides us with an analysis of

$$L^2(G/(\Gamma\cap P)\bullet A\bullet N)$$

and $\mathcal{H}(P,\Gamma\cap P)$ operates on

$$\text{Fnc}(G/(\Gamma\cap P)\bullet A\bullet N)$$

in the obvious fashion.

§4. The Fundamental Problem

Consider the following statement.

Main Conjecture (MC). *The operator $L_{G/\Gamma}^{\mathrm{dis}}(\alpha)$ is trace class for every K-finite α in $C_c^\infty(G)$.*

This conjecture is a theorem when $\mathrm{rank}(\Gamma) = 0$ (cf. TES, p. 355) or when $\mathrm{rank}(\Gamma) = 1$ (cf. Donnelly [2, p. 349]) and is undoubtedly true in general although this has yet to be proved. It is implied by various natural assumptions (cf. [15-(e)], [15-(g)]). For a short account, see [20-(b)].

Throughout the remainder of this article, MC will be admitted as a working hypothesis. Owing to the theory of the parametrix (cf. TES, p. 21), it then automatically holds for all K-finite α in $C^1(G)$.

Let now $\phi \in \mathcal{H}(G,\Gamma)$ – then

$$H(\phi) \bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha) = L_{G/\Gamma}^{\mathrm{dis}}(\alpha) \bullet H(\phi)$$

is still a trace class operator. This being so, the fundamental problem of the theory is to compute

$$\mathrm{tr}(H(\phi) \bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha)) = \mathrm{tr}(L_{G/\Gamma}^{\mathrm{dis}}(\alpha) \bullet H(\phi)).$$

Needless to say, it is enough to work just with

$$\phi = \chi_{\Gamma\varsigma\Gamma}.$$

In the event that $\varsigma \in \Gamma$, the fundamental problem reduces to the calculation of

$$\mathrm{tr}(L_{G/\Gamma}^{\mathrm{dis}}(\alpha)),$$

which, of course, is precisely the question of the Selberg trace formula itself. As we shall see, the elementary facts connected with the latter situation can be carried over without essential difficulty to the general case.

§5. The Uniform Case

Suppose that Γ is uniform in G – then the quotient space G/Γ is compact and $\forall\alpha$,

$$L_{G/\Gamma}(\alpha) = \int_G \alpha(x) L_{G/\Gamma}(x) d_G(x)$$

is an integral operator on $L^2(G/\Gamma)$ with kernel

$$K_\alpha(x,y) = \sum_{\gamma\in\Gamma} \alpha(x\gamma y^{-1}).$$

Fix a $\varsigma \in \mathrm{Com}(\Gamma)$ – then

$$\begin{aligned}
H(\Gamma\varsigma\Gamma) \bullet (L_{G/\Gamma}(\alpha)f)(x) &= \sum_\iota L_{G/\Gamma}(\alpha) f(x\varsigma_\iota) \\
&= \sum_\iota \int_{G/\Gamma} K_\alpha(x\varsigma_\iota, y) f(y) d_G(y) \\
&= \int_{G/\Gamma} \Big(\sum_\iota \sum_{\gamma\in\Gamma} \alpha(x\varsigma_\iota \gamma y^{-1})\Big) f(y) d_G(y) \\
&= \int_{G/\Gamma} \Big(\sum_{\gamma\in\Gamma\varsigma\Gamma} \alpha(x\gamma y^{-1})\Big) f(y) d_G(y).
\end{aligned}$$

This means that

$$H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}(\alpha)$$

is an integral operator on $L^2(G/\Gamma)$ with kernel

$$K_\alpha(\varsigma; x, y) = \sum_{\gamma\in\Gamma\varsigma\Gamma} \alpha(x\gamma y^{-1}).$$

One may now perform the usual manipulations to conclude that

$$\begin{aligned}
\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}(\alpha)) &= \int_{G/\Gamma} K_\alpha(\varsigma; x, x) d_G(x) \\
&= \sum_{\{\gamma\}_\Gamma} \mathrm{vol}(G_\gamma/\Gamma_\gamma) \bullet \int_{G/G_\gamma} \alpha(x\gamma x^{-1}) d_{G/G_\gamma}(x).
\end{aligned}$$

Here, the sum is taken over the Γ-conjugacy classes $\{\gamma\}_\Gamma$ in $\Gamma\varsigma\Gamma$. Moreover, as is customary,

$$\begin{cases} G_\gamma & = \text{centralizer of } \gamma \text{ in } G \\ \Gamma_\gamma & = \text{centralizer of } \gamma \text{ in } \Gamma. \end{cases}$$

To interpret this conclusion, we need a simple remark (cf. Moscovici [12, p. 337]), viz.: Let $\gamma \in \Gamma\varsigma\Gamma$ – then γ is semisimple. To be complete, let us run through the argument. It suffices to prove that the set

$$\{x\gamma x^{-1} : x \in G\}$$

is closed in G. For this purpose, suppose that $x_n\gamma x_n^{-1} \to x$. Write $G = \Omega \bullet \Gamma$, Ω compact. Let $x_n = \omega_n\gamma_n$ $(\omega_n \in \Omega, \gamma_n \in \Gamma)$ – then we can and will assume that $\omega_n \to \omega \in \Omega$, thence

$$\begin{aligned} x_n\gamma x_n^{-1} &= \omega_n\gamma_n\gamma\gamma_n^{-1}\omega_n^{-1} \to x \\ \Rightarrow \gamma_n\gamma\gamma_n^{-1} &\to \omega^{-1}x\omega \\ \Rightarrow \gamma_n\gamma\gamma_n^{-1} &= \omega^{-1}x\omega \quad (n \gg 0). \end{aligned}$$

I.e.: x is a G-conjugate of γ.

Accordingly,

$$\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}(\alpha))$$

is given in terms of orbital integrals

$$\int_{G/G_\gamma} \alpha(x\gamma x^{-1}) d_{G/G_\gamma}(x)$$

per the $\gamma \in \Gamma\varsigma\Gamma$, all of which are semisimple. The Fourier transform, in the sense of Harish-Chandra, of an orbital integral with respect to a semisimple element of G is known, thanks to the work of Herb [5], thus one does in fact have an explicit formula for

$$\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}(\alpha)),$$

thereby providing a positive solution to the fundamental problem in the uniform case.

Denote by

$$L^2(G/\Gamma; U)$$

the isotypic component of $L^2(G/\Gamma)$ corresponding to $U \in \widehat{G}$ – then $H(\Gamma\varsigma\Gamma)$ leaves each such invariant. Agreeing that $|_U$ signifies restriction to

$$L^2(G/\Gamma; U),$$

we can say that

$$\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}(\alpha)) = \sum_{U \in \widehat{G}} \mathrm{tr}\big(H(\Gamma\varsigma\Gamma)\,|_U \bullet L_{G/\Gamma}(\alpha)\,|_U\big).$$

The calculation of the local traces

$$\operatorname{tr}\left(H(\Gamma\varsigma\Gamma)\,\big|_U \bullet L_{G/\Gamma}(\alpha)\,\big|_U\right)$$

is a difficult problem of some importance, about which little is known. We shall give one example.

Suppose that $\operatorname{rank}(G) = \operatorname{rank}(K)$, so that the discrete series $\widehat{G}_d$ for G is not empty. Fix an integrable $U_0 \in \widehat{G}_d$ operating on a Hilbert space $\mathfrak{H}_0$ and let

$$\alpha_0(x) = d_{U_0}\overline{(U_0(x)v_0, v_0)} \quad (v_0 \in \mathfrak{H}_0, v_0 \ K\text{-finite}, \|v_0\| = 1),$$

d_{U_0} the formal degree of U_0 – then (cf. Osborne [14, p. 47]), $\forall U \neq U_0$,

$$U(\alpha_0) = 0,$$

implying that $\forall U \neq U_0$,

$$L_{G/\Gamma}(\alpha_0)\,\big|_U = 0.$$

Therefore

$$\operatorname{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}(\alpha_0)) = \operatorname{tr}\left(H(\Gamma\varsigma\Gamma)\,\big|_{U_0} \bullet L_{G/\Gamma}(\alpha_0)\,\big|_{U_0}\right).$$

Let $m(U_0, \Gamma)$ be the multiplicity of U_0 in $L^2(G/\Gamma)$. Write

$$\mathfrak{I}(U_0, L_{G/\Gamma}\,\big|_{U_0})$$

for the set of all

$$T : \mathfrak{H}_0 \to L^2(G/\Gamma; U_0)$$

intertwining U_0 and $L_{G/\Gamma}\,\big|_{U_0}$ – then

$$\mathfrak{I}(U_0, L_{G/\Gamma}\,\big|_{U_0})$$

is canonically an $m(U_0, \Gamma)$-dimensional Hilbert space. $H(\Gamma\varsigma\Gamma)$ determines an endomorphism

$$H(\Gamma\varsigma\Gamma : U_0)$$

of

$$\mathfrak{I}(U_0, L_{G/\Gamma}\,\big|_{U_0})$$

via the rule

$$T \to H(\Gamma\varsigma\Gamma)\,\big|_{U_0} \bullet T.$$

Plainly,

$$\operatorname{tr}\left(H(\Gamma\varsigma\Gamma)\mid_{U_0} \bullet L_{G/\Gamma}(\alpha_0)\mid_{U_0}\right) = \operatorname{tr}(H(\Gamma\varsigma\Gamma : U_0)).$$

On the other hand, by the Selberg principle,

$$\int_{G/G_\gamma} \alpha_0(x\gamma x^{-1}) d_{G/G_\gamma}(x) = 0$$

unless γ is elliptic. There are but finitely many elliptic Γ-conjugacy classes $\{\gamma\}_\Gamma$ in $\Gamma\varsigma\Gamma$. If

$$(E)\sum_{\{\gamma\}_\Gamma}$$

is the sum over these, then

$$\operatorname{tr}(H(\Gamma\varsigma\Gamma : U_0)) = (E)\sum_{\{\gamma\}_\Gamma} \operatorname{vol}(G_\gamma/\Gamma_\gamma) \bullet \int_{G/G_\gamma} \alpha_0(x\gamma x^{-1}) d_{G/G_\gamma}(x).$$

Owing to classical theorems of Harish-Chandra and Langlands, the orbital integrals on the right can be evaluated in closed terms. The familiar specifics need not be detailed. All told, then, we end up with an explicit expression for

$$\operatorname{tr}(H(\Gamma\varsigma\Gamma : U_0)).$$

One can use it to recover certain standard results from the theory of modular correspondences. This is because the duality theorem (cf. Maurin and Maurin [11-(b)] or Olshanskii [13]) guarantees that

$$\mathfrak{I}(U_0, L_{G/\Gamma}\mid_{U_0})$$

is isometrically isomorphic to $\mathfrak{A}(U_0, \Gamma)$, the set of all continuous Γ-invariant linear functionals on the space of differentiable vectors for U_0, a set that admits various concrete realizations under specialization of the assumptions (cf. Mackey [10] or Maurin and Maurin [11-(a)]). In particular, when $\varsigma \in \Gamma$,

$$H(\Gamma\varsigma\Gamma : U_0) = \mathrm{ID}$$

and

$$\operatorname{tr}(H(\Gamma\varsigma\Gamma : U_0)) = m(U_0, \Gamma),$$

leading to the formula of Langlands [9-(a), p. 255] for the multiplicity $m(U_0, \Gamma)$.

§6. The Nonuniform Case

Suppose that Γ is nonuniform in G – then the quotient space G/Γ is not compact but has finite volume and $\forall \alpha$,

$$L_{G/\Gamma}(\alpha) = \int_G \alpha(x) L_{G/\Gamma}(x) d_G(x)$$

is an integral operator on $L^2(G/\Gamma)$ with kernel

$$K_\alpha(x,y) = \sum_{\gamma \in \Gamma} \alpha(x\gamma y^{-1})$$

but this time it is no longer necessarily true that

$$K_\alpha \in L^2(G/\Gamma \times G/\Gamma).$$

Write

$$K_\alpha^{\text{dis}}(x,y) = K_\alpha(x,y) - K_\alpha^{\text{con}}(x,y).$$

Then, on the basis of MC (cf. §4),

$$K_\alpha^{\text{dis}}(x,y) \in L^2(G/\Gamma \times G/\Gamma)$$

and

$$\text{tr}(L_{G/\Gamma}^{\text{dis}}(\alpha)) = \int_{G/\Gamma} K_\alpha^{\text{dis}}(x,x) d_G(x).$$

In this connection, recall that the kernel $K_\alpha^{\text{con}}(x,y)$ of $L_{G/\Gamma}^{\text{con}}(\alpha)$ is given by

$$\sideset{}{'}\sum_{C} K_\alpha(x,y:C),$$

where

$$K_\alpha(x,y:C) = \sum_{\boldsymbol{O}} K_\alpha(x,y:\boldsymbol{O};C)$$

and

$$K_\alpha(x,y:\boldsymbol{O};C) = \frac{1}{(2\pi)^\ell} \bullet \frac{1}{*(C)} \bullet \sum_{i=1}^{r} \int_{\text{Re}(\Lambda_i)=0} K_\alpha(x,y:\boldsymbol{O}_i,\Lambda_i)|d\Lambda_i|,$$

with

$$K_\alpha(x,y:\boldsymbol{O}_i,\Lambda_i)$$

$$= \sum_{m,n} \mathbf{C}_{mn}(\alpha : \boldsymbol{O}_i, \Lambda_i) \bullet \mathbf{E}(P_i|A_i : \mathbf{e}_m^i : \Lambda_i : x)\overline{\mathbf{E}(P_i|A_i : \mathbf{e}_n^i : \Lambda_i : y)}.$$

The details are spelled out in TES (pp. 356–357).

Fix a $\varsigma \in \mathrm{Com}(\Gamma)$ – then

$$H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}(\alpha)$$

is an integral operator on $L^2(G/\Gamma)$ with kernel

$$K_\alpha(\varsigma; x, y) = \sum_{\gamma \in \Gamma\varsigma\Gamma} \alpha(x\gamma y^{-1}).$$

In addition,

$$\begin{cases} H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha) \\ H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}^{\mathrm{con}}(\alpha) \end{cases}$$

are integral operators on $L^2(G/\Gamma)$ with kernels

$$\begin{cases} K_\alpha^{\mathrm{dis}}(\varsigma; x, y) \\ K_\alpha^{\mathrm{con}}(\varsigma; x, y). \end{cases}$$

Of course,

$$K_\alpha^{\mathrm{dis}}(\varsigma; x, y) = K_\alpha(\varsigma; x, y) - K_\alpha^{\mathrm{con}}(\varsigma; x, y).$$

The description of

$$K_\alpha^{\mathrm{con}}(\varsigma; x, y)$$

is a fairly straightforward matter, modulo a little bit of preparation to which we shall now direct our attention.

Let $\boldsymbol{\Phi}_i \in \boldsymbol{\mathcal{E}}(\delta, \boldsymbol{O}_i)$ – then attached to $\boldsymbol{\Phi}_i$ is the Eisenstein series

$$\mathbf{E}(P_i|A_i : \boldsymbol{\Phi}_i : \Lambda_i : x).$$

That being, there exists a map

$$\mathbf{H}(\varsigma : P_i|A_i : \Lambda_i) : \boldsymbol{\mathcal{E}}(\delta, \boldsymbol{O}_i) \to \boldsymbol{\mathcal{E}}(\delta, \boldsymbol{O}_i),$$

characterized by the relation

$$H(\Gamma\varsigma\Gamma)\mathbf{E}(P_i|A_i : \boldsymbol{\Phi}_i : \Lambda_i :?)(x)$$

$$= \mathbf{E}(P_i|A_i : \mathbf{H}(\varsigma : P_i|A_i : \Lambda_i)\boldsymbol{\Phi}_i : \Lambda_i : x).$$

It turns out that $\mathbf{H}(\varsigma)$ is an entire function of Λ_i, slowly increasing on $\mathrm{Re}(\Lambda_i) = 0$.

We remark that for cuspidal Eisenstein series, the existence of $\mathbf{H}(\varsigma)$ can be established directly at level 1 by means of the theory of the $L(z)$ spaces (cf. Harish-Chandra [3, p. 91]), then at level n by induction. The general case is handled by the residue taking process. An example may be found in the Appendix.

Because $H(\Gamma\varsigma\Gamma)$ intertwines $L_{G/\Gamma}$, one should expect that $\mathbf{H}(\varsigma)$ intertwines $\mathbf{Ind}_{M_i\bullet A_i\bullet N_i}^{G}$. Indeed, on the one hand,

$$\begin{aligned}
&L_{G/\Gamma}(\alpha)\bullet H(\Gamma\varsigma\Gamma)\mathbf{E}(P_i|A_i:\mathbf{\Phi}_i:\Lambda_i:?)\\
=&L_{G/\Gamma}(\alpha)\mathbf{E}(P_i|A_i:\mathbf{H}(\varsigma:P_i|A_i:\Lambda_i)\mathbf{\Phi}_i:\Lambda_i:?)\\
=&\mathbf{E}(P_i|A_i:\mathbf{Ind}_{M_i\bullet A_i\bullet N_i}^{G}((\mathcal{O}_i,\Lambda_i))(\alpha)\bullet\mathbf{H}(\varsigma:P_i|A_i:\Lambda_i)\mathbf{\Phi}_i:\Lambda_i:?),
\end{aligned}$$

while, on the other hand,

$$\begin{aligned}
&H(\Gamma\varsigma\Gamma)\bullet L_{G/\Gamma}(\alpha)\mathbf{E}(P_i|A_i:\mathbf{\Phi}_i:\Lambda_i:?)\\
=&H(\Gamma\varsigma\Gamma)\mathbf{E}(P_i|A_i:\mathbf{Ind}_{M_i\bullet A_i\bullet N_i}^{G}((\mathcal{O}_i,\Lambda_i))(\alpha)\mathbf{\Phi}_i:\Lambda_i:?)\\
=&\mathbf{E}(P_i|A_i:\mathbf{H}(\varsigma:P_i|A_i:\Lambda_i)\bullet\mathbf{Ind}_{M_i\bullet A_i\bullet N_i}^{G}((\mathcal{O}_i,\Lambda_i))(\alpha)\mathbf{\Phi}_i:\Lambda_i:?).
\end{aligned}$$

So (cf. TES, p. 313),

$$\begin{aligned}
&\mathbf{H}(\varsigma:P_i|A_i:\Lambda_i)\bullet\mathbf{Ind}_{M_i\bullet A_i\bullet N_i}^{G}((\mathcal{O}_i,\Lambda_i))(\alpha)\\
&=\mathbf{Ind}_{M_i\bullet A_i\bullet N_i}^{G}((\mathcal{O}_i,\Lambda_i))(\alpha)\bullet\mathbf{H}(\varsigma:P_i|A_i:\Lambda_i).
\end{aligned}$$

There is also a connection between $\mathbf{H}(\varsigma)$ and the $\mathbf{c}$-functions. Thus, using the functional equations, we have

$$\begin{aligned}
&\mathbf{E}(P_j|A_j:\mathbf{c}(P_j|A_j:P_i|A_i:w_{ji}:\Lambda_i)\bullet\mathbf{H}(\varsigma:P_i|A_i:\Lambda_i)\\
&\qquad\times\mathbf{c}(P_i|A_i:P_j|A_j:w_{ji}^{-1}:w_{ji}\Lambda_i)\mathbf{\Phi}_j:w_{ji}\Lambda_i:?)\\
=&\mathbf{E}(P_i|A_i:\mathbf{H}(\varsigma:P_i|A_i:\Lambda_i)\bullet\mathbf{c}(P_i|A_i:P_j|A_j:w_{ji}^{-1}:w_{ji}\Lambda_i)\mathbf{\Phi}_j:\Lambda_i:?)\\
=&H(\Gamma\varsigma\Gamma)\mathbf{E}(P_i|A_i:\mathbf{c}(P_i|A_i:P_j|A_j:w_{ji}^{-1}:w_{ji}\Lambda_i)\mathbf{\Phi}_j:\Lambda_i:?)\\
=&H(\Gamma\varsigma\Gamma)\mathbf{E}(P_j|A_j:\mathbf{c}(P_j|A_j:P_i|A_i:w_{ji}:\Lambda_i)\\
&\qquad\times\mathbf{c}(P_i|A_i:P_j|A_j:w_{ji}^{-1}:w_{ji}\Lambda_i)\mathbf{\Phi}_j:w_{ji}\Lambda_i:?)\\
=&H(\Gamma\varsigma\Gamma)\mathbf{E}(P_j|A_j:\mathbf{\Phi}_j:w_{ji}\Lambda_i:?)\\
=&\mathbf{E}(P_j|A_j:\mathbf{H}(\varsigma:P_j|A_j:w_{ji}\Lambda_i)\mathbf{\Phi}_j:w_{ji}\Lambda_i:?).
\end{aligned}$$

So (cf. TES, p. 313),

$$\begin{aligned}
&\mathbf{c}(P_j|A_j:P_i|A_i:w_{ji}:\Lambda_i)\bullet\mathbf{H}(\varsigma:P_i|A_i:\Lambda_i)\\
=&\mathbf{H}(\varsigma:P_j|A_j:w_{ji}\Lambda_i)\bullet\mathbf{c}(P_j|A_j:P_i|A_i:w_{ji}:\Lambda_i).
\end{aligned}$$

It is then easy to prove that

$$K_\alpha^{\mathrm{con}}(\varsigma : x, y) = \sideset{}{'}\sum_{C} K_\alpha(\varsigma; x, y : C),$$

where

$$K_\alpha(\varsigma; x, y : C) = \sum_{\boldsymbol{O}} K_\alpha(\varsigma; x, y : \boldsymbol{O}; C)$$

and

$$K_\alpha(\varsigma; x, y : \boldsymbol{O}; C) = \frac{1}{(2\pi)^\ell} \bullet \frac{1}{*(C)} \bullet \sum_{i=1}^{r} \int_{\mathrm{Re}(\Lambda_i)=0} K_\alpha(\varsigma; x, y : \boldsymbol{O}_i, \Lambda_i) |d\Lambda_i|,$$

with

$$K_\alpha(\varsigma; x, y : \boldsymbol{O}_i, \Lambda_i)$$

$$= \sum_{m,n} \mathbf{C}_{mn}(\alpha : \boldsymbol{O}_i, \Lambda_i) \mathbf{E}(P_i|A_i : \mathbf{H}(\varsigma : P_i|A_i : \Lambda_i)\mathbf{e}_m^i : \Lambda_i : x)\overline{\mathbf{E}(P_i|A_i : \mathbf{e}_n^i : \Lambda_i : y)}.$$

Now bring in the truncation operator $Q^{\mathbf{H}}$ (cf. [15-(b)]) – then $\forall \mathbf{H} \in \mathfrak{a}_Q$,

$$\mathrm{tr}(Q^{\mathbf{H}} \circ H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}^{\mathrm{con}}(\alpha) \circ Q^{\mathbf{H}})$$

is equal to

$$\int_{G/\Gamma} Q_{(1)}^{\mathbf{H}} Q_{(2)}^{\mathbf{H}} K_\alpha^{\mathrm{con}}(\varsigma; x, x) d_G(x)$$

or still

$$\sideset{}{'}\sum_{C} \frac{1}{(2\pi)^\ell} \bullet \frac{1}{*(C)} \bullet \sum_{\boldsymbol{O}} \sum_{i=1}^{r}$$

$$\times \int_{\mathrm{Re}(\Lambda_i)=0} \sum_{m,n} \mathbf{C}_{mn}(\alpha : \boldsymbol{O}_i, \Lambda_i) \bullet (?_m, ?_n) |d\Lambda_i|,$$

$(?_m, ?_n)$ being

$$\int_{G/\Gamma} Q^{\mathbf{H}} \mathbf{E}(P_i|A_i : \mathbf{H}(\varsigma : P_i|A_i : \Lambda_i)\mathbf{e}_m^i : \Lambda_i : x) Q^{\mathbf{H}} \overline{\mathbf{E}(P_i|A_i : \mathbf{e}_n^i : \Lambda_i : x)} d_G(x).$$

This sets the stage for the calculation of

$$\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha)).$$

While the difficulties are considerable, it nevertheless seems to be a certainty that a positive solution to the fundamental problem in the nonuniform case will eventually be obtainable by utilizing variants of the methods developed by Arthur and Osborne and Warner for the derivation of the Selberg trace formula itself. At present, the contribution from the continuous spectrum to

$$\mathrm{tr}(L_{G/\Gamma}^{\mathrm{dis}}(\alpha))$$

is known (cf. [15-(g)]). To obtain the same for

$$\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha)),$$

it is first necessary to get an analogue of the main result from [15-(f)]. This is indeed possible but will be dealt with elsewhere.

Appendix

Here, by way of an example, we shall briefly consider the simplest case, viz. the situation when $\mathrm{rank}(\Gamma) = 1$. Since a treatment in extenso would be quite lengthy, thus out of place, we shall settle for a sketch of how things go, leaving aside some of the details and omitting altogether a pursuit of the matter to the bitter end.

Agreeing to use without comment the notation of [15-(a)], let us assume for simplicity that Γ has one cusp only. Fix a $\varsigma \in \mathrm{Com}(\Gamma)$ – then

$$\varsigma P \varsigma^{-1} \quad \text{and} \quad \varsigma^{-1} P \varsigma$$

are both Γ-percuspidal, hence Γ-conjugate to P. So: $\exists \gamma_\varsigma \in \Gamma$ such that

$$\varsigma^{-1} P \varsigma = \gamma_\varsigma P \gamma_\varsigma^{-1} \Rightarrow P = \varsigma\gamma_\varsigma P \gamma_\varsigma^{-1}\varsigma^{-1} \Rightarrow \varsigma\gamma_\varsigma \in P.$$

I.e.: One can always multiply ς on the right by an element from Γ to force ς into P. In the decomposition

$$\Gamma\varsigma\Gamma = \coprod_\iota \varsigma_\iota \Gamma$$

there is therefore no loss of generality in assuming that $\varsigma_\iota = m_\iota a_\iota n_\iota \in P$, giving

$$\Gamma\varsigma\Gamma \cap P = \coprod_\iota \varsigma_\iota (\Gamma \cap P).$$

Let $\{\varsigma_{\iota_0}\}$ be a subset of $\{\varsigma_\iota\}$ for which

$$\Gamma\varsigma\Gamma \cap P = \coprod_{\iota_0} (\Gamma \cap P)\varsigma_{\iota_0}(\Gamma \cap P).$$

Then

$$(\Gamma \cap P)\varsigma_{\iota_0}(\Gamma \cap P) = \coprod_\iota \varsigma_\iota (\Gamma \cap P),$$

where the ς_ι on the right have the property that

$$(\Gamma \cap P)\varsigma_{\iota_0}(\Gamma \cap P) = (\Gamma \cap P)\varsigma_\iota(\Gamma \cap P).$$

The injection

$$\Gamma \cap P \backslash \Gamma\varsigma\Gamma \cap P / \Gamma \cap P \hookrightarrow \mathcal{H}(P, \Gamma \cap P)$$

produces from the ς_{ι_0} a finite sum of Hecke operators, namely

$$\sum_{\iota_0} H((\Gamma \cap P)\varsigma_{\iota_0}(\Gamma \cap P)),$$

a bounded linear transformation on

$$L^2(G/(\Gamma\cap P)\bullet A\bullet N).$$

Recall now the following statement from reduction theory (cf. [15(a), p. 21]): Let C be a compact subset of G – then there is a number $\varepsilon_C>0$ such that if

$$a(t)\gamma a(-t)\in C$$

for some $\gamma\in\Gamma$ and some $t<\log\varepsilon_C$, then necessarily $\gamma\in\Gamma\cap P$. The replacement for this assertion in the current setting is: Let C be a compact subset of G – then there is a number $\varepsilon_C>0$ such that if

$$a(t)\varsigma\gamma a(-t)\in C$$

for some $\gamma\in\Gamma$ and some $t<\log\varepsilon_C$, then necessarily $\varsigma\gamma\in P$. In fact, suppose first that $\varsigma\in P$ – then

$$a(t)\varsigma\gamma a(-t)=a(t)\varsigma a(-t)\bullet a(t)\gamma a(-t).$$

Since

$$a(t)\varsigma a(-t)=m_\varsigma a_\varsigma a(t)n_\varsigma a(-t)$$

stays bounded for $t\ll 0$, we can apply the lemma of reduction to push γ into $\Gamma\cap P$, thence $\varsigma\gamma\in P$. In general, there is a $\gamma_\varsigma\in\Gamma$ such that $\varsigma\gamma_\varsigma\in P$. This said, write

$$a(t)\varsigma\gamma a(-t)=a(t)\varsigma\gamma_\varsigma\bullet\gamma_\varsigma^{-1}\gamma a(-t)$$

and then reason as above. In passing, note that ε_C ostensibly depends on ς. We can, however, come up with an ε_C that works simultaneously for all of the ς_ι, a uniformity that is sufficient for the applications.

Let α be a K-finite function in $C_c^\infty(G)$ – then

$$\begin{aligned}\operatorname{tr}(H(\Gamma\varsigma\Gamma)\bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha)) &= \lim_{H\to-\infty}\operatorname{tr}(Q^H\circ H(\Gamma\varsigma\Gamma)\bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha)\circ Q^H)\\ &= \lim_{H\to-\infty}\left[\int_{G/\Gamma}Q_{(1)}^H Q_{(2)}^H K_\alpha(\varsigma;x,x)d_G(x)-\int_{G/\Gamma}Q_{(1)}^H Q_{(2)}^H K_\alpha^{\mathrm{con}}(\varsigma;x,x)d_G(x)\right].\end{aligned}$$

Each of these integrals contains a singularity which must be isolated and cancelled.

We have (cf. [15-(a), p. 15])

$$\begin{cases}\int_{G/\Gamma}Q_{(1)}^H Q_{(2)}^H K_\alpha(\varsigma;x,x)d_G(x) &= \int_{G/\Gamma}Q_{(1)}^H K_\alpha(\varsigma;x,x)d_G(x)\\ \int_{G/\Gamma}Q_{(1)}^H Q_{(2)}^H K_\alpha(\varsigma;x,x)d_G(x) &= \int_{G/\Gamma}Q_{(2)}^H K_\alpha(\varsigma;x,x)d_G(x).\end{cases}$$

In contrast to what was done in [15-(a), p. 35] when $\varsigma \in \Gamma$, it is better to work with the second of these relations rather than with the first. That being, let $x \in \mathfrak{S}_{t_0,\omega_0}$ – then, by definition,

$$Q_{(2)}^{H} K_\alpha(\varsigma; x, x) = \begin{cases} K_\alpha(\varsigma; x, x) - \int_{N/N\cap\Gamma} K_\alpha(\varsigma; x, xn) d_N(n) & \text{if } x \in \mathfrak{S}_{t_H,\omega_0} \\ K_\alpha(\varsigma; x, x) & \text{if } x \notin \mathfrak{S}_{t_H,\omega_0}. \end{cases}$$

Because α is compactly supported, if $H \ll 0$, then

$$\begin{aligned}
&\int_{N/N\cap\Gamma} K_\alpha(\varsigma; x, xn) d_N(n) \\
&= \int_{N/N\cap\Gamma} \Big(\sum_{\gamma\in\Gamma\varsigma\Gamma} \alpha(x\gamma n^{-1} x^{-1}) \Big) d_N(n) \\
&= \int_{N/N\cap\Gamma} \Big(\sum_{\gamma\in\Gamma\varsigma\Gamma/\Gamma\cap N} \ \sum_{\eta\in\Gamma\cap N} \alpha(x\gamma\eta^{-1} n^{-1} x^{-1}) \Big) d_N(n) \\
&= \int_{N} \Big(\sum_{\gamma\in\Gamma\varsigma\Gamma/\Gamma\cap N} \alpha(x\gamma n^{-1} x^{-1}) \Big) d_N(n) \\
&= \int_{N} \Big(\sum_{\gamma\in\Gamma\varsigma\Gamma/\Gamma\cap N} \alpha(x\gamma n x^{-1}) \Big) d_N(n) \\
&= \int_{N} \Big(\sum_{\gamma\in\Gamma\varsigma\Gamma/\Gamma\cap P} \ \sum_{\delta\in\Gamma\cap P/\Gamma\cap N} \alpha(x\gamma\delta n x^{-1}) \Big) d_N(n) \\
&= \int_{N} \Big(\sum_{\gamma\in\Gamma\varsigma\Gamma\cap P/\Gamma\cap P} \ \sum_{\delta\in\Gamma_M} \alpha(x\gamma\delta n x^{-1}) \Big) d_N(n) \\
&= \int_{N} \Big(\sum_{\iota} \sum_{\delta\in\Gamma_M} \alpha(x\varsigma_\iota\delta n x^{-1}) \Big) d_N(n).
\end{aligned}$$

Proceeding at this point exactly as in [15-(a), pp. 36–38], fix $H_0 \ll 0$. Assuming that $H < H_0$, we get

$$\begin{aligned}
&\int_{G/\Gamma} Q_{(2)}^{H} K_\alpha(\varsigma; x, x) d_G(x) \\
&= \int_{G/\Gamma} (Q_{(2)}^{H} - Q_{(2)}^{H_0}) K_\alpha(\varsigma; x, x) d_G(x) + \int_{G/\Gamma} Q_{(2)}^{H_0} K_\alpha(\varsigma; x, x) d_G(x) \\
&= \int_{\Omega_{H_0}} \Big(\sum_{\gamma\in\Gamma\varsigma\Gamma} \alpha(x\gamma x^{-1}) \Big) d_G(x) + \mathrm{o}(H_0) \\
&+ \frac{\lambda(H_0)}{|\lambda|} \bullet \int_{M/\Gamma_M} \Big(\sum_{\iota} \sum_{\delta\in\Gamma_M} \alpha^P(m m_\iota a_\iota \delta m^{-1}) \Big) d_M(m) \\
&- \frac{\lambda(H)}{|\lambda|} \bullet \int_{M/\Gamma_M} \Big(\sum_{\iota} \sum_{\delta\in\Gamma_M} \alpha^P(m m_\iota a_\iota \delta m^{-1}) \Big) d_M(m),
\end{aligned}$$

where

$$\alpha^P(ma) = \int_N \alpha_K(man)d_N(n).$$

As usual, $o(H_0)$ is some function of H_0 such that

$$\lim_{H_0 \to -\infty} o(H_0) = 0,$$

thus is effectively ignorable. The integral over M/Γ_M may be viewed as the trace of a certain Hecke operator in $\mathcal{H}(M, \Gamma_M)$.

One must now go back and look more closely at

$$\int_{G/\Gamma} Q_{(1)}^H Q_{(2)}^H K_\alpha^{\text{con}}(\varsigma; x, x) d_G(x)$$

or still (cf. §6)

$$\frac{1}{4\pi} \bullet \sum_{\mathcal{O}} \int_{\text{Re}(\Lambda)=0} \sum_{m,n} \mathbf{C}_{mn}(\alpha : \mathcal{O}, \Lambda) \bullet (?_m, ?_n)|d\Lambda|,$$

$(?_m, ?_n)$ being

$$\int_{G/\Gamma} Q^H \mathbf{E}(P|A : \mathbf{H}(\varsigma : P|A : \Lambda)\mathbf{e}_m : \Lambda : x) Q^H \overline{\mathbf{E}(P|A : \mathbf{e}_n : \Lambda : x)} d_G(x).$$

Owing to the Langlands inner product formula (but suppressing the orbit type from the notation), $(?_m, ?_n)$ is equal to the sum of

$$-2\frac{\lambda(H)}{|\lambda|} \bullet (\mathbf{H}(\varsigma : P|A : \Lambda)\mathbf{e}_m, \mathbf{e}_n)$$

and

$$(\mathbf{c}(P|A : \Lambda)^* \frac{d}{d\Lambda}\mathbf{c}(P|A : \Lambda) \bullet \mathbf{H}(\varsigma : P|A : \Lambda)\mathbf{e}_m, \mathbf{e}_n)$$

and

$$-\frac{1}{2\Lambda(H_\lambda)}\{t_H^{2\Lambda(H_\lambda)/|\lambda|} \bullet (\mathbf{H}(\varsigma : P|A : \Lambda)\mathbf{e}_m, \mathbf{c}(P|A : \Lambda)\mathbf{e}_n)$$
$$-t_H^{-2\Lambda(H_\lambda)/|\lambda|} \bullet (\mathbf{H}(\varsigma : P|A : \Lambda)\mathbf{e}_m, \mathbf{c}(P|A : \Lambda)^*\mathbf{e}_n)\},$$

t_H the exponential of $\lambda(H)$. Consequently,

$$\sum_{m,n} \mathbf{C}_{mn}(\alpha : \mathcal{O}, \Lambda) \bullet (?_m, ?_n)$$

is equal to the sum of

$$-2\frac{\lambda(H)}{|\lambda|} \bullet \text{tr}(\mathbf{H}(\varsigma : P|A : \Lambda) \bullet \mathbf{Ind}_P^G((\mathcal{O}, \Lambda))(\alpha))$$

and

$$\mathrm{tr}(\mathbf{H}(\varsigma : P|A : \Lambda) \bullet \mathbf{Ind}_P^G((\mathcal{O}, \Lambda))(\alpha) \bullet \mathbf{c}(P|A : \Lambda)^* \frac{d}{d\Lambda}\mathbf{c}(P|A : \Lambda))$$

and

$$-\frac{1}{2\Lambda(H_\lambda)}\{t_H^{2\Lambda(H_\lambda)/|\lambda|} \bullet \mathrm{tr}(\mathbf{H}(\varsigma : P|A : \Lambda) \bullet \mathbf{Ind}_P^G((\mathcal{O}, \Lambda))(\alpha) \bullet \mathbf{c}(P|A : \Lambda)^*)$$

$$-t_H^{-2\Lambda(H_\lambda)/|\lambda|} \bullet \mathrm{tr}(\mathbf{H}(\varsigma : P|A : \Lambda) \bullet \mathbf{Ind}_P^G((\mathcal{O}, \Lambda))(\alpha) \bullet \mathbf{c}(P|A : \Lambda))\}.$$

In order to treat the singular term, we shall need a formula for $\mathbf{H}(\varsigma)$. Let $\mathbf{\Phi} \in \boldsymbol{\mathcal{E}}(\delta, \mathcal{O})$ – then

$$H(\Gamma\varsigma\Gamma)\mathbf{E}(P|A : \mathbf{\Phi} : \Lambda :?)(x) = \mathbf{E}(P|A : \mathbf{H}(\varsigma : P|A : \Lambda)\mathbf{\Phi} : \Lambda : x).$$

To make the determination, it will be enough to confine our attention to $\mathfrak{S}_{t_0,\omega_0}$, assuming that $\mathrm{Re}(\Lambda) < -\rho$ (in the obvious sense). Let **C-T** stand for constant term – then, with

$$x = ka(t)mn \in \mathfrak{S}_{t_0,\omega_0} = K \bullet A[t_0] \bullet \omega_0,$$

we have

$$\begin{aligned} &H(\Gamma\varsigma\Gamma)\mathbf{E}(P|A : \mathbf{\Phi} : \Lambda :?)(x) \\ &= \mathbf{C\text{-}T}(H(\Gamma\varsigma\Gamma)\mathbf{E}(P|A : \mathbf{\Phi} : \Lambda :?))(x) \\ &+\Big(H(\Gamma\varsigma\Gamma)\mathbf{E}(P|A : \mathbf{\Phi} : \Lambda :?)(x) - \mathbf{C\text{-}T}(H(\Gamma\varsigma\Gamma)\mathbf{E}(P|A : \mathbf{\Phi} : \Lambda :?))(x)\Big) \\ &= (\mathbf{H}(\varsigma : P|A : \Lambda)\mathbf{\Phi})(km)a(t)^{\Lambda-\rho} \\ &+(\mathbf{c}(P|A : \Lambda) \bullet \mathbf{H}(\varsigma : P|A : \Lambda)\mathbf{\Phi})(km)a(t)^{-\Lambda-\rho} + o(t) \quad (t \to -\infty). \end{aligned}$$

Therefore

$$\lim_{t\to-\infty} a(t)^{\rho-\Lambda}(H(\Gamma\varsigma\Gamma)\mathbf{E}(P|A : \mathbf{\Phi} : \Lambda :?)(x)) = (\mathbf{H}(\varsigma : P|A : \Lambda)\mathbf{\Phi})(km).$$

But also

$$\begin{aligned} &H(\Gamma\varsigma\Gamma)\mathbf{E}(P|A : \mathbf{\Phi} : \Lambda :?)(x) \\ &=\sum_\iota \sum_{\gamma\in\Gamma/\Gamma\cap P} a(x\varsigma_\iota\gamma)^{\Lambda-\rho} \bullet \mathbf{\Phi}(x\varsigma_\iota\gamma) \\ &=\sum_\iota \sum_{\gamma_\iota\in\Gamma_\iota/\Gamma_\iota\cap P} a(x\gamma_\iota\varsigma_\iota)^{\Lambda-\rho} \bullet \mathbf{\Phi}(x\gamma_\iota\varsigma_\iota) \end{aligned}$$

if for short

$$\Gamma_\iota = \varsigma_\iota\Gamma\varsigma_\iota^{-1}.$$

Put

$$\Phi_\iota = \Phi \circ R_{\varsigma_\iota}.$$

Then

$$\sum_{\gamma_\iota \in \Gamma_\iota / \Gamma_\iota \cap P} a(x\gamma_\iota \varsigma_\iota)^{\Lambda-\rho} \bullet \Phi_\iota(x\gamma_\iota)$$

$$= a_\iota^{\Lambda-\rho} \bullet \mathbf{E}(P|A : \Phi_\iota : \Lambda : x).$$

This means that

$$H(\Gamma\varsigma\Gamma)\mathbf{E}(P|A : \Phi : \Lambda : ?)(x) = \sum_\iota a_\iota^{\Lambda-\rho} \bullet \mathbf{E}(P|A : \Phi_\iota : \Lambda : x),$$

so too

$$\lim_{t \to -\infty} a(t)^{\rho-\Lambda}\Big(H(\Gamma\varsigma\Gamma)\mathbf{E}(P|A : \Phi : \Lambda : ?)(x)\Big) = \sum_\iota a_\iota^{\Lambda-\rho} \bullet \Phi_\iota(km).$$

In other words, qua an endomorphism of $\mathcal{E}(\delta, \mathcal{O})$,

$$\mathbf{H}(\varsigma : P|A : \Lambda)\Phi = \sum_\iota a_\iota^{\Lambda-\rho} \bullet \Phi \circ R_{\varsigma_\iota}.$$

By analytic continuation, the formula retains its validity for all Λ and, in addition, admits a ready interpretation:

$$\mathbf{H}(\varsigma : P|A : \Lambda)\Phi = \sum_{\iota_0} a_{\iota_0}^{\Lambda-\rho} \bullet H((\Gamma \cap P)\varsigma_{\iota_0}(\Gamma \cap P))\Phi,$$

a linear combination of Hecke operators vis-à-vis P. Naturally, one must observe that

$$(\Gamma \cap P)\varsigma_{\iota_0}(\Gamma \cap P) = (\Gamma \cap P)\varsigma_\iota(\Gamma \cap P)$$

$$\Rightarrow a_{\iota_0} = a_\iota,$$

a triviality.

To discuss

$$\sum_{\mathcal{O}} \mathrm{tr}(\mathbf{H}(\varsigma : P|A : \Lambda) \bullet \mathbf{Ind}_P^G((\mathcal{O}, \Lambda))(\alpha)),$$

we argue as on p. 42 of [15-(a)], eliminating the sum over $\mathcal{O}$ by induction in stages and then using the corresponding formula for the kernel, thereby reducing the evaluation to that of

$$\int_{M/\Gamma_M} \Big[\sum_\iota a_\iota^{\Lambda-\rho} \int_A \Big(\sum_{\delta \in \Gamma_M} \alpha^P(mm_\iota a\delta m^{-1})\Big) a^{-\Lambda+\rho} d_A(a)\Big] d_M(m).$$

In the context at hand, the mechanism of Fourier transformation is

$$\begin{cases} \widehat{f}(\Lambda) &= \int_A f(a)a^{-\Lambda+\rho}d_A(a) \\ f(a) &= \frac{1}{2\pi} \bullet \int_{\mathrm{Re}(\Lambda)=0} \widehat{f}(\Lambda)a^{\Lambda-\rho}|d\Lambda|. \end{cases}$$

Accordingly,

$$\frac{1}{4\pi} \bullet \int_{\mathrm{Re}(\Lambda)=0} \Big(\sum_{\mathcal{O}} \mathrm{tr}(\mathbf{H}(\varsigma : P|A : \Lambda) \bullet \mathbf{Ind}_P^G((\mathcal{O},\Lambda))(\alpha))\Big)|d\Lambda|$$

equals

$$\frac{1}{2} \bullet \int_{M/\Gamma_M} \Big(\sum_{\iota} \sum_{\delta\in\Gamma_M} \alpha^P(mm_\iota a_\iota \delta m^{-1})\Big) d_M(m).$$

By inspection, the cancellation is then immediate.

The Dini analysis offers no new surprises at least if one admits an assumption on the c-functions akin to that on p. 49 of [15-(a)].

The upshot of all this is the conclusion that mod $o(H_0)$

$$\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha))$$

is equal to

$$\int_{\Omega_{H_0}} \Big(\sum_{\gamma\in\Gamma\varsigma\Gamma} \alpha(x\gamma x^{-1})\Big) d_G(x)$$

$$+\frac{\lambda(H_0)}{|\lambda|} \bullet \int_{M/\Gamma_M} \Big(\sum_{\iota} \sum_{\delta\in\Gamma_M} \alpha^P(mm_\iota a_\iota \delta m^{-1})\Big) d_M(m)$$

$$-\frac{1}{4\pi} \bullet \int_{\mathrm{Re}(\Lambda)=0} \Big(\sum_{\mathcal{O}} \mathrm{tr}(\mathbf{H}(\varsigma : P|A : \Lambda) \bullet \mathbf{Ind}_P^G((\mathcal{O},\Lambda))(\alpha) \bullet \mathbf{c}(P|A:\Lambda)^* \frac{d}{d\Lambda}\mathbf{c}(P|A:\Lambda))\Big)|d\Lambda|$$

$$-\frac{1}{4} \bullet \sum_{\mathcal{O}} \mathrm{tr}(\mathbf{H}(\varsigma : P|A : 0) \bullet \mathbf{Ind}_P^G((\mathcal{O},0))(\alpha) \bullet \mathbf{c}(P|A:0)).$$

The third and fourth terms represent the contribution to

$$\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha))$$

arising from the continuous spectrum, thus require no additional analysis. As for the first and second, they contain the contribution to

$$\mathrm{tr}(H(\Gamma\varsigma\Gamma) \bullet L_{G/\Gamma}^{\mathrm{dis}}(\alpha))$$

associated with the Γ-conjugacy classes in $\Gamma\varsigma\Gamma$ and we can proceed, in essence, along the same lines employed in [15-(a), pp. 56–80] when $\varsigma \in \Gamma$ to cancel out the $\lambda(H_0)$ ending up, after substantial elaboration, with orbital integrals (weighted and unweighted) and so forth. Of course, one must start by drawing up a classification of the elements of $\Gamma\varsigma\Gamma$. This is not difficult (cf. [15-(a), §5]). Still, there is a fair amount of detail involved and since it is not especially illuminating, we shall not stop to provide it, being content instead to close with a few simple remarks.

Needless to say, we can and will assume that $\Gamma\varsigma\Gamma \neq \Gamma 1 \Gamma$. Moreover, we might just as well suppose that

$$\Gamma\varsigma\Gamma \cap Z_G = \emptyset.$$

For otherwise, if

$$\Gamma\varsigma\Gamma \cap Z_G \neq \emptyset,$$

then it is permissible to take ς central, giving

$$\sum_{\gamma\in\Gamma\varsigma\Gamma} \alpha(x\gamma y^{-1}) = \sum_{\gamma\in\Gamma} \alpha \circ L_\varsigma(x\gamma y^{-1}).$$

The calculation is therefore covered by the Selberg trace formula applied to $\alpha \circ L_\varsigma$ rather than α.

Put

$$(\Gamma\varsigma\Gamma)_S = \{\gamma \in \Gamma\varsigma\Gamma : \{\gamma\}_\Gamma \cap P = \emptyset\}.$$

Then Lemma 5.5 on p. 20 of [15-(a)] remains valid if Γ_S is replaced by $(\Gamma\varsigma\Gamma)_S$. It follows as there that the elements of $(\Gamma\varsigma\Gamma)_S$ are semisimple. Furthermore, if $\gamma \in (\Gamma\varsigma\Gamma)_S$, then Γ_γ is a uniform lattice in G_γ. Denote by $(\Gamma\varsigma\Gamma)_P$ the complement to $(\Gamma\varsigma\Gamma)_S$ in $\Gamma\varsigma\Gamma$ – then

$$\begin{aligned}\int_{\Omega_{H_0}}\Big(\sum_{\gamma\in\Gamma\varsigma\Gamma}\alpha(x\gamma x^{-1})\Big)d_G(x) &= (S)\sum_{\{\gamma\}_\Gamma}\mathrm{vol}(G_\gamma/\Gamma_\gamma)\bullet\int_{G/G_\gamma}\alpha(x\gamma x^{-1})d_G(x)\\ &\quad+\int_{\Omega_{H_0}}\Big(\sum_{\gamma\in(\Gamma\varsigma\Gamma)_P}\alpha(x\gamma x^{-1})\Big)d_G(x),\end{aligned}$$

the symbol

$$(S)\sum_{\{\gamma\}_\Gamma}$$

standing for a sum over the Γ-conjugacy classes of the elements of $(\Gamma\varsigma\Gamma)_S$. It remains to delineate $(\Gamma\varsigma\Gamma)_P$. In this regard, things run pretty much as expected so long as one takes into account the fact that $\Gamma\varsigma\Gamma \cap A$ may very well be nonempty, in contrast to what is true for Γ. One must also remember that the elements of $\mathrm{Com}(\Gamma_M)$ are semisimple (cf. §5).

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