EXPLICIT FORMULAS FROM THE CONTINUOUS SPECTRUM

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The purpose of this note is to announce the results of our investigation into the role played by the continuous spectrum in the development of the Selberg trace formula vis-à-vis a pair (G, Γ). For the sake of simplicity, we shall restrict ourselves to a "rank-2" situation, a case in point being when

$$\begin{cases} G = \mathbf{SL}(3, \mathbf{R}) \\ \Gamma = \mathbf{SL}(3, \mathbf{Z}). \end{cases}$$

Full details (in all generality) will appear elsewhere.

Let G be a reductive Lie group, Γ a lattice in G, both subject to the usual conditions (cf. [6, p. 62]). As is well-known, there is a decomposition of $L^2(G/\Gamma)$ into the orthogonal direct sum of

$$L^2_{\text{dis}}(G/\Gamma): \quad \text{the discrete spectrum}$$

and

$$L^2_{\text{con}}(G/\Gamma): \quad \text{the continuous spectrum.}$$

Consider the following statement:

Main Conjecture (MC). *The operator $L^{\text{dis}}_{G/\Gamma}(\alpha)$ is trace class for every K-finite α in $C_c^\infty(G)$.*

This conjecture is a theorem when rank $(\Gamma) = 0$ (cf. [6, p. 355]) or when rank $(\Gamma) = 1$ (cf. Donnelly [3, p. 349]) and is undoubtedly true in general although this has yet to be proved.

It is implied by various natural assumptions (cf. [7-(b)], [7-(d)]). For a short account, see [10-(b)].

Throughout the remainder of this note, MC will be admitted as a working hypothesis. Owing to the theory of the parametrix (cf. [6, p. 21]), it then automatically holds for all K-finite α in $C^1(G)$.

These points made, the fundamental problem of the theory is to compute

$$\operatorname{tr}(L_{G/\Gamma}^{\text{dis}}(\alpha))$$

in explicit terms. Thanks to the considerations to be found in [7-(c), §8], the problem can be divided into two parts:

(1) Determine the contribution to the trace arising from the conjugacy classes.

(2) Determine the contribution to the trace arising from the continuous spectrum.

[Note: Naturally, when rank $(\Gamma) = 0$, (2) is irrelevant, so only (1) is of interest, an elementary matter.]

Our approach dictates that the second issue be addressed first. The essence of the method of attack can be found already in [7-(a)], the key being the cancellation principle. There, of course, rank $(\Gamma) = 1$ and all the Arthur polynomials are linear, so everything, by comparison, is fairly simple. The situation when rank $(\Gamma) > 1$ is far more complicated. Nevertheless, it is still possible to arrive at an explicit determination, the basis for the cancellation being a certain remarkable "addition" property enjoyed by the Arthur polynomials, combined with a multidimensional Dini calculus. The way it works is this. Each proper G-conjugacy class C of Γ-cuspidal split parabolic subgroups of G makes a contribution

$$\mathbf{Con}(\alpha : \Gamma : C)$$

to the trace, the total contribution to the trace furnished by the continuous spectrum being the sum

$$\mathbf{Con-Sp}(\alpha:\Gamma)=\sum_{C}\mathbf{Con}(\alpha:\Gamma:C).$$

Accordingly, fix a C containing $P=M\bullet A\bullet N$, say — then

$$\mathbf{Con}(\alpha:\Gamma:C)=\sum_{\mathcal{O}}\sum_{w\in W(A)}\mathbf{Con}(\alpha:\Gamma:C:\mathcal{O}:w),$$

the actual form of the contribution

$$\mathbf{Con}(\alpha:\Gamma:C:\mathcal{O}:w)$$

depending on w through

$$\mathrm{rank}(1-w),$$

the orbit type $\mathcal{O}$ having a passive part in the overall procedure.

To provide some motivation for [7-(d)], we shall explicate here the position when $\mathrm{rank}(\Gamma)=2$. Before doing this, though, it will be a good idea to recall how things go when $\mathrm{rank}(\Gamma)=1$. For use below, denote by $*(C)$ the number of chambers in A (cf. [6, p. 104]).

Fixing $\mathcal{O}$, let us suppose that $\mathrm{rank}(\Gamma)=1$ – then $\#(W(A))=2$. Thus, there are two terms appearing in the contribution from the continuous spectrum.

$\boxed{w=1}$ In this case,

$$\mathbf{Con}(\alpha:\Gamma:C:\mathcal{O}:1)$$

is equal to

$$-\frac{1}{2\pi}\bullet\frac{1}{*(C)}\bullet\sum_{w\in W(A)}$$

$$\times \int_{\mathrm{Re}(\Lambda)=0} \mathbf{tr}\Big(\mathbf{Ind}_P^G((\mathcal{O},\Lambda))(\alpha)$$

$$\bullet \mathbf{c}(P|A : P|A : w : \Lambda)^* \frac{d}{d\Lambda}\mathbf{c}(P|A : P|A : w : \Lambda)\Big)|d\Lambda|.$$

[Note: Since the c-function attached to the trivial element of $W(A)$ is a constant, the contribution is concentrated entirely in the c-function of the nontrivial element of $W(A)$. Still, this mode of expression possesses an inherent symmetry that can be generalized.]

$\boxed{w \neq 1}$ In this case,

$$\mathbf{Con}(\alpha : \Gamma : C : \mathcal{O} : w)$$

is equal to

$$-\frac{1}{2\pi} \bullet 2\pi \bullet \frac{1}{*(C)} \bullet \frac{1}{|\det(1-w)|}$$

$$\times \mathbf{tr}\Big(\mathbf{Ind}_P^G((\mathcal{O},0))(\alpha) \bullet \mathbf{c}(P|A : P|A : w : 0)\Big).$$

[Note: Since

$$w \neq 1 \Rightarrow |\det(1-w)| = 2,$$

the prefacing constant is $1/4$. The "$1/2\pi$" is inherent in the Fourier inversion formula; the "2π" is inherent in the Dini calculus. Because $1-w$ is nonsingular, they cancel.]

Keeping the orbit type fixed, assume now that $\mathrm{rank}(\Gamma) = 2$. There are then two G-conjugacy classes C' and C'' of maximal Γ-cuspidal split parabolic subgroups of G and one G-conjugacy class C of minimal Γ-cuspidal split parabolic subgroups of G. It will be best to discuss each level separately.

$\boxed{C', C''}$ Two cases can occur.

(I) Suppose that $P' \in C'$, $P'' \in C''$ are associate (e.g., $\mathbf{A}_2$) – then $W(A') = \{1\}$, $W(A'') = \{1\}$ and

$$\begin{cases} W(A'', A') = \{w'\} \\ W(A', A'') = \{w''\}. \end{cases}$$

In this case,

$$\mathbf{Con}(\alpha : \Gamma : C' : \mathcal{O}' : 1)$$

is equal to

$$-\frac{1}{2\pi} \bullet \frac{1}{*(C')}$$

$$\times \int_{\mathrm{Re}(\Lambda')=0} \mathrm{tr}\Big(\mathbf{Ind}_{P'}^{G}((\mathcal{O}', \Lambda'))(\alpha)$$

$$\bullet \mathbf{c}(P''|A'' : P'|A' : w' : \Lambda')^* \frac{d}{d\Lambda'} \mathbf{c}(P''|A'' : P'|A' : w' : \Lambda')\Big) |d\Lambda'|$$

and

$$\mathbf{Con}(\alpha : \Gamma : C'' : \mathcal{O}'' : 1)$$

is equal to

$$-\frac{1}{2\pi} \bullet \frac{1}{*(C'')}$$

$$\times \int_{\mathrm{Re}(\Lambda'')=0} \mathrm{tr}\Big(\mathbf{Ind}_{P''}^{G}((\mathcal{O}'', \Lambda''))(\alpha)$$

$$\bullet \mathbf{c}(P'|A' : P''|A'' : w'' : \Lambda'')^* \frac{d}{d\Lambda''} \mathbf{c}(P'|A' : P''|A'' : w'' : \Lambda'')\Big) |d\Lambda''|.$$

(II) Suppose that $P' \in C', P'' \in C''$ are not associate (e.g., $\mathbf{A}_1 \times \mathbf{A}_1$, $\mathbf{B}_2, \mathbf{G}_2$) – then

$$\begin{cases} W(A') = \{1, w'\} \\ W(A'') = \{1, w''\}. \end{cases}$$

In this case,

$$\mathbf{Con}(\alpha : \Gamma : C' : \mathcal{O}' : 1)$$

is equal to

$$-\frac{1}{2\pi}\bullet\frac{1}{*(C')}\bullet\sum_{w'\in W(A')}$$

$$\times\int_{\mathrm{Re}(\Lambda')=0}\mathrm{tr}\Big(\mathbf{Ind}_{P'}^{G}((\mathcal{O}',\Lambda'))(\alpha)$$

$$\bullet\mathbf{c}(P'|A':P'|A':w':\Lambda')^{*}\frac{d}{d\Lambda'}\mathbf{c}(P'|A':P'|A':w':\Lambda')\Big)|d\Lambda'|$$

and

$$\mathbf{Con}(\alpha:\Gamma:C':\mathcal{O}':w')$$

is equal to

$$-\frac{1}{2\pi}\bullet 2\pi\bullet\frac{1}{*(C')}\bullet\frac{1}{|\det(1-w')|}$$

$$\times\mathrm{tr}\Big(\mathbf{Ind}_{P'}^{G}((\mathcal{O}',0))(\alpha)\bullet\mathbf{c}(P'|A':P'|A':w':0)\Big),$$

while

$$\mathbf{Con}(\alpha:\Gamma:C'':\mathcal{O}'':1)$$

is equal to

$$-\frac{1}{2\pi}\bullet\frac{1}{*(C'')}\bullet\sum_{w''\in W(A'')}$$

$$\times\int_{\mathrm{Re}(\Lambda'')=0}\mathrm{tr}\Big(\mathbf{Ind}_{P''}^{G}((\mathcal{O}'',\Lambda''))(\alpha)$$

$$\bullet\mathbf{c}(P''|A'':P''|A'':w'':\Lambda'')^{*}\frac{d}{d\Lambda''}\mathbf{c}(P''|A'':P''|A'':w'':\Lambda'')\Big)|d\Lambda''|$$

and

$$\mathbf{Con}(\alpha:\Gamma:C'':\mathcal{O}'':w'')$$

is equal to

$$-\frac{1}{2\pi}\bullet 2\pi\bullet\frac{1}{*(C'')}\bullet\frac{1}{|\det(1-w'')|}$$

$$\times\mathrm{tr}\Big(\mathbf{Ind}_{P''}^{G}((\mathcal{O}'',0))(\alpha)\bullet\mathbf{c}(P''|A'':P''|A'':w'':0)\Big).$$

[Note: At the maximal level, therefore, the contribution to the trace is entirely analogous to what obtains when $\mathrm{rank}(\Gamma) = 1$, including the interpretation of the constants.]

$\boxed{C}$ Given $w \in W(A)$, there are three possibilities:

$$\begin{cases} \mathrm{rank}(1-w) = 0 \\ \mathrm{rank}(1-w) = 1 \\ \mathrm{rank}(1-w) = 2. \end{cases}$$

The two extreme cases are the easiest to treat and will be dealt with first.

Let λ_1 and λ_2 be the simple roots; let λ^1 and λ^2 be their duals. Generically, write

$$\hat{\lambda} = \frac{\lambda}{||\lambda||}.$$

$\boxed{\mathrm{rank}(1-w) = 0}$ This requirement implies that $w = 1$. Introduce

$$P_P^G(H) = \frac{1}{2}\{(H, \hat{\lambda}_1)(H, \hat{\lambda}^2) + (H, \hat{\lambda}^1)(H, \hat{\lambda}_2)\}.$$

Then P_P^G is an Arthur polynomial. As such, it is homogeneous of degree 2. Denote by D_P^G the associated differential operator. In this case,

$$\mathbf{Con}(\alpha : \Gamma : C : \mathcal{O} : 1)$$

is equal to

$$-\frac{1}{(2\pi)^2} \bullet \frac{1}{*(C)} \bullet \sum_{w \in W(A)}$$

$$\times \int_{\mathrm{Re}(\Lambda)=0} \mathrm{tr}\Big(\mathbf{Ind}_P^G((\mathcal{O}, \Lambda))(\alpha)$$

$$\bullet \mathbf{c}(P|A : P|A : w : \Lambda)^* D_P^G \mathbf{c}(P|A : P|A : w : \Lambda)\Big)|d\Lambda|.$$

[Note: The similarity with the "$w = 1$" contribution when $\text{rank}(\Gamma) = 1$ is quite striking. In particular, the constants have the "right" interpretation and the derivative is "logarithmic" in character. Needless to say, in the sum over $w \in W(A)$, the term corresponding to $w = 1$ is, a priori, zero.]

$\boxed{\text{rank}(1 - w) = 2}$ This requirement implies that $1 - w$ is nonsingular. In this case,

$$\mathbf{Con}(\alpha : \Gamma : C : \mathcal{O} : w)$$

is equal to

$$-\frac{1}{(2\pi)^2} \bullet (2\pi)^2 \bullet \frac{1}{*(C)} \bullet \frac{1}{|\det(1 - w)|}$$

$$\times \text{tr}\Big(\mathbf{Ind}_P^G((\mathcal{O}, 0))(\alpha) \bullet \mathbf{c}(P|A : P|A : w : 0)\Big).$$

[Note: Again, the resemblance to the "$w \neq 1$" contribution when $\text{rank}(\Gamma) = 1$ is immediately apparent. Once more, the "$1/(2\pi)^2$" is inherent in the Fourier inversion formula; the $(2\pi)^2$ is inherent in the Dini calculus. Because $1 - w$ is nonsingular, they cancel.]

$\boxed{\text{rank}(1 - w) = 1}$ This requirement implies that w is a reflection, say $w = w_\lambda$, where, without loss of generality, λ is a positive short root. The extra "1/2" that arises in what follows has its origin in a change of variables, which can be traced back to the fact that

$$(1 - w_\lambda)(\lambda) = 2\lambda.$$

Put

$$*(C(\lambda)) = 2$$

and let D_λ be the differential operator corresponding to $-\hat{\lambda}$. We distinguish two cases.

(I) $\lambda \not\perp \lambda_1$ and $\lambda \not\perp \lambda_2$. Write

$$\begin{cases} \theta_1 & \text{for the angle between} \quad \lambda \quad \text{and} \quad \lambda_1 \\ \theta_2 & \text{for the angle between} \quad \lambda \quad \text{and} \quad \lambda_2. \end{cases}$$

In this case,

$$\mathbf{Con}(\alpha : \Gamma : C : \boldsymbol{O} : w_\lambda)$$

is equal to

$$-\frac{1}{(2\pi)^2} \bullet 2\pi \bullet \frac{1}{*(C(\lambda))} \bullet \frac{1}{|\det((1-w_\lambda)|\mathrm{Ker}(1-w_\lambda)^\perp)|} \bullet \frac{1}{2}$$

$$\bullet \frac{\sin(\theta_1+\theta_2)}{\cos(\theta_1)\cos(\theta_2)}$$

$$\times D_\lambda \Big|_{\Lambda'=0} \int_{\mathrm{Ker}(1-w_\lambda)} \mathrm{tr}\Big(\mathbf{Ind}_P^G((\boldsymbol{O}, \Lambda + \Lambda'))(\alpha)$$

$$\bullet \mathbf{c}(P|A : P|A : w_\lambda : \Lambda + \Lambda')\Big)|d\Lambda|.$$

[Note: The constant

$$\frac{\sin(\theta_1+\theta_2)}{\cos(\theta_1)\cos(\theta_2)}$$

is strictly positive or strictly negative.]

(II) $\lambda \perp \lambda_1$ or $\lambda \perp \lambda_2$. Let $i = 1$ or 2 and suppose that $\lambda \perp \lambda_i$. In this case,

$$\mathbf{Con}(\alpha : \Gamma : C : \boldsymbol{O} : w_\lambda)$$

is equal to the sum of a pair of terms, namely:

(II_1) Call w_i the simple reflection in λ_i – then the first term is

$$-\frac{1}{(2\pi)^2} \bullet 2\pi \bullet \frac{1}{*(C(\lambda))} \bullet \frac{1}{|\det((1-w_\lambda)|\mathrm{Ker}(1-w_\lambda)^\perp)|} \bullet \frac{1}{2}$$

$$\times \int_{\mathrm{Ker}(1-w_\lambda)} \mathrm{tr}\Big(\mathbf{Ind}_P^G((\mathcal{O},\Lambda))(\alpha)$$

$$\bullet \mathbf{c}(P|A : P|A : w_i w_\lambda : \Lambda)^* \frac{d}{d\Lambda}\mathbf{c}(P|A : P|A : w_i : \Lambda)\Big)|d\Lambda|.$$

[Note: Here, the c-function enters as a "hybrid" logarithmic derivative.]

(II_2) Call θ_{12} the angle between λ_1 and λ_2 – then the second term is

$$-\frac{1}{(2\pi)^2}\bullet 2\pi \bullet \frac{1}{*(C(\lambda))}\bullet \frac{1}{|\det((1-w_\lambda)|\mathrm{Ker}(1-w_\lambda)^\perp)|}\bullet \frac{1}{2}\bullet \cot(\pi-\theta_{12})$$

$$\times D_\lambda \Big|_{\Lambda'=0} \int_{\mathrm{Ker}(1-w_\lambda)} \mathrm{tr}\Big(\mathbf{Ind}_P^G((\mathcal{O},\Lambda+\Lambda'))(\alpha)$$

$$\bullet \mathbf{c}(P|A : P|A : w_\lambda : \Lambda+\Lambda')\Big)|d\Lambda|.$$

[Note: Since (λ_1,λ_2) is ≤ 0, the cotangent of $\pi-\theta_{12}$ is ≥ 0 and can $= 0$ (e.g., in $\mathbf{A}_1 \times \mathbf{A}_1$).]

We remark that the "2π" supra is the Dini constant, hence does not cancel the "$1/(2\pi)^2$", the Fourier constant. Also, $\forall \lambda$,

$$|\det((1-w_\lambda)|\mathrm{Ker}(1-w_\lambda)^\perp)| = 2.$$

To have a specific illustration of all this, take

$$\begin{cases} G = \mathbf{SL}(3,\mathbf{R}) \\ \Gamma = \mathbf{SL}(3,\mathbf{Z}). \end{cases}$$

Then $\#(W(A)) = 6$. Apart from $w = 1$, there are two rotations, w' and w'', and three reflections, w_1, w_2, and w_3. Regarding the latter, only case I applies and we accordingly pick up a sum

$$\sum_{i=1}^{3} \mathbf{Con}(\alpha : \Gamma : C : \mathcal{O} : w_i)$$

of three "orthogonal derivatives".

The appearance of

$$D_\lambda \Big|_{\Lambda'=0} \int_{\mathrm{Ker}(1-w_\lambda)} \mathrm{tr}\Big(\mathbf{Ind}_P^G((\boldsymbol{O}, \Lambda+\Lambda'))(\alpha)$$

$$\bullet \mathbf{c}(P|A : P|A : w_\lambda : \Lambda+\Lambda')\Big)|d\Lambda|$$

is not a total surprise, if only because in higher rank derivatives of Dirac distributions are produced by the Dini calculus in the presence of quadratic denominators (via the two roots). Indeed, if

$$\delta(\mathrm{Ker}(1-w_\lambda))$$

is the Dirac distribution concentrated on $\mathrm{Ker}(1-w_\lambda)$, then our "orthogonal derivative" is, up to a constant, the result of applying

$$\delta'(\mathrm{Ker}(1-w_\lambda))$$

to

$$\mathrm{tr}\Big(\mathbf{Ind}_P^G((\boldsymbol{O}, ?))(\alpha) \bullet \mathbf{c}(P|A : P|A : w_\lambda : ?)\Big).$$

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