G, Γ, G/Γ

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CONTENTS

§1. *ASSOCIATIVE ALGEBRAS*

1: DEFINITION An associative algebra over C is a finite dimensional vector space A over C equipped with a bilinear map

$$\mu: A \times A \to A, \quad (x,y) \to \mu(x,y) \equiv xy$$

such that $(xy)z = x(yz)$.

2: DEFINITION An associative algebra A is said to be unital if there exists an element $e \in A$ with the property that $xe = ex = x$ for all $x \in A$.

[Note: Such an e is called an identity element and is denoted by 1_A.]

3: N.B. Identity elements are unique.

4: EXAMPLE Let V be a finite dimensional vector space over C — then $\text{Hom}(V)$ (the set of all C-linear maps of V) is a unital associative algebra over C (multiplication being composition of linear transformations and identity element id_V).

Let A be an associative algebra over C.

5: DEFINITION A representation of A is a pair (ρ,V), where V is a finite dimensional vector space over C and $\rho: A \to \text{Hom}(V)$ is a morphism of associative algebras.

[Note: If A is unital, then it will be assumed that $\rho(1_A) = \text{id}_V$, thus is a morphism of unital associative algebras.]

6: DEFINITION Let (ρ,V) be a representation of A — then a linear subspace $U \subset V$ is said to be ρ-invariant if $\forall\, x \in A$, $\rho(x)U \subset U$.

7: N.B. A ρ-invariant subspace $U \subset V$ gives rise to two representations of A, viz. by restricting to U and passing to the quotient V/U.

8: DEFINITION A representation (ρ,V) of A is irreducible if $V \neq \{0\}$ and if the only ρ-invariant subspaces are $\{0\}$ and V.

9: NOTATION Given a representation (ρ,V) of A, put

$$\mathrm{Ker}(\rho) = \{x \in A : \rho(x) = 0\}.$$

10: N.B. $\mathrm{Ker}(\rho)$ is a two-sided ideal in A.

11: DEFINITION A representation (ρ,V) of A is faithful if $\mathrm{Ker}(\rho) = \{0\}$.

12: DEFINITION Let (ρ,V), (σ,W) be representations of A -- then an intertwining operator is a $\underline{C}$-linear map $T:V \to W$ such that $T\rho(x) = \sigma(x)T$ for all $x \in A$.

13: NOTATION $I_A(\rho,\sigma)$ is the set of intertwining operators between (ρ,V) and (σ,W).

14: EXAMPLE Let (ρ,V) be a representation of A and suppose that $U \subset V$ is a ρ-invariant subspace -- then the inclusion map $U \to V$ is an intertwining operator, as is the quotient map $V \to V/U$.

15: DEFINITION Representations (ρ,V) and (σ,W) of A are equivalent if there exists an invertible operator in $I_A(\rho,\sigma)$, in which case we write

$$(\rho,V) \approx (\sigma,W) \qquad (\text{or } \rho \approx \sigma).$$

16: NOTATION $\hat{A}$ is the set of equivalence classes of irreducible representations of A.

<u>17:</u> EXAMPLE Take A = Hom(V), where V is a finite dimensional complex vector space --- then up to equivalence, the only irreducible representation of Hom(V) is the representation ρ given by

$$\rho(T)v = Tv \quad (T \in \mathrm{Hom}(V)).$$

§2. *REPRESENTATION THEORY*

Let A be a unital associative algebra over $\mathbb{C}$.

1: THEOREM Let (ρ,V), (σ,W) be irreducible representations of A -- then

$$\dim I_A(\rho,\sigma) = \begin{cases} 1 \text{ if } (\rho,V) \approx (\sigma,W) \\ 0 \text{ if } (\rho,V) \not\approx (\sigma,W). \end{cases}$$

2: THEOREM Let (ρ,V) be an irreducible representation of A -- then $\rho(A) = \operatorname{Hom}(V)$.

3: DEFINITION A representation (ρ,V) of A is completely reducible if for every ρ-invariant subspace $V_1 \subset V$ there exists a ρ-invariant subspace $V_2 \subset V$ such that $V = V_1 \oplus V_2$.

4: LEMMA Suppose that (ρ,V) is a representation of A -- then (ρ,V) is completely reducible iff there is a decomposition

$$V = V_1 \oplus \cdots \oplus V_s,$$

where each V_i is ρ-invariant and irreducible.

5: LEMMA Suppose that (ρ,V) is a representation of A -- then (ρ,V) is completely reducible iff there is a decomposition

$$V = U_1 + \cdots + U_t,$$

where each U_j is ρ-invariant and irreducible.

6: DEFINITION Let V be a finite dimensional vector space over $\mathbb{C}$. Given a subset $\mathcal{S}$ of Hom(V), put

$$\text{Com}(\mathcal{S}) = \{T \in \text{Hom}(V): Ts = sT \ \forall\ s \in \mathcal{S}\},$$

the <u>commutant</u> of $\mathcal{S}$.

<u>7:</u> <u>N.B.</u> Com($\mathcal{S}$) is a unital associative algebra over $\mathbb{C}$.

<u>8:</u> THEOREM Suppose that V is a finite dimensional vector space over $\mathbb{C}$ and let $\mathcal{V} \subset \text{Hom}(V)$ be an associative algebra over $\mathbb{C}$ with identity id_V. Assume: V is completely reducible per the canonical action of $\mathcal{V}$ -- then

$$\text{Com}(\text{Com}(\mathcal{V})) = \mathcal{V}.$$

[Note: A priori,

$$\mathcal{V} \subset \text{Com}(\text{Com}(\mathcal{V})).]$$

<u>9:</u> NOTATION Let (ρ, V) be a completely reducible representation of A. Given $\delta \in \hat{A}$, put

$$V_\delta = \sum_{U \subset V: [U] = \delta} U,$$

the subspaces U being ρ-invariant and irreducible, [U] standing for the equivalence class in $\hat{A}$ determined by U.

<u>10:</u> THEOREM Let (ρ, V) be a completely reducible representation of A and let

$$V = V_1 \oplus \cdots \oplus V_s$$

be a decomposition, where each V_i is ρ-invariant and irreducible -- then $\forall\ \delta \in \hat{A}$,

$$V_\delta = \oplus_{[V_i]=\delta} V_i,$$

thus

$$V = \oplus_{\delta \in \hat{A}} V_\delta .$$

[Note: An empty sum is taken to be zero.]

<u>11:</u> DEFINITION The decomposition

$$V = \oplus_{\delta \in \hat{A}} V_\delta$$

is the <u>primary decomposition</u> of V and V_δ is the δ-<u>isotypic</u> subspace of V.

<u>12:</u> DEFINITION The cardinality $m_V(\delta)$ of

$$\{i : [V_i] = \delta\}$$

is the <u>multiplicity</u> of δ in V.

<u>13:</u> NOTATION Given $\delta \in \hat{A}$, let $U(\delta)$ be an element in the class δ.

<u>14:</u> LEMMA

$$m_V(\delta) = \dim I_A(U(\delta), V) = \dim I_A(V, U(\delta)).$$

§3. *CHARACTERS*

Let A be a unital associative algebra over $\mathbf{C}$.

<u>1:</u> DEFINITION Let (ρ,V) be a representation of A -- then its <u>character</u> is the linear functional

$$\chi_\rho : A \to \mathbf{C}$$

given by the prescription

$$\chi_\rho(x) = \operatorname{tr}(\rho(x)) \quad (x \in A).$$

<u>2:</u> LEMMA

$$\chi_\rho(1_A) = \dim V.$$

<u>3:</u> LEMMA $\forall\, x,y \in A$,

$$\chi_\rho(xy) = \chi_\rho(yx).$$

<u>4:</u> DEFINITION Let (ρ,V) be a representation of A -- then a <u>composition series</u> for ρ is a sequence of ρ-invariant subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_s = V$$

such that

$$\{0\} \neq V_i/V_{i-1} \quad (i = 1,\ldots,s)$$

is irreducible.

<u>5:</u> LEMMA Composition series exist.

<u>6:</u> DEFINITION The <u>semisimplification</u> of (ρ,V) is the direct sum

$$V_{ss} = \bigoplus_{i=1}^{s} (V_i/V_{i-1})$$

equipped with the canonical operations.

7: DEFINITION The irreducible quotients V_i/V_{i-1} are the composition factors of (ρ,V).

Let ρ_{ss} be the representation of A per V_{ss} and let ρ_i be the representation of A per V_i/V_{i-1}.

8: LEMMA

$$\chi_{\rho_{ss}} = \sum_{i=1}^{s} \chi_{\rho_i} = \chi_\rho.$$

9: LEMMA Suppose that (σ_1,U_1), ..., (σ_r,U_r) are irreducible representations of A. Assume: (σ_k,U_k) is not equivalent to (σ_ℓ,U_ℓ) $(k \neq \ell)$ -- then the set

$$\{\chi_{\sigma_1},\ldots,\chi_{\sigma_r}\}$$

is linearly independent.

10: SCHOLIUM The composition factors in a composition series for ρ are unique up to isomorphism and order of appearance and (ρ_{ss},V_{ss}) is uniquely determined by χ_ρ up to isomorphism.

§4. *SIMPLE AND SEMISIMPLE ALGEBRAS*

Let A be a unital associative algebra over $\mathbf{C}$.

1: DEFINITION A is simple if the only two-sided ideals in A are $\{0\}$ and A.

2: LEMMA If V is a finite dimensional vector space over $\mathbf{C}$, then $\mathrm{Hom}(V)$ is simple.

3: THEOREM If A is simple, then there is a finite dimensional vector space V over $\mathbf{C}$ such that $A \approx \mathrm{Hom}(V)$.

4: DEFINITION A is semisimple if it is a finite direct sum of simple algebras.

Accordingly, if A is semisimple, then there is a finite set L, finite dimensional complex vector spaces V_λ $(\lambda \in L)$, and an isomorphism

$$\phi: A \rightarrow \bigoplus_{\lambda \in L} \mathrm{Hom}(V_\lambda).$$

Denote by E_λ the element

$$0 \oplus \cdots \oplus \mathrm{id}_{V_\lambda} \oplus \cdots \oplus 0$$

and define a representation $(\rho_\lambda, V_\lambda)$ by the prescription

$$\rho_\lambda(x) = \phi(x)E_\lambda \quad (x \in A).$$

5: LEMMA The $(\rho_\lambda, V_\lambda)$ are irreducible.

6: THEOREM Every irreducible representation of A is equivalent to some $(\rho_\lambda, V_\lambda)$.

7: N.B. Therefore

$$\hat{A} \longleftrightarrow L,$$

so the term "λ-isotypic subspace" makes sense.

Put

$$e_\lambda = \phi^{-1}(E_\lambda).$$

Then e_λ is a central idempotent and

$$\sum_{\lambda \in L} e_\lambda = 1_A.$$

8: THEOREM Suppose that A is semisimple and let (ρ, V) be a representation of A -- then its λ-isotypic subspace is $\rho(e_\lambda)V$ and

$$V = \bigoplus_{\lambda \in L} \rho(e_\lambda)V$$

is the primary decomposition of V.

9: LEMMA Let A be a unital associative algebra over $\mathbf{C}$ and let (ρ, V) be a completely reducible representation of A -- then $\rho(A)$ is semisimple.

10: THEOREM Let A be a unital associative algebra over $\mathbf{C}$ -- then the following conditions are equivalent:

1. The left regular representation (L, A) of A is completely reducible ($L(x)y = xy$).

2. Every representation of A is completely reducible.

3. A is a semisimple algebra.

[1 => 3:L(A) is semisimple (cf. #9). On the other hand, A ≈ L(A), L being faithful.

3 => 2: Quote #8 and §2, #4.

2 => 1: Obvious.]

<u>11:</u> THEOREM Every representation of a semisimple algebra is uniquely determined by its character up to isomorphism.

§1. *GROUP ALGEBRAS*

<u>1:</u> NOTATION If X is a finite set, then $|X|$ is the cardinality of X and C(X) is the vector space of complex valued functions on X.

<u>2:</u> <u>N.B.</u> The functions $\{\delta_x : x \in X\}$, where

$$\delta_x(y) = \begin{cases} 1 & (x = y) \\ 0 & (x \neq y), \end{cases}$$

constitute a basis for C(X). Therefore

$$\dim C(X) = |X|$$

and every $f \in C(X)$ admits a decomposition

$$f = \sum_{x \in X} f(x)\delta_x.$$

In particular: If 1_X is the function on X which is $\equiv 1$, then

$$1_X = \sum_{x \in X} \delta_x.$$

Let G be a finite group.

<u>3:</u> DEFINITION Given $f,g \in C(G)$, their <u>convolution</u> $f * g$ is the element of C(G) defined by the rule

$$(f * g)(x) = \sum_{y \in G} f(xy^{-1})g(y)$$

$$= \sum_{y \in G} f(y)g(y^{-1}x).$$

[Note: $\forall\ x,y \in G$,

$$\delta_x * \delta_y = \delta_{xy}.]$$

4: LEMMA C(G) is an associative algebra over $\mathbf{C}$.

5: N.B. If e is the identity in G, then δ_e is the identity in C(G), which is therefore unital.

6: LEMMA The center of C(G) consists of those f such that

$$f(x) = f(yxy^{-1}) \quad (x,y \in G).$$

[Note: In other words, the center of C(G) consists of those f that are constant on conjugacy classes, the so-called class functions.]

E.g.: $\forall\ x \in G$, the function

$$\sum_{y\in G} \delta_{yxy^{-1}}$$

is a class function.

[Given z in G,

$$\begin{aligned}(\sum_{y\in G} \delta_{yxy^{-1}})*\delta_z &= \sum_{y\in G} \delta_{yxy^{-1}z} \\ &= \sum_{y\in G} \delta_{zyx(zy)^{-1}z} \\ &= \sum_{y\in G} \delta_{zyxy^{-1}} \\ &= \delta_z*(\sum_{y\in G} \delta_{yxy^{-1}}).]\end{aligned}$$

7: DEFINITION A representation of G is a pair (π, V), where V is a finite dimensional vector space over $\mathbb{C}$ and $\pi: G \to GL(V)$ is a morphism of groups.

8: SCHOLIUM Let V be a finite dimensional vector space over $\mathbb{C}$.

- Every representation $\pi: G \to GL(V)$ extends to a representation ρ of $C(G)$ on V, viz.

$$\rho(f) = \sum_{x \in G} f(x)\pi(x).$$

- Every representation $\rho: C(G) \to \text{Hom}(V)$ restricts to a representation π of G on V, viz.

$$\pi(x) = \rho(\delta_x).$$

[Note: If π is given, it is customary to denote its extension "ρ" by π as well.]

9: LEMMA Let $W \subset V$ be a linear subspace -- then W is invariant under G iff W is invariant under $C(G)$.

10: LEMMA An operator $T \in \text{Hom}(V)$ commutes with the action of G iff it commutes with the action of $C(G)$.

11: THEOREM $C(G)$ is semisimple.

PROOF Let (ρ, V) be a representation of $C(G)$ and suppose that $V_1 \subset V$ is a ρ-invariant subspace. Fix a linear complement U per $V_1: V = V_1 \oplus U$. Let $P: V \to V_1$ be the corresponding projection and put

$$Q = \frac{1}{|G|} \sum_{x \in G} \pi(x) P \pi(x)^{-1}.$$

Then Q is a projection with range V_1. In addition, $\forall\, y \in G$,

$$\pi(y)Q = \frac{1}{|G|}\sum_{x\in G} \pi(yx)P\pi(x)^{-1}$$

$$= \frac{1}{|G|}\sum_{x\in G} \pi(x)P\pi(y^{-1}x)^{-1}$$

$$= \frac{1}{|G|}\sum_{x\in G} \pi(x)P\pi(x)^{-1}\pi(y)$$

$$= Q\pi(y).$$

Consequently, $\forall\, v \in V$,

$$\pi(y)(id_V - Q)v = \pi(y)(v - Qv)$$

$$= \pi(y)v - \pi(y)Qv$$

$$= \pi(y)v - Q\pi(y)v$$

$$= (id_V - Q)\pi(y)v,$$

thus the range V_2 of $id_V - Q$ is a ρ-invariant complement per V_1. It therefore follows that every representation of $C(G)$ is completely reducible, hence $C(G)$ is semisimple (cf. I, §4, #10).

<u>12:</u> DEFINITION

- The left translation representation L of G on C(G) is the prescription

$$L(x)f(y) = f(x^{-1}y) \quad (\Rightarrow L(x)f = \delta_x * f).$$

- The right translation representation of G on C(G) is the prescription

$$R(x)f(y) = f(yx) \quad (\Rightarrow R(x)f = f * \delta_{x^{-1}}).$$

<u>13:</u> <u>N.B.</u> Since C(G) is semisimple, both L and R are completely reducible.

14: REMARK There is also a representation $\pi_{L,R}$ of $G \times G$ on $C(G)$, namely

$$(\pi_{L,R}(x_1,x_2)f)(x) = f(x_1^{-1}xx_2).$$

And it too is completely reducible ($C(G \times G)$ is semisimple).

15: DEFINITION Let (π_1,V_1), (π_2,V_2) be representations of G -- then an intertwining operator is a $\mathbb{C}$-linear map $T:V_1 \to V_2$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$.

16: NOTATION $I_G(\pi_1,\pi_2)$ is the set of intertwining operators between (π_1,V_1) and (π_2,V_2).

17: N.B. On the basis of the definitions,

$$I_G(\pi_1,\pi_2) = I_{C(G)}(\rho_1,\rho_2).$$

18: LEMMA Let (π_1,V_1), (π_2,V_2) be irreducible representations of G and let $T \in I_G(\pi_1,\pi_2)$ -- then either T is zero or it is an isomorphism.

19: LEMMA Suppose that (π,V) is an irreducible representation of G and suppose that $T \in I_G(\pi,\pi)$ -- then T is a scalar multiple of id_V.

20: DEFINITION Representations (π_1,V_1) and (π_2,V_2) of G are equivalent if there exists an invertible operator in $I_G(\pi_1,\pi_2)$, in which case we write

$$(\pi_1,V_1) \approx (\pi_2,V_2) \quad (\text{or } \pi_1 \approx \pi_2).$$

21: NOTATION $\hat{G}$ is the set of equivalence classes of irreducible representations of G.

[Note: By convention, the zero representation of G on $V = \{0\}$ is not to be viewed as irreducible.]

<u>22:</u> <u>N.B.</u> There is a one-to-one correspondence

$$\hat{G} \approx \widehat{C(G)}.$$

In the sequel, Π stands for an element of $\hat{G}$ with representation space $V(\Pi)$ of dimension d_Π. Without loss of generality, it can be assumed moreover that Π is unitary with respect to a G-invariant inner product $< \ , \ >_\Pi$ on $V(\Pi)$.

[Recall the argument. Start with an inner product $< \ , \ >$ on $V(\Pi)$ and put

$$<v_1,v_2>_\Pi = \frac{1}{|G|} \sum_{x \in G} <\Pi(x)v_1,\Pi(x)v_2>.]$$

APPENDIX

Let (π_1,V_1), (π_2,V_2) be unitary representations of G. Suppose that there exists an invertible

$$T \in I_G(\pi_1,\pi_2).$$

Then there exists a unitary

$$U \in I_G(\pi_1,\pi_2).$$

[Let $T = U|T|$ be the polar decomposition of T -- then $\forall\ x \in G$,

$$|T|\pi_1(x) = \pi_1(x)|T|.$$

Therefore

$$U\pi_1(x)U^{-1} = T|T|^{-1}\pi_1(x)|T|T^{-1}$$

$$= T\pi_1(x)T^{-1} = \pi_2(x).]$$

§2. CONTRAGREDIENTS AND TENSOR PRODUCTS

1: NOTATION Given a finite dimensional vector space V over $\mathbb{C}$, let V^* be its dual and denote by

$$\begin{cases} V^* \times V \to \mathbb{C} \\ (v^*,v) \to \langle v^*,v\rangle \quad (= v^*(v)) \end{cases}$$

the evaluation pairing.

Let G be a finite group.

2: DEFINITION Suppose that $\pi:G \to GL(V)$ is a representation -- then its contragredient is the representation $\pi^*:G \to GL(V^*)$ defined by requiring that $\forall\ x \in G$,

$$\pi^*(x)v^* = v^* \circ \pi(x^{-1}) \quad (v^* \in V^*),$$

thus $\forall\ v \in V$,

$$\langle \pi^*(x)v^*,v\rangle = \langle v^*,\pi(x^{-1})v\rangle.$$

3: N.B. The identification $(V^*)^* \approx V$ leads to an equivalence $(\pi^*)^* \approx \pi$.

4: LEMMA (π,V) is irreducible iff (π^*,V^*) is irreducible.

5: CONVENTION Given $(\Pi,V(\Pi))$ in $\hat{G}$, take

$$V(\Pi^*) = V(\Pi)^*, \ \Pi^*(x) = \Pi(x^{-1})^{\intercal}.$$

6: NOTATION Given finite dimensional vector spaces V_1,V_2 over $\mathbb{C}$, let $V_1 \otimes V_2$ be their tensor product.

Let G be a finite group.

<u>7:</u> DEFINITION Suppose that $\pi_1:G \to GL(V_1)$, $\pi_2:G \to GL(V_2)$ are representations -- then their <u>tensor product</u> is the representation $\pi_1 \otimes \pi_2:G \to GL(V_1 \otimes V_2)$ defined by requiring that $\forall\ x \in G$,

$$(\pi_1 \otimes \pi_2)(x)(v_1 \otimes v_2) = \pi_1(x)v_1 \otimes \pi_2(x)v_2.$$

Let (π_1,V_1), (π_2,V_2) be representations of G -- then the prescription

$$\pi_{1,2}(x)T = \pi_2(x)T\pi_1(x^{-1}) \qquad (T \in \text{Hom}(V_1,V_2))$$

defines a representation $\pi_{1,2}$ of G on $\text{Hom}(V_1,V_2)$.

<u>8:</u> RAPPEL There is a canonical isomorphism

$$\Theta:V_2 \otimes V_1^* \approx \text{Hom}(V_1,V_2).$$

[Send $v_2 \otimes v_1^*$ to the linear transformation

$$T(v_2,v_1^*):v_1 \to v_1^*(v_1)v_2.]$$

Consider

$$\pi_2(x)v_2 \otimes \pi_1^*(x)v_1^* \in V_2 \otimes V_1^*.$$

Then the corresponding element of $\text{Hom}(V_1,V_2)$ is the assignment

$$\begin{aligned} v_1 \to\ & (\pi_1^*(x)v_1^*)(v_1)\pi_2(x)v_2 \\ &= v_1^*(\pi_1(x^{-1})v_1)\pi_2(x)v_2 \\ &= \pi_2(x)(v_1^*(\pi_1(x^{-1})v_1))v_2 \end{aligned}$$

$$= \pi_2(x)T(v_2,v_1^*)\pi_1(x^{-1})v_1.$$

<u>9:</u> LEMMA $\pi_2 \otimes \pi_1^*$ is equivalent to $\pi_{1,2}$.

[The isomorphism Θ intertwines $\pi_2 \otimes \pi_1^*$ and $\pi_{1,2}$: $\forall\ x \in G$,

$$\Theta \circ (\pi_2(x) \otimes \pi_1^*(x)) = \pi_{1,2}(x) \circ \Theta.]$$

Let G_1, G_2 be finite groups.

<u>10:</u> DEFINITION Suppose that $\pi_1: G_1 \to GL(V_1)$, $\pi_2: G \to GL(V_2)$ are representations -- then their <u>outer tensor product</u> is the representation $\pi_1 \underline{\otimes} \pi_2$: $G_1 \times G_2 \to GL(V_1 \otimes V_2)$ defined by requiring that $\forall\ x_1 \in G_1$, $\forall\ x_2 \in G_2$,

$$(\pi_1 \underline{\otimes} \pi_2)(x_1,x_2) = \pi_1(x_1) \otimes \pi_2(x_2).$$

<u>11:</u> <u>N.B.</u> If $G_1 = G_2 = G$, then the restriction of the outer tensor product $\pi_1 \underline{\otimes} \pi_2$ to the diagonal subgroup

$$\{(x,x) : x \in G\}$$

of $G \times G$ is the tensor product $\pi_1 \otimes \pi_2$.

<u>12:</u> REMARK Take $G_1 = G_2 = G$ and define a representation $\pi_{1,2}$ of $G \times G$ on $\mathrm{Hom}(V_1,V_2)$ via the prescription

$$\pi_{1,2}(x,y)T = \pi_2(x)T\pi_1(y^{-1}) \quad (T \in \mathrm{Hom}(V_1,V_2)).$$

Then $\pi_2 \underline{\otimes} \pi_1^*$ is equivalent to $\pi_{1,2}$.

13: LEMMA If π_1 and π_2 are irreducible, then $\pi_1 \underline{\otimes} \pi_2$ is irreducible.

[To begin with,

$$C(G_1 \times G_2) \approx C(G_1) \otimes C(G_2)$$

and

$$\mathrm{Hom}(V_1 \otimes V_2) \approx \mathrm{Hom}(V_1) \otimes \mathrm{Hom}(V_2).$$

Now make the passage

$$\left[\begin{array}{l} \pi_1 \to \rho_1 \\ \\ \pi_2 \to \rho_2. \end{array}\right.$$

Then

$$\left[\begin{array}{l} \rho_1(C(G_1)) = \mathrm{Hom}(V_1) \\ \\ \rho_2(C(G_2)) = \mathrm{Hom}(V_2) \end{array}\right. \qquad \text{(cf. I, §2, \#2).]}$$

Conversely:

14: THEOREM Every irreducible representation of $G_1 \times G_2$ is equivalent to an outer tensor product $\pi_1 \underline{\otimes} \pi_2$.

15: SCHOLIUM

$$\widehat{G_1 \times G_2} \approx \hat{G}_1 \times \hat{G}_2.$$

§3. *FOURIER TRANSFORMS*

Let G be a finite group.

1: DEFINITION Given $f \in C(G)$, its Fourier transform $\hat{f}$ is that element of

$$\bigoplus_{\Pi \in \hat{G}} \mathrm{Hom}(V(\Pi))$$

whose Π-component is

$$\hat{f}(\Pi) \equiv \sum_{x \in G} f(x)\Pi(x) \ (= \Pi(f)).$$

E.g.: $\forall\ x \in G$,

$$\hat{\delta}_x(\Pi) = \Pi(x).$$

2: LEMMA $\forall\ f_1, f_2 \in C(G)$,

$$\widehat{f_1 * f_2}(\Pi) = f_1(\Pi) f_2(\Pi).$$

3: EXAMPLE $\forall\ x \in G$,

$$\begin{cases} \widehat{L(x)f}(\Pi) = \widehat{\delta_x * f}(\Pi) = \Pi(x)\hat{f}(\Pi) \\ \widehat{R(x)f}(\Pi) = \widehat{f * \delta_{x^{-1}}}(\Pi) = \hat{f}(\Pi)\Pi(x^{-1}). \end{cases}$$

4: THEOREM The Fourier transform

$$\wedge : C(G) \rightarrow \bigoplus_{\Pi \in \hat{G}} \mathrm{Hom}(V(\Pi))$$

is an algebra isomorphism.

5: APPLICATION

$$|G| = \sum_{\Pi \in \hat{G}} d_{\Pi}^2.$$

[In fact,

$$\dim C(G) = |G| \text{ and } \dim \mathrm{Hom}(V_{\Pi}) = d_{\Pi}^2.]$$

As it stands, $C(G)$ is a unital associative algebra over $\mathbf{C}$. But more is true: $C(G)$ is a $*$-algebra, i.e., admits a conjugate linear antiautomorphism $f \to f^*$ given by $f^*(x) = \overline{f(x^{-1})}$ $(x \in G)$.

Each $T \in \mathrm{Hom}(V(\Pi))$ has an adjoint T^* per $< , >_{\Pi}$: $\forall\ v_1, v_2 \in V(\Pi)$,

$$\langle Tv_1, v_2 \rangle_{\Pi} = \langle v_1, T^* v_2 \rangle_{\Pi}.$$

Therefore

$$\bigoplus_{\Pi \in \hat{G}} \mathrm{Hom}(V(\Pi))$$

admits a conjugate linear antiautomorphism by using the arrow $T \to T^*$ on each summand.

6: N.B. It can and will be assumed that

$$\begin{cases} V(\Pi^*) = V(\Pi) \\ \Pi^*(x) = \Pi(x^{-1}) \end{cases} \quad \text{(cf. §2, \#5)},$$

hence in terms of adjoints

$$\Pi(x)^* = \Pi(x)^{-1} = \Pi(x^{-1}) = \Pi^*(x).$$

7: LEMMA The Fourier transform

$$\wedge : C(G) \to \bigoplus_{\Pi \in \hat{G}} \mathrm{Hom}(V(\Pi))$$

preserves the *-operations: $\forall f \in C(G)$,

$$\widehat{f^*} = (\hat{f})^*.$$

8: INVERSION FORMULA Given $f \in C(G)$, $\forall x \in G$,

$$f(x) = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi \, \mathrm{tr}(\Pi(x^{-1})\hat{f}(\Pi)).$$

In particular:

$$f(e) = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi \, \mathrm{tr}(\hat{f}(\Pi)).$$

9: PARSEVAL IDENTITY Given $f_1, f_2 \in C(G)$,

$$\sum_{x \in G} f_1(x) f_2(x^{-1}) = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi \, \mathrm{tr}(\hat{f}_1(\Pi)\hat{f}_2(\Pi)).$$

PROOF Put $f = f_1 * f_2$ -- then

$$f(e) = \sum_{x \in G} f_1(x) f_2(x^{-1}).$$

On the other hand,

$$\frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi \, \mathrm{tr}(\hat{f}_1(\Pi)\hat{f}_2(\Pi))$$

$$= \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi \, \mathrm{tr}(\widehat{f_1 * f_2}(\Pi))$$

$$= (f_1 * f_2)(e) = f(e).$$

<u>10:</u> COMPLETENESS PRINCIPLE If $f \in C(G)$ and if $\hat{f}(\Pi) = 0$ for all Π, then $f = 0$.

§4. *CLASS FUNCTIONS*

Let G be a finite group.

1: DEFINITION Let (π,V) be a representation of G -- then its character is the function

$$\chi_\pi : G \to C$$

given by the prescription

$$\chi_\pi(x) = \text{tr}(\pi(x)) \quad (x \in G).$$

2: N.B. It is clear that characters are class functions and that equivalent representations have equal characters.

3: LEMMA $\forall\ x \in G$,

$$\chi_\pi(x^{-1}) = \overline{\chi_\pi(x)}.$$

4: N.B.

$$\chi_{\pi^*} = \overline{\chi_\pi}.$$

5: LEMMA Let

$$\begin{cases} \pi_1 : G \to GL(V_1) \\ \pi_2 : G \to GL(V_2) \end{cases}$$

be representations of G -- then the character of

$$(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$$

is $\chi_{\pi_1}\chi_{\pi_2}$.

[For the record, the character of

$$(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$$

is $\chi_{\pi_1} + \chi_{\pi_2}$, implying thereby that a nonnegative integral linear combination of characters is again a character.]

6: EXAMPLE $\pi_{1,2}$ is equivalent to $\pi_2 \otimes \pi_1^*$ (cf. §2, #9), hence

$$\chi_{\pi_{1,2}} = \chi_{\pi_2 \otimes \pi_1^*} = \chi_{\pi_2}\chi_{\pi_1^*} = \chi_{\pi_2}\overline{\chi_{\pi_1}}.$$

7: DEFINITION The character of an irreducible representation is called an irreducible character.

[Note: The zero function (i.e., the additive identity of $C(G)$) is a character but it is not an irreducible character (cf. §1, #21).]

8: N.B. The irreducible characters are thus the $\chi_\Pi (\Pi \in \hat{G})$.

9: FIRST ORTHOGONALITY RELATION Let $\Pi_i, \Pi_j \in \hat{G}$ -- then

$$\frac{1}{|G|} \sum_{x \in G} \chi_i(x)\chi_j(x^{-1}) = \delta_{ij},$$

where for short

$$\chi_i = \chi_{\Pi_i}, \ \chi_j = \chi_{\Pi_j}.$$

10: NOTATION Given $x \in G$, write $C(x)$ for its conjugacy class and G_x for its centralizer.

11: RAPPEL The number of conjugates of x in G is $[G:G_x]$, i.e.,

$$|C(x)| = [G:G_x].$$

[Note: The class equation for G is the relation

$$|G| = \sum_i [G:G_{x_i}],$$

one x_i having been chosen from each conjugacy class.

12: SECOND ORTHOGONALITY RELATION Let $x_1, x_2 \in G$ -- then

$$\sum_{\Pi \in \hat{G}} \chi_\Pi(x_1)\chi_\Pi(x_2^{-1}) = \begin{cases} |G_x| & \text{if } x = x_1 = x_2 \\ 0 & \text{if } C(x_1) \neq C(x_2). \end{cases}$$

[Note:

$$|G_x| = \frac{|G|}{[G:G_x]} = \frac{|G|}{|C(x)|}.]$$

13: NOTATION Given $f,g \in C(G)$, put

$$\langle f,g \rangle_G = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)},$$

the canonical inner product on $C(G)$.

14: EXAMPLE $\forall\ \Pi_1, \Pi_2 \in \hat{G}$,

$$\langle \chi_{\Pi_1}, \chi_{\Pi_2} \rangle_G = \begin{cases} 1 & \text{if } \Pi_1 = \Pi_2 \\ 0 & \text{if } \Pi_1 \neq \Pi_2. \end{cases}$$

15: SCHOLIUM The irreducible characters form an orthonormal set, thus are linearly independent (cf. I, §3, #9).

Recall now that the Fourier transform

$$\wedge: C(G) \to \bigoplus_{\Pi \in \hat{G}} \operatorname{Hom}(V(\Pi))$$

is an algebra isomorphism. Since the center of each $\operatorname{Hom}(V(\Pi))$ consists of scalar multiples of the identity operator, it follows that an $f \in C(G)$ is a class function iff $\forall\ \Pi \in \hat{G}$,

$$\hat{f}(\Pi) = C_\Pi \operatorname{id}_{V(\Pi)} \quad (C_\Pi \in \mathbf{C}).$$

<u>16:</u> INVERSION FORMULA Given a class function $f \in C(G)$, $\forall\ x \in G$,

$$f(x) = \sum_{\Pi \in \hat{G}} \langle f, \overline{\chi_\Pi} \rangle_G \overline{\chi_\Pi(x)}.$$

PROOF

$$f(x) = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi \operatorname{tr}(\Pi(x^{-1})\hat{f}(\Pi))$$

$$= \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi C_\Pi \chi_\Pi(x^{-1})$$

$$= \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi C_\Pi \overline{\chi_\Pi(x)}.$$

Fix $\Pi_0 \in \hat{G}$ -- then

$$\langle f, \overline{\chi_{\Pi_0}} \rangle_G = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi C_\Pi \langle \overline{\chi_\Pi}, \overline{\chi_{\Pi_0}} \rangle_G$$

$$= \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi C_\Pi \overline{\langle \chi_\Pi, \chi_{\Pi_0} \rangle_G}$$

$$= \frac{1}{|G|} d_{\Pi_0} C_{\Pi_0}.$$

17: N.B. $\forall\, x \in G$,

$$f(x) = \overline{\overline{f(x)}} = \overline{\sum_{\Pi \in \hat{G}} \langle \bar{f}, \overline{\chi_\Pi} \rangle_G \, \overline{\chi_\Pi(x)}}$$

$$= \sum_{\Pi \in \hat{G}} \overline{\langle \bar{f}, \overline{\chi_\Pi} \rangle_G} \; \overline{\overline{\chi_\Pi(x)}}$$

$$= \sum_{\Pi \in \hat{G}} \langle f, \chi_\Pi \rangle_G \, \chi_\Pi(x).$$

The preceding discussion makes it clear that a class function f is a character iff $\langle f, \Pi \rangle_G$ is a nonnegative integer for all $\Pi \in \hat{G}$.

18: NOTATION CON(G) is the set of conjugacy classes of G.

19: SCHOLIUM The dimension of the space of class functions is equal to $|\mathrm{CON}(G)|$ or still, is equal to $|\hat{G}|$.

20: NOTATION Given $C \in \mathrm{CON}(G)$, let χ_C be the characteristic function of C:

$$\chi_C(x) = \begin{cases} 1 \text{ if } x \in C \\ 0 \text{ if } x \notin C. \end{cases}$$

21: LEMMA

$$\chi_C = \sum_{y \in C} \delta_y.$$

22: N.B. The χ_C $(C \in \mathrm{CON}(G))$ are a basis for the class functions on G (as are the χ_Π $(\Pi \in \hat{G})$).

23: LEMMA Let $C_1, C_2, \ldots$ be the elements of CON(G) -- then there are nonnegative integers $m_{i,j,k}$ such that

$$\chi_{C_i}\chi_{C_j} = \sum_k m_{i,j,k}\chi_{C_k}.$$

[Note: Fixing an $x_k \in C_k$, qualitatively $m_{i,j,k}$ is the number of ordered pairs (x,y) with $x \in C_i$, $y \in C_j$ and $xy = x_k$ while quantitatively

$$m_{i,j,k} = \frac{|C_i|\ |C_j|}{|G|} \sum_{\Pi \in \hat{G}} \frac{\chi_\Pi(x_i)\chi_\Pi(x_j)\overline{\chi_\Pi(x_k)}}{d_\Pi}\ .]$$

24: NOTATION Given $\Pi \in \hat{G}$, put

$$e_\Pi = \wedge^{-1}(E_\Pi) \quad (E_\Pi \in \mathrm{Hom}(V(\Pi)) \qquad (\text{cf. I, §4})).$$

25: LEMMA

$$e_\Pi = \frac{d_\Pi}{|G|} \sum_{y \in G} \chi_\Pi(y^{-1})\,\delta_y.$$

[Note: In brief,

$$e_\Pi = \frac{d_\Pi}{|G|}\chi_{\Pi^*}.]$$

26: LEMMA

$$e_{\Pi_1} * e_{\Pi_2} = \begin{cases} e_\Pi \text{ if } \Pi = \Pi_1 = \Pi_2 \\ \\ 0 \text{ if } \Pi_1 \neq \Pi_2. \end{cases}$$

27: LEMMA

$$\delta_e = \sum_{\Pi \in \hat{G}} e_\Pi.$$

§5. *DECOMPOSITION THEORY*

Let G be a finite group.

1: CONSTRUCTION Suppose that G operates on a finite set S, hence for each $x \in G$ there is given a bijection $s \to x\cdot s$ of S satisfying the identities

$$e\cdot s = s,\ x\cdot(y\cdot s) = (xy)\cdot s.$$

Let $V = \{f:S \to \mathbf{C}\}$ and define a representation $\pi:G \to GL(V)$ by

$$\pi(x)f(s) = f(x^{-1}\cdot s).$$

Then

$$\chi_\pi(x) = |\{s \in S:x\cdot s = s\}|.$$

2: EXAMPLE Take $S = G$ and write $x\cdot y = xy$ -- then the role of V is played by C(G) and the rôle of π is played by L (the left translation representation of G (cf. §1, #12)), hence

$$\chi_L(x) = |\{y \in G:xy = y\}|,$$

which is $|G|$ if $x = e$ and is 0 otherwise.

3: EXAMPLE Take $S = G$ but replace G by $G \times G$, the action being $(x_1,x_2)\cdot y = x_1yx_2^{-1}$ -- then the associated representation is $\pi_{L,R}$ (cf. §1, #14) and

$$\begin{aligned}\chi_{\pi_{L,R}}(x_1,x_2) &= |\{y \in G:x_1yx_2^{-1} = y\}|\\ &= |\{y \in G:x_1 = yx_2y^{-1}\}|\\ &= |G_{x_1}|\end{aligned}$$

if x_1 and x_2 are conjugate and is 0 otherwise.

4: DEFINITION Let (π, V) be a representation of G -- then by complete reducibility, there is a direct sum decomposition

$$\pi = \bigoplus_{\Pi \in \hat{G}} m(\Pi, \pi)\Pi,$$

the nonnegative integer $m(\Pi, \pi)$ being the multiplicity of Π in π.

5: LEMMA $\forall \ \Pi \in \hat{G}$,

$$m(\Pi, \pi) = \langle \chi_\Pi, \chi_\pi \rangle_G.$$

6: N.B.

$$\dim I_G(\Pi, \pi) = m(\Pi, \pi).$$

7: REMARK The operator

$$P_\Pi = \frac{d_\Pi}{|G|} \sum_{x \in G} \overline{\chi_\Pi(x)} \pi(x)$$

is the projection onto the Π-isotypic subspace of V.

8: THEOREM Each $\Pi \in \hat{G}$ is contained in L with multiplicity d_Π.

PROOF In fact,

$$\begin{aligned} m(\Pi, L) &= \langle \chi_\Pi, \chi_L \rangle_G \\ &= \frac{1}{|G|} \sum_{x \in G} \chi_\Pi(x) \overline{\chi_L(x)} \\ &= \frac{1}{|G|} \chi_\Pi(e) |G| \\ &= \chi_\Pi(e) = d_\Pi. \end{aligned}$$

9: N.B. It is a corollary that

$$|G| = \sum_{\Pi \in \hat{G}} d_{\Pi}^2 \quad \text{(cf. §3, #5)}.$$

10: LEMMA Let (π_1, V_1), (π_2, V_2) be representations of G. Assume: $\chi_{\pi_1} = \chi_{\pi_2}$ -- then $(\pi_1, V_1) \approx (\pi_2, V_2)$.

PROOF $\forall \; \Pi \in \hat{G}$,

$$\langle \chi_{\Pi}, \chi_{\pi_1} \rangle_G = \langle \chi_{\Pi}, \chi_{\pi_2} \rangle_G$$

or still, $\forall \; \Pi \in \hat{G}$,

$$m(\Pi, \pi_1) = m(\Pi, \pi_2),$$

from which the assertion.

11: IRREDUCIBILITY CRITERION A representation $\pi: G \to GL(V)$ is irreducible iff $\langle \chi_{\pi}, \chi_{\pi} \rangle_G = 1$.

PROOF The necessity is implied by the first orthogonality relations and the sufficiency follows upon noting that

$$\langle \chi_{\pi}, \chi_{\pi} \rangle_G = \sum_{\Pi \in \hat{G}} m(\Pi, \pi)^2.$$

Let G_1, G_2 be finite groups and let

$$\begin{cases} \Pi_1 : G_1 \to GL(V_1) \\ \Pi_2 : G_2 \to GL(V_2) \end{cases}$$

be irreducible representations of G_1, G_2 -- then the character $\chi_{\Pi_1 \underline{\otimes} \Pi_2}$ of

$$(\Pi_1 \underline{\otimes} \Pi_2, V_1 \otimes V_2)$$

is the function

$$(x_1,x_2) \to \chi_{\Pi_1}(x_1)\chi_{\Pi_2}(x_2) \quad (x_1 \in G_1,\ x_2 \in G_2).$$

12: LEMMA $\Pi_1 \underline{\otimes} \Pi_2$ is irreducible (cf. §2, #13).

PROOF It is a question of applying the irreducibility criterion. Thus

$$\langle \chi_{\Pi_1 \underline{\otimes} \Pi_2}, \chi_{\Pi_1 \underline{\otimes} \Pi_2} \rangle_{G \times G}$$

$$= \frac{1}{|G_1 \times G_2|} \sum_{(x_1,x_2) \in G_1 \times G_2} \chi_{\Pi_1}(x_1)\chi_{\Pi_2}(x_2)\overline{\chi_{\Pi_1}(x_1)\chi_{\Pi_2}(x_2)}$$

$$= \frac{1}{|G_1|} \sum_{x_1 \in G_1} \chi_{\Pi_1}(x_1)\overline{\chi_{\Pi_1}(x_1)} \cdot \frac{1}{|G_2|} \sum_{x_2 \in G_2} \chi_{\Pi_2}(x_2)\overline{\chi_{\Pi_2}(x_2)}$$

$$= \langle \chi_{\Pi_1}, \chi_{\Pi_2} \rangle_{G_1} \cdot \langle \chi_{\Pi_2}, \chi_{\Pi_2} \rangle_{G_2}$$

$$= 1.$$

13: REMARK The cardinality of $\widehat{G_1 \times G_2}$ is $|\mathrm{CON}(G_1 \times G_2)|$ (cf. §4, #19).

But

$$|\mathrm{CON}(G_1 \times G_2)| = |\mathrm{CON}(G_1)||\mathrm{CON}(G_2)|$$

and the preceding considerations produce

$$|\mathrm{CON}(G_1)||\mathrm{CON}(G_2)|$$

pairwise distinct irreducible characters of $G_1 \times G_2$. Therefore every irreducible representation of $G_1 \times G_2$ is equivalent to an outer tensor product $\Pi_1 \underline{\otimes} \Pi_2$, where $\Pi_1 \in \hat{G}_1$, $\Pi_2 \in \hat{G}_2$ (cf. §2, #14).

§6. *INTEGRABILITY*

1: DEFINITION An algebraic integer is a complex number λ which is a root of a polynomial of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

where $a_i \in \mathbb{Z}$ $(0 \le i \le n-1)$.

[Note: Equivalently, an algebraic integer is a complex number λ which is a zero of

$$\det(A - XI)$$

for some square matrix A with entries in $\mathbb{Z}$.]

2: N.B. The rational algebraic integers are precisely the elements of $\mathbb{Z}$.

3: LEMMA If μ,ν are algebraic integers, then $\mu + \nu$ and $\mu\nu$ are also algebraic integers.

Therefore the set of algebraic integers is a subring of $\mathbb{C}$.

4: EXAMPLE Roots of unity are algebraic integers.

Let G be a finite group.

5: LEMMA Let (π,V) be a representation of G, χ_π its character -- then $\forall\ x \in G$, $\chi_\pi(x)$ is an algebraic integer.

[This is because $\chi_\pi(x)$ is a finite sum of roots of unity.]

The center of C(G) (i.e., the class function) is a unital commutative associative algebra over $\mathbb{C}$, thus its irreducible representations are just homomorphisms into C and are indexed by the $\Pi \in \hat{G}$, say ω_Π with

$$\omega_{\Pi_1}(e_{\Pi_2}) = \delta_{\Pi_1,\Pi_2}.$$

[Note: The e_Π $(\Pi \in \hat{G})$ are a basis for the class functions on G.]

<u>6:</u> THEOREM $\forall\ C \in \text{CON}(G)$, $\omega_\Pi(C)$ is an algebraic integer.

PROOF In the notation of §4, #23,

$$\chi_{C_i}\chi_{C_j} = \sum_k m_{i,j,k}\chi_{C_k},$$

hence

$$\begin{aligned}\omega_\Pi(\chi_{C_i})\omega_\Pi(\chi_{C_j}) \\ &= \omega_\Pi(\chi_{C_i}\chi_{C_j}) \\ &= \sum_k m_{i,j,k}\omega_\Pi(\chi_{C_k})\end{aligned}$$

=>

$$\begin{aligned}\sum_k (m_{i,j,k} - \delta_{jk}\omega_\Pi(\chi_{C_i}))\omega_\Pi(\chi_{C_k}) \\ &= 0.\end{aligned}$$

But this means that $\omega_\Pi(\chi_{C_i})$ is an eigenvalue of the matrix A_i whose $(j,k)^{th}$ entry is $m_{i,j,k}$ or still, is a zero of

$$\det(A_i - XI),$$

thus is an algebraic integer.

<u>7:</u> LEMMA $\forall\ C \in \text{CON}(G)$,

$$\omega_\Pi(\chi_C) = \frac{|C|}{d_\Pi}\chi_\Pi(x) \qquad (x \in C).$$

PROOF Owing to §4, #25,

$$e_\Pi = \frac{d_\Pi}{|G|}\sum_{y\in G}\chi_\Pi(y^{-1})\delta_y,$$

so

$$\frac{|G|}{d_\Pi}\chi_\Pi(x)e_\Pi = \chi_\Pi(x)\sum_{y\in G}\chi_\Pi(y^{-1})\delta_y$$

=>

$$\sum_{\Pi\in\hat{G}}\frac{|G|}{d_\Pi}\chi_\Pi(x)e_\Pi = \sum_{\Pi\in\hat{G}}\chi_\Pi(x)\sum_{y\in G}\chi_\Pi(y^{-1})\delta_y$$

$$= \sum_{y\in G}\left(\sum_{\Pi\in\hat{G}}\chi_\Pi(x)\chi_\Pi(y^{-1})\right)\delta_y$$

$$= \sum_{y\in C}|G_x|\delta_y \qquad (\text{cf. §4, \#12})$$

$$= |G_x|\sum_{y\in C}\delta_y$$

$$= |G_x|\chi_C \qquad (\text{cf. §4, \#21}).$$

Now fix $\Pi_0 \in \hat{G}$ -- then

$$\omega_{\Pi_0}(\chi_C) = \omega_{\Pi_0}\left(\frac{1}{|G_x|}\sum_{\Pi\in\hat{G}}\frac{|G|}{d_\Pi}\chi_\Pi(x)e_\Pi\right)$$

$$= \frac{|G|}{|G_x|}\sum_{\Pi\in\hat{G}}\frac{1}{d_\Pi}\chi_\Pi(x)\omega_{\Pi_0}(e_\Pi)$$

$$= \frac{|G|}{|G_x|}\frac{\chi_{\Pi_0}(x)}{d_{\Pi_0}} = \frac{|C|}{d_{\Pi_0}}\chi_{\Pi_0}(x).$$

Consequently, $\forall\ C \in \mathrm{CON}(G)$,

$$\frac{|C|}{d_\Pi} \chi_\Pi(x) \qquad (x \in C)$$

is an algebraic integer.

8: THEOREM $\forall\ \Pi \in \hat{G}$,

$$\frac{|G|}{d_\Pi} \in Z.$$

PROOF In view of §4, #9,

$$|G| = \sum_{x \in G} \chi_\Pi(x)\chi_\Pi(x^{-1}).$$

Given $C \in \mathrm{CON}(G)$, fix an $x_C \in C$ -- then

$$|G| = \sum_{C \in \mathrm{CON}(G)} |C|\chi_\Pi(x_C)\chi_\Pi(x_C^{-1})$$

=>

$$\frac{|G|}{d_\Pi} = \sum_{C \in \mathrm{CON}(G)} \left(\frac{|C|}{d_\Pi} \chi_\Pi(x_C)\right)\chi_\Pi(x_C^{-1}),$$

hence $\frac{|G|}{d_\Pi}$ is a rational algebraic integer, hence is an integer.

In other words, the d_Π divide $|G|$.

9: THEOREM If A is an abelian normal subgroup of G, then the d_Π divide $[G:A]$.

10: APPLICATION Let $Z(G)$ be the center of G -- then the d_Π divide $[G:Z(G)]$.

§7. *INDUCED CLASS FUNCTIONS*

Let G be a finite group, $\Gamma \subset G$ a subgroup.

1: NOTATION $CL(G)$ is the subspace of $C(G)$ comprised of the class functions and $CL(\Gamma)$ is the subspace of $C(\Gamma)$ comprised of the class functions.

2: NOTATION Extend a function $\phi \in C(\Gamma)$ to a function $\overset{\circ}{\phi} \in C(G)$ by writing

$$\overset{\circ}{\phi}(x) = \begin{cases} \phi(x) \text{ if } x \in \Gamma \\ \\ 0 \text{ if } x \notin \Gamma. \end{cases}$$

3: NOTATION Given a class function $\phi \in CL(\Gamma)$, put

$$(i_{\Gamma \to G}\phi)(x) = \frac{1}{|\Gamma|} \sum_{y \in G} \overset{\circ}{\phi}(yxy^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{y \in G, yxy^{-1} \in \Gamma} \phi(yxy^{-1}).$$

4: LEMMA

$$i_{\Gamma \to G}\phi \in CL(G),$$

the induced class function.

5: N.B. Therefore

$$i_{\Gamma \to G}: CL(\Gamma) \to CL(G).$$

[Note:

$$i_{\Gamma \to G}c\phi = c\,i_{\Gamma \to G}\phi \ (c \in \mathbf{C}), \ i_{\Gamma \to G}(\phi_1 + \phi_2) = i_{\Gamma \to G}\phi_1 + i_{\Gamma \to G}\phi_2$$

but in general,

$$i_{\Gamma \to G}(\phi_1\phi_2) \neq (i_{\Gamma \to G}\phi_1)(i_{\Gamma \to G}\phi_2).]$$

The arrow of restriction $C(G) \to C(\Gamma)$ leads to a map

$$r_{G \to \Gamma}: CL(G) \to CL(\Gamma).$$

And:

<u>6:</u> FROBENIUS RECIPROCITY Let $\phi \in CL(\Gamma)$, $\psi \in CL(G)$ -- then

$$\langle i_{\Gamma \to G}\phi, \psi\rangle_G = \langle \phi, r_{G \to \Gamma}\psi\rangle_\Gamma.$$

PROOF

$$\langle i_{\Gamma \to G}\phi, \psi\rangle_G = \frac{1}{|G|} \sum_{x \in G} (i_{\Gamma \to G}\phi)(x)\overline{\psi(x)}$$

$$= \frac{1}{|G|}\frac{1}{|\Gamma|} \sum_{x \in G} \sum_{y \in G} \overset{\circ}{\phi}(yxy^{-1})\overline{\psi(x)}$$

$$= \frac{1}{|\Gamma|}\frac{1}{|G|} \sum_{y \in G} \sum_{x \in G} \overset{\circ}{\phi}(x)\overline{\psi(y^{-1}xy)}$$

$$= \frac{1}{|G|} \sum_{y \in G} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \phi(\gamma)\overline{\psi(\gamma)}$$

$$= \frac{1}{|G|} \sum_{y \in G} \langle \phi, r_{G \to \Gamma}\psi\rangle_\Gamma$$

$$= \langle \phi, r_{G \to \Gamma}\phi\rangle_\Gamma.$$

<u>7:</u> APPLICATION If ϕ is a character of Γ, then $i_{\Gamma \to G}\phi$ is a character of G.

[If χ is a character of G, then $r_{G \to \Gamma}\chi$ is a character of Γ, hence

$$<\phi, r_{G \to \Gamma}\chi_\Pi>_\Gamma$$

is a nonnegative integer for all $\Pi \in \hat{G}$ or still,

$$<i_{\Gamma \to G}\phi, \chi_\Pi>_G$$

is a nonnegative integer for all $\Pi \in \hat{G}$ which implies that $i_{\Gamma \to G}\phi$ is a character of G (cf. §4, #17 ff.).

8: LEMMA Let $\phi \in CL(\Gamma)$, $\psi \in CL(G)$ -- then

$$i_{\Gamma \to G}((r_{G \to \Gamma}\psi)\phi) = \psi(i_{\Gamma \to G}\phi).$$

PROOF From the definitions,

$$\begin{aligned} &i_{\Gamma \to G}((r_{G \to \Gamma}\psi)\phi)(x) \\ &= \frac{1}{|\Gamma|} \sum_{y \in G} \overset{\circ}{\overline{r_{G \to \Gamma}\psi}}\,(yxy^{-1})\,\overset{\circ}{\phi}(yxy^{-1}) \\ &= \frac{1}{|\Gamma|} \sum_{y \in G} \psi(yxy^{-1})\,\overset{\circ}{\phi}(yxy^{-1}) \\ &= \frac{1}{|\Gamma|} \sum_{y \in G} \psi(x)\,\overset{\circ}{\phi}(yxy^{-1}) \\ &= \psi(x)\,\frac{1}{|\Gamma|} \sum_{y \in G} \overset{\circ}{\phi}(yxy^{-1}) \\ &= \psi(x)\,(i_{\Gamma \to G}\phi)(x). \end{aligned}$$

9: APPLICATION The image of $i_{\Gamma \to G}$ is an ideal in CL(G).

Write

$$G = \coprod_{k=1}^{n} x_k\Gamma.$$

10: LEMMA For any $\phi \in CL(\Gamma)$,

$$(i_{\Gamma \to G}\phi)(x) = \sum_{k=1}^{n} \overset{\circ}{\phi}(x_k^{-1} x x_k).$$

PROOF In fact,

$$\begin{aligned}(i_{\Gamma \to G}\phi)(x) &= \frac{1}{|\Gamma|} \sum_{y \in G} \overset{\circ}{\phi}(y x y^{-1}) \\ &= \frac{1}{|\Gamma|} \sum_{y \in G} \overset{\circ}{\phi}(y^{-1} x y) \\ &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{k=1}^{n} \overset{\circ}{\phi}(\gamma^{-1} x_k^{-1} x x_k \gamma).\end{aligned}$$

There are then two possibilities.

- $\gamma^{-1} x_k^{-1} x x_k \gamma \notin \Gamma$

$$\Rightarrow x_k^{-1} x x_k \notin \Gamma$$

$$\Rightarrow \phi^{\circ}(\gamma^{-1} x_k^{-1} x x_k \gamma) = 0 = \overset{\circ}{\phi}(x_k^{-1} x x_k).$$

- $\gamma^{-1} x_k^{-1} x x_k \gamma \in \Gamma$

$$\Rightarrow x_k^{-1} x x_k \in \Gamma$$

$$\begin{aligned}\Rightarrow \phi^{\circ}(\gamma^{-1} x_k^{-1} x x_k \gamma) &= \phi(\gamma^{-1} x_k^{-1} x x_k \gamma) \\ &= \phi(x_k^{-1} x x_k) = \overset{\circ}{\phi}(x_k^{-1} x x_k).\end{aligned}$$

Therefore the sum $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma}$ disappears, leaving

$$\sum_{k=1}^{n} \overset{\circ}{\phi}(x_k^{-1} x x_k).$$

[Note: If instead,

$$G = \coprod_{k=1}^{n} \Gamma x_k,$$

then for any $\phi \in CL(\Gamma)$,

$$(i_{\Gamma \to G}\phi)(x) = \sum_{k=1}^{n} \overset{\circ}{\phi}(x_k x x_k^{-1}).]$$

<u>11:</u> EXAMPLE Let S be a transitive G-set, π the associated representation (cf. §5, #1). Fix a point $s \in S$ and let G_s be its stabilizer -- then

$$\chi_\pi = i_{G_s \to G} 1_{G_s},$$

where $1_{G_s} \in CL(G_s)$ is $\equiv 1$.

[Take $S = \{1,\ldots,n\}$ and $s = 1$. Write

$$G = \coprod_{k=1}^{n} x_k G_s$$

with $x_k \cdot 1 = k$ -- then

$$(i_{G_s \to G} 1_{G_s})(x) = \sum_{k=1}^{n} \overset{\circ}{1}_{G_s}(x_k^{-1} x x_k)$$

$$= \sum_{k, x_k^{-1} x x_k \in G_s} 1$$

$$= \sum_{k, (x_k^{-1} x x_k) \cdot 1 = 1} 1$$

$$= \sum_{k, (x x_k) \cdot 1 = x_k \cdot 1} 1$$

$$= \sum_{k,x \cdot k = k} 1$$

$$= |\{k \in S : x \cdot k = k\}|$$

$$= \chi_\pi(x) \quad \text{(cf. §5, \#1).]}$$

[Note: Here is a "for instance". Take $S = G/\Gamma$ and write

$$G/\Gamma = \coprod_{k=1}^{n} x_k \Gamma.$$

Then G/Γ is a transitive G-set and

$$G_{x_k \Gamma} = x_k \Gamma x_k^{-1}.$$

In particular: Take $x_k = 1$ to get

$$\chi_\pi = i_{\Gamma \to G} 1_\Gamma,$$

thus at a given $x \in G$, $(i_{\Gamma \to G} 1_\Gamma)(x)$ is the number of left cosets of Γ in G fixed by x.]

<u>12:</u> LEMMA Suppose that $\Gamma_1 \subset \Gamma_2 \subset G$. Let $\phi_1 \in CL(\Gamma_1)$ -- then

$$i_{\Gamma_2 \to G}(i_{\Gamma_1 \to \Gamma_2} \phi_1) = i_{\Gamma_1 \to G} \phi_1.$$

PROOF Both sides of the putative equality are class functions, thus it suffices to show that

$$\langle i_{\Gamma_2 \to G}(i_{\Gamma_1 \to \Gamma_2} \phi_1), \chi_\Pi \rangle_G = \langle i_{\Gamma_1 \to G} \phi_1, \chi_\Pi \rangle_G$$

for all $\Pi \in \hat{G}$. But the LHS equals

$$\langle i_{\Gamma_1 \to \Gamma_2} \phi_1, r_{G \to \Gamma_2} \chi_\Pi \rangle_{\Gamma_2}$$

$$= \langle \phi_1, r_{\Gamma_2 \to \Gamma_1}(r_{G \to \Gamma_2} \chi_\Pi) \rangle_{\Gamma_1}$$

$$= \langle \phi_1, \mathrm{res}_{G \to \Gamma_1} \chi_\Pi \rangle_{\Gamma_1}$$

$$= \langle i_{\Gamma_1 \to G} \phi_1, \chi_\Pi \rangle_G,$$

which is the RHS.

13: NOTATION Given $x \in G$, put

$$\Gamma^x = x\Gamma x^{-1} = \{x\gamma x^{-1} : \gamma \in \Gamma\}.$$

The range of

$$i_{\Gamma \to G} : CL(\Gamma) \to CL(G)$$

is contained in the subspace S_Γ of $CL(G)$ consisting of those class functions $f \in CL(G)$ that vanish on

$$G - \bigcup_{x \in G} \Gamma^x.$$

14: LEMMA

$$i_{\Gamma \to G} CL(\Gamma) = S_\Gamma.$$

PROOF Assume not, thus

$$i_{\Gamma \to G} CL(\Gamma) \neq S_\Gamma.$$

Then there exists a nonzero $f \in S_\Gamma$ which is orthogonal to all functions in $i_{\Gamma \to G} CL(\Gamma) : \forall\ \phi \in CL(\Gamma)$,

$$\langle i_{\Gamma \to G} \phi, f \rangle_G = 0$$

or still, $\forall\ \phi \in CL(\Gamma)$,

$$\langle \phi, r_{G \to \Gamma} f \rangle_\Gamma = 0.$$

Now take $\phi = r_{G \to \Gamma} f$ to get

$$\langle r_{G \to \Gamma} f, r_{G \to \Gamma} f \rangle_\Gamma = 0,$$

hence $r_{G \to \Gamma} f = 0$, i.e., f vanishes on Γ. But $f \in CL(G)$, so $\forall\, x \in G$, f vanishes on Γ^x. Since $f \in S_\Gamma$, it then follows that f vanishes on G: $f \equiv 0$, contradicting the supposition that f is nonzero.

<u>15:</u> APPLICATION The image of $i_{\Gamma \to G}$ is an ideal in CL(G) (cf. #9).

Let $\phi \in CL(\Gamma)$. Given $x \in G$, define $\phi^x \in C(\Gamma^x)$ by

$$\begin{aligned} \phi^x(y) &= \phi(x^{-1}yx) \quad (y = x\gamma x^{-1}, \gamma \in \Gamma) \\ &= \phi(x^{-1}x\gamma x^{-1}x) \\ &= \phi(\gamma). \end{aligned}$$

<u>16:</u> LEMMA

$$\phi^x \in CL(\Gamma^x).$$

PROOF Let

$$y_1 = x\gamma_1 x^{-1},\ y_2 = x\gamma_2 x^{-1}.$$

Then

$$\begin{aligned} &\phi^x(y_1 y_2 y_1^{-1}) \\ &= \phi(x^{-1} y_1 y_2 y_1^{-1} x) \\ &= \phi(x^{-1}(x\gamma_1 x^{-1})(x\gamma_2 x^{-1})(x\gamma_1 x^{-1})^{-1} x) \\ &= \phi(x^{-1}(x\gamma_1 x^{-1})(x\gamma_2 x^{-1})(x\gamma_1^{-1} x^{-1}) x) \end{aligned}$$

$$= \phi(\gamma_1 \gamma_2 \gamma_1^{-1})$$

$$= \phi(\gamma_2) = \phi^x(y_2).$$

17: LEMMA $\forall\, x \in G$ and $\forall\, \phi \in CL(\Gamma)$,

$$i_{\Gamma^x \to G} \phi^x = i_{\Gamma \to G} \phi.$$

PROOF Write

$$G = \coprod_{k=1}^{n} x_k \Gamma = \coprod_{k=1}^{n} x x_k x^{-1} \Gamma^x.$$

Then (cf. #10)

$$(i_{\Gamma^x \to G} \phi^x)(y) = \sum_{k=1}^{n} \overset{\circ}{\phi}{}^x((x x_k x^{-1})^{-1} y (x x_k x^{-1}))$$

$$= \sum_{k=1}^{n} \overset{\circ}{\phi}{}^x(x x_k^{-1} x^{-1} y x x_k x^{-1})$$

$$= \sum_{k=1}^{n} \overset{\circ}{\phi}(x^{-1}(x x_k^{-1} x^{-1} y x x_k x^{-1}) x)$$

$$= \sum_{k=1}^{n} \overset{\circ}{\phi}(x_k^{-1} x^{-1} y x x_k)$$

$$= (i_{\Gamma \to G} \phi)(x^{-1} y x)$$

$$= (i_{\Gamma \to G} \phi)(y) \quad \text{(cf. \#4).}$$

§8. *MACKEY THEORY*

Let G be a finite group, let $\Gamma_1, \Gamma_2 \subset G$ be subgroups, and let

$$G = \bigcup_{s \in S} \Gamma_1 s \Gamma_2$$

be a double coset decomposition of G. Given $s \in S$, put

$$\Gamma_2(s) = \Gamma_2^s \cap \Gamma_1 \quad (= s\Gamma_2 s^{-1} \cap \Gamma_1).$$

1: LEMMA Let

$$\Gamma_1 = \bigcup_{t \in T(s)} t\Gamma_2(s)$$

be a left coset decomposition of Γ_1 -- then

$$\Gamma_1 s \Gamma_2 = (\bigcup_{t \in T(s)} t\Gamma_2(s)) s\Gamma_2$$

$$= \bigcup_{t \in T(s)} t\Gamma_2(s) s\Gamma_2$$

$$= \bigcup_{t \in T(s)} t\Gamma_2(s)(s\Gamma_2 s^{-1})s$$

$$= \bigcup_{t \in T(s)} t(s\Gamma_2 s^{-1})s$$

$$= \bigcup_{t \in T(s)} ts\Gamma_2$$

is a partition of $\Gamma_1 s \Gamma_2$.

PROOF Suppose that

$$t_1 s\Gamma_2 \cap t_2 s\Gamma_2 \neq \emptyset \quad (t_1 \neq t_2),$$

so

$$t_1 s = t_2 s \gamma_2 \quad (\gamma_2 \in \Gamma_2).$$

Then

$$t_1 = t_2 s \gamma_2 s^{-1} \Rightarrow t_2^{-1} t_1 \in \Gamma_2^s.$$

Meanwhile

$$t_1, t_2 \in \Gamma_1 \Rightarrow t_2^{-1} t_1 \in \Gamma_1.$$

Therefore

$$t_2^{-1} t_1 \in \Gamma_2^s \cap \Gamma_1 = \Gamma_2(s)$$

$$\Rightarrow t_1 = t_2.$$

Let $R(s) = \{ts : t \in T(s)\} \equiv T(s)s$ and let

$$R = \bigcup_{s \in S} R(s).$$

2: LEMMA R is a set of left coset representatives of Γ_2 in G.

PROOF Let $x \in G$ -- then

$$x \in \Gamma_1 s \Gamma_2 \quad (\exists\ s \in S)$$

$$\Rightarrow x = ts\gamma_2 \quad (\exists\ t \in T(s))$$

$$\Rightarrow x = r\gamma_2 \quad (r \in R(s),\ r = ts).$$

Therefore

$$G = \bigcup_{r \in R} r\Gamma_2.$$

Suppose now that

$$x \in r\Gamma_2 \cap r'\Gamma_2.$$

Then

$$x = r\gamma_2 = r'\gamma_2' \quad (r \in R(s),\ r' \in R(s'))$$

$$\Rightarrow \left[\begin{array}{ll} x = ts\gamma_2 & (t \in T(s)) \\ & \\ x = t's'\gamma_2' & (t' \in T(s')). \end{array}\right.$$

$$\text{But} \left[\begin{array}{l} t \in T(s) \Rightarrow t \in \Gamma_1 \\ \\ t' \in T(s') \Rightarrow t' \in \Gamma_1 \end{array}\right. \Rightarrow x \in \Gamma_1 s\Gamma_2 \cap \Gamma_1 s'\Gamma_2 \Rightarrow s = s'$$

$$\Rightarrow ts\gamma_2 = t's\gamma_2'$$

$$\Rightarrow ts = t's\gamma_2'\gamma_2^{-1} = t's\gamma_2''$$

$$\Rightarrow t = t' \Rightarrow r = r'.$$

Given $\phi \in CL(\Gamma_2)$, put

$$\phi_s = r_{\Gamma_2^s \to \Gamma_2(s)} \phi^s.$$

Here, by definition (cf. §7, #16), $\phi^s \in CL(\Gamma_2^s)$, where

$$\phi^s(y) = \phi(\gamma_2) \quad (y = s\gamma_2 s^{-1},\ \gamma_2 \in \Gamma_2).$$

3: THEOREM Under the above assumptions,

$$r_{G \to \Gamma_1}(i_{\Gamma_2 \to G}\phi) = \sum_{s \in S} i_{\Gamma_2(s) \to \Gamma_1}\phi_s.$$

PROOF Since

$$G = \coprod_{r \in R} r\Gamma_2,$$

$\forall\ x \in G$,

$$(i_{\Gamma_2 \to G}\phi)(x) = \sum_{r \in R} \mathring{\phi}(r^{-1}xr) \quad \text{(cf. §7, \#10)},$$

so $\forall\ \gamma_1 \in \Gamma_1$,

$$(r_{G \to \Gamma_1}(i_{\Gamma_2 \to G}\phi))(\gamma_1)$$

$$= \sum_{r \in R} \mathring{\phi}(r^{-1}\gamma_1 r)$$

$$= \sum_{r \in R,\, r^{-1}\gamma_1 r \in \Gamma_2} \phi(r^{-1}\gamma_1 r)$$

$$= \sum_{s \in S,\, r \in R(s),\, r^{-1}\gamma_1 r \in \Gamma_2} \phi(r^{-1}\gamma_1 r)$$

$$= \sum_{s \in S,\, t \in T(s),\, s^{-1}t^{-1}\gamma_1 ts \in \Gamma_2} \phi(s^{-1}t^{-1}\gamma_1 ts)$$

$$= \sum_{s \in S,\, t \in T(s),\, t^{-1}\gamma_1 t \in \Gamma_2^s} \phi^s(t^{-1}\gamma_1 t)$$

$$= \sum_{s \in S} \sum_{t \in T(s)} \phi_s(t^{-1}\gamma_1 t)$$

$$= \sum_{s \in S} (i_{\Gamma_2(s) \to \Gamma_1} \phi_s)(\gamma_1) \qquad \text{(cf. §7, \#10).}$$

4: LEMMA Let $\psi \in CL(\Gamma_1)$, $\phi \in CL(\Gamma_2)$ -- then

$$<i_{\Gamma_1 \to G}\psi, i_{\Gamma_2 \to G}\phi>_G$$

$$= \sum_{s \in S} <r_{\Gamma_1 \to \Gamma_2(s)}\psi, \phi_s>_{\Gamma_2(s)}.$$

PROOF Taking into account §7, #6,

$$<i_{\Gamma_1 \to G}\psi, i_{\Gamma_2 \to G}\phi>_G$$

$$= <\psi, r_{G \to \Gamma_1}(i_{\Gamma_2 \to G}\phi)>_{\Gamma_1}$$

$$= <\psi, \sum_{s \in S} i_{\Gamma_2(s) \to \Gamma_1}\phi_s>_{\Gamma_1}$$

$$= \sum_{s \in S} <\psi, i_{\Gamma_2(s) \to \Gamma_1}\phi_s>_{\Gamma_1}$$

$$= \sum_{s \in S} \overline{<i_{\Gamma_2(s) \to \Gamma_1}\phi_s, \psi>_{\Gamma_1}}$$

$$= \sum_{s \in S} \overline{<\phi_s, r_{\Gamma_1 \to \Gamma_2(s)}\psi>_{\Gamma_2(s)}}$$

$$= \sum_{s \in S} <r_{\Gamma_1 \to \Gamma_2(s)}\psi, \phi_s>_{\Gamma_2(s)}.$$

5: NOTATION Given a subgroup $\Gamma \subset G$, let 1_Γ stand for the function $\Gamma \to \mathbf{C}$ which is $\equiv 1$, that is, the character of the trivial one-dimensional representation of Γ.

6: EXAMPLE Take $\Gamma_1 = \Gamma_2 = \Gamma$ -- then

$$\langle i_{\Gamma \to G} 1_\Gamma, i_{\Gamma \to G} 1_\Gamma \rangle_G$$

$$= |\Gamma \backslash G / \Gamma|.$$

Therefore $i_{\Gamma \to G} 1_\Gamma$ is not irreducible if $|\Gamma \backslash G / \Gamma| > 1$ (cf. §5, #11).

[Note: $i_{\Gamma \to G} 1_\Gamma$ is a character of G (cf. §7, #7).]

§9. *INDUCED REPRESENTATIONS*

Let G be a finite group, $\Gamma \subset G$ a subgroup.

<u>1:</u> CONSTRUCTION Let (θ,E) be a unitary representation of Γ and denote by $E_{\Gamma,\theta}^G$ the space of all E-valued functions f on G such that $f(x\gamma) = \theta(\gamma^{-1})\ f(x)$ $(x \in G,\ \gamma \in \Gamma)$ -- then the prescription

$$(\mathrm{Ind}_{\Gamma,\theta}^G(x)f)(y) = f(x^{-1}y)$$

defines a representation $\mathrm{Ind}_{\Gamma,\theta}^G$ of G on $E_{\Gamma,\theta}^G$, the representation of G <u>induced</u> by θ.

<u>2:</u> <u>N.B.</u> The inner product

$$\langle f,g\rangle_\theta = \frac{1}{|G|} \sum_{x\in G} \langle f(x),g(x)\rangle_E$$

equips $E_{\Gamma,\theta}^G$ with the structure of a Hilbert space and $\mathrm{Ind}_{\Gamma,\theta}^G$ is a unitary representation.

<u>3:</u> EXAMPLE Take $\Gamma = \{e\}$ and take θ to be the trivial representation of Γ on $E = \mathbf{C}$ -- then $E_{\Gamma,\theta}^G = C(G)$ and

$$\mathrm{Ind}_{\Gamma,\theta}^G = L,$$

the left translation representation of G (cf. §1, #12).

<u>4:</u> EXAMPLE Take $\Gamma = G$ and let (π,V) be a unitary representation of G. Define a linear bijection

$$T:V_{G,\pi}^G \to V$$

by sending f to f(e) -- then $\forall\, x \in G$,

$$T(\mathrm{Ind}_{G,\pi}^{G}(x)f) = (\mathrm{Ind}_{G,\pi}^{G}(x)f)(e)$$

$$= f(x^{-1}e) = f(x^{-1}) = f(ex^{-1})$$

$$= \pi(x)f(e) = \pi(x)(Tf).$$

Therefore

$$T \circ \mathrm{Ind}_{G,\pi}^{G} = \pi \circ T.$$

I.e.:

$$T \in I_G(\mathrm{Ind}_{G,\pi}^{G},\pi)$$

is an invertible intertwining operator, thus $\mathrm{Ind}_{G,\pi}^{G}$ is equivalent to π.

[Note: T is unitary. In fact,

$$\langle f,g\rangle_\pi = \frac{1}{|G|}\sum_{x\in G}\langle f(x),g(x)\rangle_V$$

$$= \frac{1}{|G|}\sum_{x\in G}\langle f(ex),g(ex)\rangle_V$$

$$= \frac{1}{|G|}\sum_{x\in G}\langle \pi(x^{-1})f(e),\pi(x^{-1})g(e)\rangle_V$$

$$= \frac{1}{|G|}\sum_{x\in G}\langle f(e),g(e)\rangle_V$$

$$= \langle f(e),g(e)\rangle_V$$

$$= \langle T(f),T(g)\rangle_V.]$$

<u>5:</u> LEMMA The dimension of $E_{\Gamma,\theta}^{G}$ equals

$$\frac{|G|}{|\Gamma|}\dim E.$$

PROOF Write

$$G = \coprod_{k=1}^{n} x_k\Gamma,$$

where $n = \frac{|G|}{|\Gamma|}$, and define a bijection

$$\Lambda:E_{\Gamma,\theta}^{G} \to \bigoplus_{k=1}^{n} E$$

by the stipulation that

$$\Lambda f = (f(x_1),\ldots,f(x_n)),$$

from which the assertion.

For any character χ of G and for any conjugacy class $C \in \text{CON}(G)$, write $\chi(C)$ for the common value of $\chi(x)$ $(x \in C)$ (and analogously if G is replaced by Γ).

Fixing C, the intersection $C \cap \Gamma$ is a union of elements of $\text{CON}(\Gamma)$, say

$$C \cap \Gamma = \bigcup_{\ell} C_\ell.$$

[Note: If $C \cap \Gamma = \emptyset$, then the sum that follows is empty and its value is 0.]

<u>6:</u> THEOREM Set $\pi = \text{Ind}_{\Gamma,\theta}^{G}$ -- then

$$\chi_\pi(C) = \frac{|G|}{|\Gamma|} \sum_{\ell} \frac{|C_\ell|}{|C|} \chi_\theta(C_\ell).$$

PROOF If χ_C is the characteristic function of C (cf. §4, #20), then $\chi_C = \sum_{y \in C} \delta_y$ (cf. §4, #21). Denoting by ρ the canonical extension of π to $C(G)$, it thus follows that

$$\chi_\pi(C) = \frac{1}{|C|} \text{tr}(\rho(\chi_C)).$$

Fix an orthonormal basis $\phi_1, \ldots, \phi_m$ in E and in $E^G_{\Gamma,\theta}$, let

$$f_j(x) = \begin{cases} \left(\frac{|G|}{|\Gamma|}\right)^{1/2} \theta(\gamma^{-1})\phi_j & (x = \gamma \in \Gamma) \\ 0 & (x \notin \Gamma). \end{cases}$$

- The $f_j (1 \leq j \leq m)$ are an orthonormal set in $E^G_{\Gamma,\theta}$.
- The $\rho(x_k)f_j \quad (1 \leq k \leq n, 1 \leq j \leq m)$ are an orthonormal basis for $E^G_{\Gamma,\theta}$.

Proceeding

$$\begin{aligned} \operatorname{tr}(\rho(\chi_C)) &= \sum_{k=1}^{n} \sum_{j=1}^{m} \langle \rho(\chi_C)\rho(x_k)f_j, \rho(x_k)f_j \rangle_\theta \\ &= \sum_{k=1}^{n} \sum_{j=1}^{m} \langle \rho(x_k^{-1})\rho(\chi_C)\rho(x_k)f_j, f_j \rangle_\theta \\ &= \sum_{k=1}^{n} \sum_{j=1}^{m} \langle \rho(\chi_C)f_j, f_j \rangle_\theta \\ &= n \sum_{j=1}^{m} \langle \rho(\chi_C)f_j, f_j \rangle_\theta \\ &= [G:\Gamma] \sum_{j=1}^{m} \langle \rho(\chi_C)f_j, f_j \rangle_\theta \\ &= \frac{|G|}{|\Gamma|} \sum_{j=1}^{m} \langle \rho(\chi_C)f_j, f_j \rangle_\theta. \end{aligned}$$

But

$$\langle \rho(\chi_C)f_j, f_j \rangle_\theta = \langle \sum_{y \in C} \rho(\delta_y)f_j, f_j \rangle_\theta$$

$$= \sum_{y \in C} \langle \rho(\delta_y) f_j, f_j \rangle_\theta$$

$$= \sum_{y \in C \cap \Gamma} \langle \rho(\delta_y) f_j, f_j \rangle_\theta .$$

Therefore

$$\mathrm{tr}(\rho(\chi_C)) = \frac{|G|}{|\Gamma|} \sum_{j=1}^{m} \sum_{y \in C \cap \Gamma} \langle \rho(\delta_y) f_j, f_j \rangle_\theta$$

$$= \frac{|G|}{|\Gamma|} \sum_{j=1}^{m} \sum_{\ell} \sum_{\gamma \in C_\ell} \langle \rho(\gamma) f_j, f_j \rangle_\theta .$$

But $\forall\ \gamma_0 \in \Gamma$,

$$\langle \rho(\gamma_0) f_j, f_j \rangle_\theta = \frac{1}{|G|} \sum_{\gamma \in \Gamma} \langle f_j(\gamma_0^{-1}\gamma), f_j(\gamma) \rangle_E$$

$$= \frac{1}{|G|} \sum_{\gamma \in \Gamma} \frac{|G|}{|\Gamma|} \langle \theta(\gamma^{-1}\gamma_0) \phi_j, \theta(\gamma^{-1}) \phi_j \rangle_E$$

$$= \frac{1}{|G|} \sum_{\gamma \in \Gamma} \frac{|G|}{|\Gamma|} \langle \theta(\gamma^{-1}) \theta(\gamma_0) \phi_j, \theta(\gamma^{-1}) \phi_j \rangle_E$$

$$= \frac{1}{|G|} \sum_{\gamma \in \Gamma} \frac{|G|}{|\Gamma|} \langle \theta(\gamma_0) \phi_j, \phi_j \rangle_E = \langle \theta(\gamma_0) \phi_j, \phi_j \rangle_E .$$

Therefore

$$\mathrm{tr}(\rho(\chi_C))$$

$$= \frac{|G|}{|\Gamma|} \sum_{j=1}^{m} \sum_{\ell} \sum_{\gamma \in C_\ell} \langle \theta(\gamma) \phi_j, \phi_j \rangle_E$$

$$= \frac{|G|}{|\Gamma|} \sum_{\ell} \sum_{\gamma \in C_\ell} \sum_{j=1}^{m} \langle \theta(\gamma) \phi_j, \phi_j \rangle_E$$

$$= \frac{|G|}{|\Gamma|} \sum_{\ell} \sum_{\gamma \in C_\ell} \mathrm{tr}(\theta(\gamma))$$

$$= \frac{|G|}{|\Gamma|} \sum_{\ell} \sum_{\gamma \in C_\ell} \chi_\theta(\gamma)$$

$$= \frac{|G|}{|\Gamma|} \sum_{\ell} |C_\ell| \chi_\theta(C_\ell).$$

I.e.:

$$\chi_\pi(C) = \frac{|G|}{|\Gamma|} \sum_{\ell} \frac{|C_\ell|}{|C|} \chi_\theta(C_\ell).$$

[Note: If θ is the trivial representation of Γ on $E = \mathbf{C}$, then $\chi_\theta = 1_\Gamma$ (the function $\equiv 1$) and matters reduce to

$$\chi_\pi(C) = \frac{|G|}{|\Gamma|} \frac{|C \cap \Gamma|}{|C|} .]$$

<u>7:</u> <u>N.B.</u> Take $C = \{e\}$:

$$\chi_\pi(e) = \frac{|G|}{|\Gamma|} \chi_\theta(e)$$

$$\Rightarrow$$

$$\dim E_{\Gamma,\theta}^G = \frac{|G|}{|\Gamma|} \dim E \text{ (cf. \#5)}.$$

<u>8:</u> LEMMA Set $\pi = \text{Ind}_{\Gamma,\theta}^G$ -- then for any class function $f \in CL(G)$,

$$\langle \chi_\pi, f \rangle_G = \langle \chi_\theta, f|\Gamma \rangle_\Gamma.$$

PROOF

$$\langle \chi_\pi, f \rangle_G = \frac{1}{|G|} \sum_{x \in G} \chi_\pi(x) \overline{f(x)}$$

$$= \frac{1}{|G|} \sum_{C \in CON(G)} |C| \chi_\pi(C) \overline{f(C)}$$

$$= \frac{1}{|G|} \sum_{C \in \text{CON}(G)} |C| \frac{|G|}{|\Gamma|} \sum_{\ell} \frac{|C_\ell|}{|C|} \chi_\theta(C_\ell)\overline{f(C)}$$

$$= \frac{1}{|\Gamma|} \sum_{C \in \text{CON}(G)} \sum_{\ell} |C_\ell| \chi_\theta(C_\ell)\overline{f(C)}$$

$$= \frac{1}{|\Gamma|} \sum_{C_\ell \in \text{CON}(\Gamma)} |C_\ell| \chi_\theta(C_\ell)\overline{f(C_\ell)}$$

$$= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_\theta(\gamma)\overline{f(\gamma)} = \langle \chi_\theta, f|\Gamma\rangle_\Gamma.$$

[Note: One cannot simply quote §7, #6... .]

9: APPLICATION Take $f = \chi_\Pi$ ($\Pi \in \hat{G}$) and suppose that θ is irreducible -- then the multiplicity of Π in $\text{Ind}_{\Gamma,\theta}^{G}$ equals the multiplicity of θ in the restriction of Π to Γ (cf. §5, #5).

10: THEOREM Set $\pi = \text{Ind}_{\Gamma,\theta}^{G}$ -- then

$$i_{\Gamma \to G}\chi_\theta = \chi_\pi.$$

PROOF The function

$$i_{\Gamma \to G}\chi_\theta$$

is a class function on G, as is χ_π, thus it suffices to show that $\forall\ \Pi \in \hat{G}$,

$$\langle i_{\Gamma \to G}\chi_\theta, \chi_\Pi\rangle_G = \langle \chi_\pi, \chi_\Pi\rangle_G.$$

But

$$\langle i_{\Gamma \to G}\chi_\theta, \chi_\Pi\rangle_G$$

$$= \frac{1}{|G|} \sum_{x \in G} (i_{\Gamma \to G}\chi_\theta)(x)\overline{\chi_\Pi(x)}$$

$$= \frac{1}{|G|} \frac{1}{|\Gamma|} \sum_{x \in G} \sum_{y \in G} \overset{\circ}{\chi}_{\theta}(yxy^{-1}) \overline{\chi_{\Pi}(x)}$$

$$= \frac{1}{|G|} \frac{1}{|\Gamma|} \sum_{x \in G} \sum_{y \in G} \overset{\circ}{\chi}_{\theta}(x) \overline{\chi_{\Pi}(y^{-1}xy)}$$

$$= \frac{1}{|G|} \frac{1}{|\Gamma|} \sum_{x \in G} \sum_{y \in G} \overset{\circ}{\chi}_{\theta}(x) \overline{\chi_{\Pi}(x)}$$

$$= \frac{1}{|\Gamma|} \sum_{x \in G} \overset{\circ}{\chi}_{\theta}(x) \overline{\chi_{\Pi}(x)}$$

$$= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_{\theta}(\gamma) \overline{\chi_{\Pi}(\gamma)}$$

$$= \langle \chi_{\theta}, \chi_{\Pi} | \Gamma \rangle_{\Gamma} = \langle \chi_{\pi}, \chi_{\Pi} \rangle_{G} \quad \text{(cf. \#8)}.$$

11: N.B. It is this result that provides the link with the machinery developed in §7 and §8.

Suppose that $\Gamma_1 \subset \Gamma_2 \subset G$ are subgroups. Let (θ_1, E_1) be a unitary representation of Γ_1 -- then one can form $\text{Ind}_{\Gamma_1,\theta}^{G}$. On the other hand, one can first form $\theta_2 = \text{Ind}_{\Gamma_1,\theta_1}^{\Gamma_2}$ and then form $\text{Ind}_{\Gamma_2,\theta_2}^{G}$.

12: INDUCTION IN STAGES

$$\pi_1 \equiv \text{Ind}_{\Gamma_1,\theta_1}^{G} \approx \text{Ind}_{\Gamma_2,\theta_2}^{G} \equiv \pi_2.$$

[Apply §7, #12:

$$\chi_{\pi_1} = \chi_{\pi_2}.]$$

[Note: Characters determine representations up to equivalence (cf. §5, #10).]

13: LEMMA If (θ_1, E_1), (θ_2, E_2) are unitary representations of Γ, then

$$\mathrm{Ind}_{\Gamma,\theta_1 \oplus \theta_2}^{G} \approx \mathrm{Ind}_{\Gamma,\theta_1}^{G} \oplus \mathrm{Ind}_{\Gamma,\theta_2}^{G}.$$

14: N.B. Consequently, $\mathrm{Ind}_{\Gamma,\theta}^{G}$ cannot be irreducible unless θ itself is irreducible (cf. §10, #3).

Let G_1, G_2 be finite groups, let $\Gamma_1 \subset G_1$, $\Gamma_2 \subset G_2$ be subgroups.

Put

$$G = G_1 \times G_2,\ \Gamma = \Gamma_1 \times \Gamma_2.$$

15: LEMMA If

$$\left[\begin{array}{l} \chi_1 \text{ is a character of } \Gamma_1 \\ \chi_2 \text{ is a character of } \Gamma_2, \end{array}\right.$$

then $\chi_1\chi_2$ is a character of Γ and

$$i_{\Gamma \to G}\chi_1\chi_2 = (i_{\Gamma_1 \to G_1}\chi_1)(i_{\Gamma_2 \to G_2}\chi_2).$$

§10. IRREDUCIBILITY OF $\text{Ind}_{\Gamma,\theta}^{G}$

Let G be a finite group.

1: DEFINITION Let (π_1,V_1), (π_2,V_2) be unitary representations of G -- then π_1 and π_2 are disjoint if they have no common nonzero unitarily equivalent subrepresentations.

2: LEMMA π_1 and π_2 are disjoint iff χ_{π_1} and χ_{π_2} are orthogonal:

$$\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle_G = 0.$$

3: THEOREM Let Γ be a subgroup of G, (θ,E) an irreducible unitary representation of Γ -- then $\text{Ind}_{\Gamma,\theta}^{G}$ is irreducible iff for every $x \in G-\Gamma$, the unitary representations

$$\gamma \to \theta(\gamma), \ \gamma \to \theta(x^{-1}\gamma x)$$

of the subgroup

$$\Gamma(x) = \Gamma^x \cap \Gamma \quad (\Gamma^x = x\Gamma x^{-1})$$

are disjoint.

PROOF Set $\pi = \text{Ind}_{\Gamma,\theta}^{G}$ -- then on general grounds, π is irreducible iff $\langle \chi_\pi, \chi_\pi \rangle_G = 1$ (cf. §5, #11).

I.e.: Iff

$$\langle i_{\Gamma \to G}\chi_\theta, i_{\Gamma \to G}\chi_\theta \rangle_G = 1 \quad \text{(cf. §9, \#10)}$$

or still, iff

$$\langle \chi_\theta, r_{G \to \Gamma}(i_{\Gamma \to G}\chi_\theta) \rangle_\Gamma = 1 \quad \text{(cf. §7, \#6)}$$

or still, iff

$$\langle \chi_\theta, \sum_{s\in S} i_{\Gamma(s)\to\Gamma}(\chi_\theta)_s\rangle_\Gamma$$

$$= \sum_{s\in S} \langle \chi_\theta, i_{\Gamma(s)\to\Gamma}(\chi_\theta)_s\rangle_\Gamma = 1 \quad (\text{cf. §8, \#3}).$$

Here $S = \Gamma\backslash G/\Gamma$ and it can be assumed that one element of the sum is $s = e$ in which case $(\chi_\theta)_s = \chi_\theta$, $\Gamma(s) = \Gamma$, hence

$$\langle \chi_\pi, \chi_\pi\rangle_G$$

$$= \langle \chi_\theta, \chi_\theta\rangle_\Gamma + \sum_{\substack{s\in\Gamma\backslash G/\Gamma \\ s\notin\Gamma}} \langle \chi_\theta, i_{\Gamma(s)\to\Gamma}(\chi_\theta)_s\rangle_\Gamma$$

$$= \langle \chi_\theta, \chi_\theta\rangle_\Gamma + \sum_{\substack{s\in\Gamma\backslash G/\Gamma \\ s\notin\Gamma}} \langle r_{\Gamma\to\Gamma(s)}\chi_\theta, (\chi_\theta)_s\rangle_{\Gamma(s)} \quad (\text{cf. §7, \#6})$$

$$= 1 + \sum_{\substack{s\in\Gamma\backslash G/\Gamma \\ s\notin\Gamma}} \langle r_{\Gamma\to\Gamma(s)}\chi_\theta, (\chi_\theta)_s\rangle_{\Gamma(s)} \quad (\text{cf. §5, \#11}).$$

Each term

$$\langle r_{\Gamma\to\Gamma(s)}\chi_\theta, (\chi_\theta)_s\rangle_{\Gamma(s)}$$

is nonnegative and per $\Gamma(s)$,

$$\left[\begin{array}{l} r_{\Gamma\to\Gamma(s)}\chi_\theta \text{ is the character of } \gamma\to\theta(\gamma) \\ (\chi_\theta)_s \text{ is the character of } \gamma\to\theta(s^{-1}\gamma s). \end{array}\right.$$

If now $\pi = \mathrm{Ind}_{\Gamma,\theta}^G$ is irreducible, then $\langle \chi_\pi, \chi_\pi\rangle_G = 1$, thus $\forall\ s \in \Gamma\backslash G/\Gamma$ $(s \notin \Gamma)$,

$$r_{\Gamma\to\Gamma(s)}\chi_\theta \text{ and } (\chi_\theta)_s$$

are orthogonal. Since S can be chosen so that it contains any given element of $G - \Gamma$, the disjointness claim is manifest. Conversely, the orthogonality of

$$r_{\Gamma \to \Gamma(s)}\chi_\theta \text{ and } (\chi_\theta)_s$$

$\forall\ s \in \Gamma\backslash G/\Gamma\ (s \notin \Gamma)$ forces $\langle\chi_\pi, \chi_\pi\rangle_G = 1$.

§11. *BURNSIDE RINGS*

Let G be a finite group.

1: DEFINITION Let $\chi_1,\ldots,\chi_t$ be the characters of the irreducible unitary representations of G -- then the character ring X(G) is the free abelian group on generators $\chi_1,\ldots,\chi_t$ under pointwise addition and multiplication with unit 1_G (cf. §8, #5).

[Note: Recall that

$$t = |\hat{G}| = |CON(G)| = \dim CL(G).]$$

2: N.B. The pointwise sum or product of two characters is a character and the canonical arrow

$$X(G) \otimes_Z C \to CL(G)$$

is an isomorphism.

3: DEFINITION An element of X(G) is called a virtual character.

4: LEMMA A class function $f \in CL(G)$ is a virtual character iff $\langle f,\chi_\Pi\rangle_G \in Z$ for all $\Pi \in \hat{G}$.

5: REMARK The values of a virtual character are algebraic integers (cf. §6, #5), hence X(G) is a proper subring of CL(G).

[Note: On the other hand, a class function whose values are algebraic integers need not be a virtual character.]

6: NOTATION Let $\mathcal{H}$ be a collection of subgroups of G with the property that

$$H \in \mathcal{H} \ \& \ H' \subset H \Rightarrow H' \in \mathcal{H},$$

in which case $\mathcal{H}$ is termed a <u>hereditary class</u>.

Given $\mathcal{H}$, let $X(G;\mathcal{H})$ be the additive subgroup of $X(G)$ spanned by the

$$i_{H \to G} 1_H \quad (H \in \mathcal{H}).$$

<u>7:</u> LEMMA $X(G;\mathcal{H})$ is a subring of $X(G)$.

PROOF Let $H_1, H_2 \in \mathcal{H}$ -- then the claim is that

$$(i_{H_1 \to G} 1_{H_1})(i_{H_2 \to G} 1_{H_2}) \in \mathcal{H}.$$

Put

$$\chi = (i_{H_1 \to G} 1_{H_1})$$

and write

$$\chi(i_{H_2 \to G} 1_{H_2})$$

$$= i_{H_2 \to G}((r_{G \to H_2}\chi) 1_{H_2}) \qquad (\text{cf. §7, \#8})$$

$$= i_{H_2 \to G}((r_{G \to H_2}\chi)).$$

Then, thanks to §8, #3, there are subgroups $K_1, \ldots, K_r$ of H_2 such that

$$r_{G \to H_2}\chi = \sum_{\ell=1}^{r} i_{K_\ell \to H_2} 1_{K_\ell}.$$

Therefore

$$i_{H_2 \to G}((r_{G \to H_2}\chi))$$

$$= i_{H_2 \to G}(\sum_{\ell=1}^{r} i_{K_\ell \to H_2} 1_{K_\ell})$$

$$= \sum_{\ell=1}^{r} i_{H_2 \to G}(i_{K_\ell \to H_2} 1_{K_\ell})$$

$$= \sum_{\ell=1}^{r} i_{K_\ell \to G} 1_{K_\ell} \qquad \text{(cf. §7, \#12)}$$

$$\in X(G;\mathcal{H}).$$

8: DEFINITION $X(G;\mathcal{H})$ is the Burnside ring of G associated with the hereditary class $\mathcal{H}$.

[Note: It is not a priori evident that $1_G \in X(G;\mathcal{H})$.]

9: CRITERION Let R be a ring of Z-valued functions on a finite set X under pointwise operations. Suppose that for each $x \in X$ and each prime p there exists $f \in R$ such that $f(x) \not\equiv 0 \bmod p$ -- then $1_X \in R$.

[Attach to each $x \in X$ the ideal

$$I_x = \{f(x) : f \in R\} \subset Z.$$

Then, in view of the assumption, $I_x = Z$ so there exists $f_x \in R$ such that $f_x(x) = 1$, hence

$$\prod_{x \in X} (1 - f_x) = 0.$$

Now expand the product to get 1 as a sum of elements of R.]

Let G be a finite group.

10: DEFINITION Let p be a prime -- then G is a p-group if every element $x \in G$ has order a power of p.

[Note: Every p-group is nilpotent.]

<u>11:</u> LEMMA G is a p-group iff $|G|$ is a power of p.

<u>12:</u> DEFINITION Let p be a prime -- then a subgroup P of G is a <u>Sylow p-subgroup</u> of G if it is a maximal p-subgroup of G.

<u>13:</u> THEOREM

- Sylow p-subgroups exist.
- All Sylow p-subgroups are conjugate.
- Every p-subgroup is contained in a Sylow p-subgroup.

<u>14:</u> <u>N.B.</u> The number of Sylow p-subgroups of G is a divisor of $|G|$.

<u>15:</u> DEFINITION Given a prime p, a finite group H is <u>p-elementary</u> if it is the direct product of a cyclic group C of order prime to p and a p-group P.

[Note: Accordingly, C and P are normal subgroups, $C \cap P = \{e\}$, and $H = CP$.]

<u>16:</u> LEMMA Subgroups of p-elementary groups are again p-elementary, hence the p-elementary subgroups of G constitute a hereditary class $\mathcal{E}_p(G)$.

<u>17:</u> DEFINITION A finite group H is <u>elementary</u> if it is p-elementary for some prime p.

<u>18:</u> NOTATION Put

$$\mathcal{E}(G) = \bigcup_p \mathcal{E}_p(G).$$

<u>19:</u> <u>N.B.</u> Since $\mathcal{E}(G)$ is a hereditary class, one can form its Burnside ring $X(G;\mathcal{E}(G))$.

20: DEFINITION Given a prime p, a group H is p-semielementary if it is the semidirect product of a cyclic subgroup C of order prime to p and a p-group P.

[Note: Accordingly, C is a normal subgroup, $C \cap P = \{e\}$, and $H = CP$.]

21: LEMMA Subgroups of p-semielementary groups are again p-semielementary, hence the p-semielementary subgroups of G constitute a hereditary class $\mathcal{SE}_p(G)$.

22: DEFINITION A finite group H is semielementary if it is p-semielementary for some prime p.

23: NOTATION Put

$$\mathcal{SE}(G) = \bigcup_p \mathcal{SE}_p(G).$$

24: N.B. Since $\mathcal{SE}(G)$ is a hereditary class, one can form its Burnside ring $X(G;\mathcal{SE}(G))$.

25: LEMMA

$$1_G \in X(G;\mathcal{SE}(G)),$$

i.e., there exist integers a_H $(H \in \mathcal{SE}(G))$ such that

$$1_G = \sum_{H \in \mathcal{SE}(G)} a_H (i_H \to {}_G 1_H).$$

PROOF It suffices to show that the ring $X(G;\mathcal{SE}(G))$ satisfies the assumptions of #9: For every $x \in G$ and for every prime p, there exists a group $H_{x,p} \equiv H \in \mathcal{SE}(G)$ such that

$$(i_H \to {}_G 1_H)(x) \not\equiv 0 \bmod p.$$

This said, factor the order of x as $p^a n$ $(p \nmid n)$ and let $C = \langle x^{p^a} \rangle$ (hence $|C| = n$, hence is prime to p). Let N be the normalizer of C in G, let P be a Sylow p-subgroup of N containing x, and let $H_{x,p} \equiv H = CP$ -- then H is p-semielementary and the claim is that

$$(i_{H \to G} 1_H)(x) \not\equiv 0 \bmod p.$$

By definition,

$$(i_{H \to G} 1_H)(x) = \frac{1}{|H|} \sum_{y \in G, yxy^{-1} \in H} 1_H(yxy^{-1}).$$

But

$$yxy^{-1} \in H \Rightarrow yCy^{-1} \subset H$$

$$\Rightarrow yCy^{-1} = C \Rightarrow y \in N.$$

Therefore

$$(i_{H \to G} 1_H)(x) = (i_{H \to N} 1_H)(x),$$

the term on the right being the number of left cosets of H in N fixed by x (cf. §7, #11). Since C is a normal subgroup of N and since $C \subset H$, it follows that C must fix the left cosets of H in N. Thus the x-orbits have cardinality dividing p^a, thus each nontrivial x-orbit has cardinality divisible by p. On the other hand, the number of left cosets of H in N is prime to p (H contains a Sylow p-subgroup of N). Combining these facts then leads to the conclusion that the number of left cosets of H in N fixed by x is prime to p, i.e.,

$$(i_{H \to G} 1_H)(x) \not\equiv \bmod p.$$

<u>26:</u> DEFINITION A <u>monomial character</u> of a finite group is a character of degree 1.

27: DEFINITION A finite group H is said to be an M-group if each irreducible character of H is induced by a monomial character of a subgroup of H.

28: THEOREM Suppose that H is a finite group which is a semidirect product of an abelian normal subgroup and a nilpotent group (in particular, a p-group) -- then H is an M-group.

29: APPLICATION p-elementary groups and p-semielementary groups are M-groups.

§12. *BRAUER THEORY*

Let G be a finite group.

1: CHARACTERIZATION OF CHARACTERS A class function $f \in CL(G)$ is a virtual character (i.e., belongs to $X(G)$) iff for every $H \in E(G)$,

$$r_{G \to H} f \in X(H).$$

2: INDUCTION PRINCIPLE A class function $f \in CL(G)$ is a virtual character (i.e., belongs to $X(G)$) iff there exist elementary subgroups H_i, monomial characters λ_i of H_i, and integers a_i $(1 \leq i \leq n)$ such that

$$f = \sum_{i=1}^{n} a_i (i_{H_i \to G} \lambda_i).$$

These are the main results. Turning to their proofs, let R be the ring with unit 1_G whose elements are the class functions f on G such that

$$r_{G \to H} f \in X(H)$$

for all $H \in E(G)$ and let L be the subgroup of $X(G)$ spanned over Z by characters of the form $i_{H \to G}\lambda$, where λ is a monomial character of some $H \in E(G)$.

3: LEMMA Statements 1 and 2 are equivalent to $L = R$.

[Note: Obviously,

$$L \subset X(G) \subset R.]$$

4: LEMMA L is an ideal in R.

PROOF Let $\Lambda \in L$, say

$$\Lambda = \sum_i a_i (i_{H_i \to G} \lambda_i),$$

and let $\psi \in R$ -- then

$$\psi\Lambda = \sum_i a_i \psi(i_{H_i \to G}\lambda_i)$$

$$= \sum_i a_i (i_{H_i \to G}((r_{G \to H_i}\psi)\lambda_i)) \quad \text{(cf. §7, #8)}.$$

Since

$$r_{G \to H_i}\psi \in X(H_i),$$

there exist integers b_{ij} such that

$$r_{G \to H_i}\psi = \sum_j b_{ij}\xi_{ij},$$

ξ_{ij} running through the irreducible characters of H_i, hence

$$\psi\Lambda = \sum_{i,j} a_i b_{ij} (i_{H_i \to G}\xi_{ij}).$$

But elementary groups are M-groups (cf. §11, #29), so ξ_{ij} is induced by a monomial character of some subgroup of H_i. Taking into account that $E(G)$ is a hereditary class, apply §7, #12 to conclude that $\psi\Lambda \in L$. Therefore L is an ideal in R.

[Note: Operations in R are pointwise and, of course, R is commutative.]

Matters thus reduce to showing that $1_G \in L$. To this end, suppose that it were possible to write

$$1_G = \sum_k c_k (i_{H_k \to G}\chi_k),$$

where $c_k \in Z$ and χ_k is a character of some proper subgroup H_k of G. Inductively, it can be assumed that #2 holds for H_k, hence that χ_k can be written as a Z-linear combination of induced monomial characters from elements of $E(H_k)$. But then $1_G \in L$, as desired.

[Note: Nothing need be done if G is elementary to begin with (it being automatic that $1_G \in L$).]

<u>5:</u> LEMMA If G is not elementary, then 1_G can be written as a $\mathbb{Z}$-linear combination of induced characters from proper subgroups of G.

<u>Case 1:</u> G is not semielementary, thus $G \notin SE(G)$ and the $H \in SE(G)$ are proper subgroups. The contention then follows from §11, #25.

<u>Case 2:</u> G is semielementary: $G \in SE(G)$, say $G = CP$ for some prime p. Let N be the normalizer of P in G, hence $N = (C \cap N) \times P$ is p-elementary and it can be assumed that $N \neq G$ (otherwise G is elementary and there is nothing to prove). Write

$$i_{N \to G} 1_N = a_0 1_G + \sum_{i>0} a_i \chi_i,$$

where the $\chi_i \neq 1_G$ are irreducible characters and the a_i are positive integers.

<u>6:</u> <u>N.B.</u>

$$\begin{aligned} a_0 &= \langle i_{N \to G} 1_N, 1_G \rangle_G \\ &= \langle 1_N, r_{G \to N} 1_G \rangle_N \quad \text{(cf. §7, \#6)} \\ &= \langle 1_N, 1_N \rangle_N = 1. \end{aligned}$$

<u>7:</u> <u>N.B.</u> $\chi_i(e) > 1$ for all $i > 0$.

[Suppose that $\chi_i(e) = 1$ ($\exists$ i). Write

$$\text{res}_{G \to N} \chi_i = c 1_N + \chi$$

for some character χ orthogonal to 1_N -- then

$$c = \langle 1_N, \text{res}_{G \to N}\chi_i\rangle_N$$

$$= \langle i_{N \to G}1_N, \chi_i\rangle_G \quad \text{(cf. §7, \#6)}$$

$$= a_i.$$

And

$$1 = \chi_i(e) = a_i + \chi(e)$$

$$\Rightarrow a_i = 1 \Rightarrow \text{res}_{G \to N}\chi_i = 1_N.$$

Recall now that the kernel K_i of χ_i is the proper normal subgroup of G consisting of those $x \in G$ such that $\chi_i(x) = \chi_i(e)$ or still, consisting of those $x \in G$ such that $\chi_i(x) = 1$, thus $N \subset K_i$ (since $\text{res}_{G \to N}\chi_i = 1_N$). But this is impossible: P is a Sylow p-subgroup of K_i, so $G = K_i$ (cf. infra).

[Note: Let $x \in G$ -- then both P and xPx^{-1} are Sylow p-subgroups of K_i, hence

$$kxPx^{-1}k^{-1} = P$$

for some $k \in K_i$ which implies that $kx \in N \subset K_i$, thereby forcing $x \in K_i$, so $G = K_i$.]

Return to the formula

$$i_{N \to G}1_N = 1_G + \sum_{i>0} a_i\chi_i.$$

Since $\chi_i(e) > 1$ for all $i > 0$, the χ_i are not monomial. On the other hand, G = CP is semielementary, thus is an M-group, thus each χ_i is induced by a monomial character λ_i of some proper subgroup H_i of G. Therefore

$$1_G = i_{N \to G}1_N - \sum_{i>0} a_i(i_{H_i \to G}\lambda_i),$$

which completes the proof of #5.

§13. GROUPS OF LIE TYPE

Let k be a finite field.

1: DEFINITION A k-group is a linear algebraic group defined over k.

[Note: A k-subgroup of a k-group is a subgroup which is a k-group.]

2: NOTATION Given k-groups $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$,..., denote their group of k-rational points $\underline{\underline{A}}(k)$, $\underline{\underline{B}}(k)$, $\underline{\underline{C}}(k)$..., by A, B, C,... .

Let $\underline{\underline{G}}$ be a connected reductive k-group.

3: DEFINITION G is said to be a group of Lie type.

4: N.B. G is, of course, finite and it is possible to compute $|G|$ explicitly.

5: DEFINITION A maximal closed connected solvable subgroup of $\underline{\underline{G}}$ is called a Borel subgroup.

[Note: The conditions "closed" and "connected" can be omitted from the definition.]

6: LEMMA

- Any two Borel subgroups of $\underline{\underline{G}}$ are conjugate.
- Every element of $\underline{\underline{G}}$ belongs to some Borel subgroup of $\underline{\underline{G}}$.
- Every closed subgroup of $\underline{\underline{G}}$ containing a Borel subgroup is equal to its own normalizer and is connected.
- Any two closed subgroups of $\underline{\underline{G}}$ containing the same Borel subgroup and conjugate in $\underline{\underline{G}}$ are equal.

7: N.B. Since k is finite, $\underline{\underline{G}}$ is quasi-split, hence contains a Borel subgroup defined over k.

[Note: Any two such are G-conjugate.]

Let $\underline{\underline{B}}$ be a Borel k-subgroup of $\underline{\underline{G}}$, let $\underline{\underline{T}} \subset \underline{\underline{B}}$ be a maximal torus of $\underline{\underline{G}}$ defined over k, and put

$$\underline{\underline{N}} = N_{\underline{\underline{G}}}(\underline{\underline{T}}).$$

8: LEMMA $\underline{\underline{N}}$ is a k-subgroup of $\underline{\underline{G}}$.

9: NOTATION Set

$$\underline{\underline{W}} = \underline{\underline{N}}/\underline{\underline{T}}.$$

10: LEMMA

$$W \approx N/T.$$

[Note:

$$\underline{\underline{B}} \cap \underline{\underline{N}} = \underline{\underline{T}} \Rightarrow B \cap N = T.]$$

11: LEMMA W is a finite Coxeter group.

[Note: Spelled out, W admits a finite system of generators $w_1, \ldots, w_\ell$ ($w_i \neq 1$ and $w_i \neq w_j$ for $i \neq j$) subject to the relations

$$w_i^2 = 1, \ (w_i w_j)^{m_{ij}} = 1 \quad (i \neq j),$$

where m_{ij} is the order of $w_i w_j$ $(i \neq j)$.]

12: BRUHAT LEMMA

$$G = \coprod_{w \in W} BwB.$$

13: DEFINITION A closed subgroup $\underline{\underline{P}}$ of $\underline{\underline{G}}$ is parabolic if it contains a Borel subgroup of $\underline{\underline{G}}$.

14: LEMMA Let $\underline{\underline{P}}_1$, $\underline{\underline{P}}_2$ be parabolic k-subgroups of $\underline{\underline{G}}$ -- then $\underline{\underline{P}}_1 = \underline{\underline{P}}_2$ iff $P_1 = P_2$.

15: NOTATION Given a parabolic k-subgroup of $\underline{\underline{G}}$, denote its unipotent radical by $\underline{\underline{U}}$.

[Note: Recall that $\underline{\underline{P}}$ is the normalizer of $\underline{\underline{U}}$.]

17: DEFINITION Let $\underline{\underline{P}}$ be a parabolic k-subgroup of $\underline{\underline{G}}$ -- then a closed connected reductive k-subgroup $\underline{\underline{L}}$ of $\underline{\underline{P}}$ is a Levi subgroup of $\underline{\underline{P}}$ if $\underline{\underline{P}}$ is the semidirect product $\underline{\underline{LU}}$ (hence $P = LU$).

18: LEMMA Levi subgroups of $\underline{\underline{P}}$ exist and any two such are conjugate by a unique element of U.

19: N.B. L is a group of Lie type.

20: LEMMA Let $\underline{\underline{P}}_1$, $\underline{\underline{P}}_2$ be parabolic k-subgroups of $\underline{\underline{G}}$ -- then the following conditions are equivalent.

- $P_1 \cap U_2 \subset U_1$, $P_2 \cap U_1 \subset U_2$
- $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ have a common Levi subgroup.

21: APPLICATION

$$U_1 = U_2 \Rightarrow \underline{\underline{P}}_1 = \underline{\underline{P}}_2.$$

[For under these circumstances, $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ have a common Levi subgroup $\underline{\underline{L}}$, thus

$$P_1 = LU_1 = LU_2 = P_2,$$

so one can quote #14.]

22: DEFINITION Let $\underline{\underline{P}}_1$, $\underline{\underline{P}}_2$ be parabolic k-subgroups of $\underline{\underline{G}}$ -- then $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ are said to be associate if there exists an $x \in G$ such that $\underline{\underline{P}}_1$ and $x\underline{\underline{P}}_2x^{-1}$ have a common Levi subgroup.

23: N.B. The relation determined by "to be associate" is an equivalence relstion on the set of parabolic k-subgroups of $\underline{\underline{G}}$.

24: LEMMA If $\underline{\underline{P}}_1$, $\underline{\underline{P}}_2$ are not associate, then $\forall\ x \in G$, $\underline{\underline{P}}_1$, $x\underline{\underline{P}}_2x^{-1}$ are not associate.

[If there exists $x \in G$ such that $\underline{\underline{P}}_1$ and $x\underline{\underline{P}}_2x^{-1}$ are associate, then there exists $y \in G$ such that $\underline{\underline{P}}_1$ and $yx\underline{\underline{P}}_2x^{-1}y^{-1}$ have a common Levi subgroup, thus $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ are associate, contradiction.]

25: LEMMA Let $\underline{\underline{P}}_1$, $\underline{\underline{P}}_2$ be parabolic k-subgroups of $\underline{\underline{G}}$. Assume: $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ are associate -- then $|P_1| = |P_2|$.

[There is no loss of generality in supposing that $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ have a common Levi subgroup $\underline{\underline{L}}$, thereby reducing matters to the claim that $|U_1| = |U_2|$.]

26: DESCENT Fix a parabolic k-subgroup $\underline{\underline{P}} \subset \underline{\underline{G}}$ and let $\underline{\underline{L}} \subset \underline{\underline{P}}$ be a Levi subgroup -- then there is a 1-to-1 correspondence between the set of parabolic k-subgroups of $\underline{\underline{G}}$ contained in $\underline{\underline{P}}$ and the set of parabolic k-subgroups of $\underline{\underline{L}}$.

• Given a parabolic k-subgroup $\underline{\underline{P}}' \subset \underline{\underline{P}}$, write $\underline{\underline{P}}' = \underline{\underline{L}}'\underline{\underline{U}}'$ and put $*\underline{\underline{P}} = \underline{\underline{P}}' \cap \underline{\underline{L}}$ -- then $*\underline{\underline{P}}$ is a parabolic k-subgroup of $\underline{\underline{L}}$ with unipotent radical $*\underline{\underline{U}} = \underline{\underline{U}}' \cap \underline{\underline{L}}$.

• Given a parabolic k-subgroup $*\underline{\underline{P}}$ of $\underline{\underline{L}}$, write $*\underline{\underline{P}} = *\underline{\underline{L}}*\underline{\underline{U}}$ and put $\underline{\underline{L}}' = *\underline{\underline{L}}$, $\underline{\underline{U}}' = *\underline{\underline{U}}\underline{\underline{U}}$ -- then $\underline{\underline{P}}' = \underline{\underline{L}}'\underline{\underline{U}}'$ is a parabolic k-subgroup of $\underline{\underline{G}}$ such that $\underline{\underline{P}}' \subset \underline{\underline{P}}$.

The bijection in question is the assignment $\underline{\underline{P}}' \rightarrow *\underline{\underline{P}}$.

<u>27:</u> <u>N.B.</u> $\underline{\underline{P}}'$ and $\underline{\underline{P}}''$ are conjugate by an element of G iff $\underline{\underline{P}}' \cap L$ and $\underline{\underline{P}}'' \cap L$ are conjugate by an element of L.

APPENDIX

LEMMA Suppose that $\underline{\underline{P}}_1 = \underline{\underline{L}}_1\underline{\underline{U}}_1$ and $\underline{\underline{P}}_2 = \underline{\underline{L}}_2\underline{\underline{U}}_2$ are associate -- then $\underline{\underline{L}}_1$ and $\underline{\underline{L}}_2$ are conjugate by an element of G.

[Choose $x \in G$ such that $\underline{\underline{P}}_1$ and $x\underline{\underline{P}}_2x^{-1}$ have a common Levi subgroup $\underline{\underline{L}}$. Choose $u_1 \in U_1$:

$$u_1\underline{\underline{L}}u_1^{-1} = \underline{\underline{L}}_1.$$

Choose $xu_2x^{-1} \in xU_2x^{-1}$:

$$xu_2x^{-1}\underline{\underline{L}}xu_2^{-1}x^{-1} = x\underline{\underline{L}}_2x^{-1}$$

Then

$$u_2x^{-1}\underline{\underline{L}}xu_2^{-1} = \underline{\underline{L}}_2$$

$\Rightarrow$

$$\underline{\underline{L}} = xu_2^{-1}\underline{\underline{L}}_2u_2x^{-1}$$

=>

$$\underline{\underline{L}}_1 = u_1 \underline{\underline{L}} u_1^{-1}$$

$$= u_1 x u_2^{-1} \underline{\underline{L}}_2 u_2 x^{-1} u_1^{-1} .]$$

§14. *HARISH-CHANDRA THEORY*

Let k be a finite field, $\underline{\underline{G}}$ a connected reductive k-group.

<u>1:</u> DEFINITION Let $\underline{\underline{P}}$ be a parabolic k-subgroup of $\underline{\underline{G}}$ -- then P is termed a <u>cuspidal subgroup</u> of G.

<u>2:</u> NOTATION Given a cuspidal subgroup P = LU of G and an $f \in C(G)$, let

$$f_P(x) = \sum_{u \in U} f(xu) \quad (x \in G).$$

[Note: If P = G, then

$$f_G(x) = f(x) \quad (x \in G).]$$

<u>3:</u> DEFINITION Let $f \in C(G)$ -- then f is said to be a <u>cusp form</u> if $f_P = 0$ for all $P \neq G$.

<u>4:</u> NOTATION Write ${}^0C(G)$ for the set of cusp forms and put

$${}^0CL(G) = CL(G) \cap {}^0C(G).$$

<u>5:</u> LEMMA ${}^0C(G)$ is a linear subspace of C(G).

<u>6:</u> LEMMA ${}^0C(G)$ is stable under left translations, hence is a left ideal in C(G).

<u>7:</u> REMARK If $\underline{\underline{G}}$ is a torus, then ${}^0C(G) = C(G)$.

<u>8:</u> NOTATION Given $f \in C(G)$, write $f_P \sim 0$ if

$$\sum_{\ell \in L} f_P(x\ell)\overline{\phi(\ell)} = 0$$

for all $\phi \in {}^0C(L)$ and all $x \in G$.

[Note: Bear in mind that L is a group of Lie type (cf. §13, #19).]

9: N.B. Matters are independent of the choice of L in P.

10: LANGLANDS PRINCIPLE If $f_P \sim 0$ for all cuspidal subgroups P of G (including P = G), then f = 0.

PROOF Proceed by induction on the semisimple k-rank s of $\underline{G}$, the case s = 0 being trivial (because then $\underline{G}$ is anisotropic, there is only one P, viz. P = G, and L = G, ${}^0C(L) = C(G)\ldots$). So assume that s is positive and let P = LU be for the moment a proper cuspidal subgroup, thus $U \neq \{e\}$ and the semisimple k-rank of $\underline{L}$ is strictly smaller than that of $\underline{G}$. Using now §13, #26, let *P = *L*U be a cuspidal subgroup of L -- then P' = L'U' = *LU' is a cuspidal subgroup of G contained in P. Freeze $x \in G$ and put $g(\ell) = f_P(x\ell)$ $(\ell \in L)$:

$$\begin{aligned} g_{*P}(\ell) &= \sum_{*u\in *U} g(\ell *u) \\ &= \sum_{*u\in *U} f_P(x\ell *u) \\ &= \sum_{*u\in *U} \sum_{u\in U} f(x\ell *uu) \\ &= \sum_{u'\in U'} f(x\ell u') \\ &= f_{P'}(x\ell). \end{aligned}$$

But by assumption,

$$\sum_{*\ell\in *L} f_{P'}(x\ell *\ell)\overline{\phi(*\ell)} = 0$$

for all $\phi \in {}^0C(*L)$ or still,

$$\sum_{*\ell \in *L} g_{*P}(\ell * \ell)\overline{\phi(*\ell)} = 0$$

for all $\phi \in {}^0C(*L)$. The induction hypothesis then implies that $g = 0$, hence

$$f_P(x) = g(e) = 0.$$

Therefore f is a cusp form ($x \in G$ being arbitrary), i.e., $f \in {}^0C(G)$. Finally,

$$f_G \sim 0 \Rightarrow \sum_{y \in G} f(xy)\overline{\phi(y)} = 0$$

for all $\phi \in {}^0C(G)$ and all $x \in G$. Take $x = e$ to conclude that

$$\sum_{y \in G} f(y)\overline{\phi(y)} = 0$$

for all $\phi \in {}^0C(G)$ and then take $\phi = f$ to conclude that

$$\langle f,f\rangle_G = 0 \Rightarrow f = 0.$$

<u>11:</u> NOTATION Given a cuspidal subgroup $P = LU$ of G, let $C(G;P)$ be the subspace of $C(G)$ consisting of those f such that

(i) $f(xu) = f(x)$ $(x \in G,\ u \in U)$

and

(ii) $\ell \rightarrow f(x\ell) \in {}^0C(L)$ $(x \in G,\ \ell \in L)$.

[Note: $C(G;P)$ is stable under left translations, hence is a left ideal in $C(G)$.]

<u>12:</u> EXAMPLE

$$C(G;G) = {}^0C(G).$$

<u>13:</u> SUBLEMMA Fix P -- then $\forall\ f \in C(G;P)$ and $\forall\ g \in C(G)$,

$$<f,g_P>_G = |U|<f,g>_P.$$

PROOF

$$<f,g_P>_G = \frac{1}{|G|} \sum_{x\in G} f(x)\overline{g_P(x)}$$

$$= \frac{1}{|G|} \sum_{x\in G} f(x) \sum_{u\in U} \overline{g(xu)}$$

$$= \frac{1}{|G|} \sum_{u\in U} \sum_{x\in G} f(x)\overline{g(xu)}$$

$$= \frac{1}{|G|} \sum_{u\in U} \sum_{x\in G} f(xu)\overline{g(xu)}$$

$$= \sum_{u\in U} \frac{1}{|G|} \sum_{x\in G} f(x)\overline{g(x)}$$

$$= \sum_{u\in U} <f,g>_G = |U|<f,g>_G.$$

<u>14:</u> RAPPEL Let H be a finite dimensional complex Hilbert space -- then a subset $M \subset H$ is <u>total</u> if $M_{\ell in} = H$, this being the case iff $M^{\perp} = \{0\}$.

[Note: Subspaces of H are necessarily closed... .]

Put

$$M = \bigcup_P C(G;P).$$

<u>15:</u> LEMMA $C(G)$ is spanned by the $f \in M$.

PROOF It suffices to show that if for some $g \in C(G)$, we have

$$<f,g>_G = 0$$

for all $f \in C(G;P)$ and for all cuspidal P, then $g = 0$. And to this end, it need only be established that $g_P \sim 0$ for all cuspidal P (cf. #10). So fix $x \in G$ and let $\phi \in {}^0C(L)$. Define $f \in C(G)$ as follows:

$$\begin{cases} f(y) = 0 \text{ if } y \notin xP \\ f(x\ell u) = \phi(\ell) \quad (\ell \in L,\ u \in U). \end{cases}$$

Then $f \in C(G;P)$, so

$$0 = \langle f,g \rangle_G = \frac{1}{|U|} \langle f,g_P \rangle_G$$

=>

$$\begin{aligned} 0 &= \frac{1}{|U|} \langle g_P,f \rangle_G \\ &= \frac{1}{|U|} \sum_{y \in G} g_P(y)\overline{f(y)} \\ &= \frac{1}{|U|} \sum_{y \in xP} g_P(y)\overline{f(y)} \\ &= \frac{1}{|U|} \sum_{\ell,u} g_P(x\ell u)\overline{f(x\ell u)} \\ &= \frac{1}{|U|} \sum_{\ell,u} g_P(x\ell)\overline{\phi(\ell)} \\ &= \frac{1}{|U|} \sum_{u} \sum_{\ell \in L} g_P(x\ell)\overline{\phi(\ell)} \\ &= \sum_{\ell \in L} g_P(x\ell)\overline{\phi(\ell)}. \end{aligned}$$

Therefore $g_P \sim 0$.

16: CONVENTION Cuspidal subgroups P_1, P_2 are said to be associate if this is the case of $\underline{\underline{P}}_1, \underline{\underline{P}}_2$.

17: LEMMA If P_1, P_2 are associate, then

$$C(G;P_1) = C(G;P_2).$$

18: LEMMA If P_1, P_2 are not associate, then $C(G;P_1)$, $C(G;P_2)$ are orthogonal.

Let $P_1, \ldots, P_r$ be a set of representatives for the association classes of cuspidal subgroups of G.

19: THEOREM There is an orthogonal decomposition

$$C(G) = \bigoplus_{i=1}^{r} C(G;P_i).$$

20: N.B. #17, #18 can be established without the use of representation theory but its introduction leads to another approach.

21: LEMMA Let $\Pi \in \hat{G}$ -- then χ_Π is a cusp form iff $\forall$ cuspidal $P \neq G$,

$$m(\Pi, \mathrm{Ind}_{U,\theta}^{G}) = 0,$$

where θ is the trivial representation of U on $E = \mathbf{C}$. I.e.: Iff

$$\langle \chi_\Pi, i_U \to {}_G 1_U \rangle_G = \langle r_G \to {}_U \chi_\Pi, 1_U \rangle_U$$

$$= \langle 1_U, r_G \to {}_U \chi_\Pi \rangle_U = m(\theta, \Pi|U)$$

$$= 0.$$

<u>22:</u> LEMMA Let $\Pi \in \hat{G}$ -- then χ_Π is a cusp form iff $\forall$ cuspidal $P \neq G$,

$$\sum_{u \in U} \Pi(u) = 0.$$

<u>23:</u> <u>N.B.</u> Let

$$V(\Pi)_U = \{v \in V(\Pi): \sum_{u \in U} \Pi(u)v = 0\}.$$

Then χ_Π is a cusp form iff $\forall$ cuspidal $P \neq G$,

$$V(\Pi) = V(\Pi)_U.$$

<u>24:</u> DEFINITION Let $\Pi \in \hat{G}$ -- then Π is said to be in the <u>discrete series</u> if its character χ_Π is a cusp form.

<u>25:</u> NOTATION $^0\hat{G}$ is the subset of $\hat{G}$ consisting of those Π in the discrete series.

Given $P = LU$ and $\Theta \in {}^0\hat{L}$, one can lift Θ to P and form $\mathrm{Ind}_{P,\Theta}^G$ with character

$$i_{P \to G}\chi_\Theta \quad \text{(cf. §9, \#10).}$$

<u>26:</u> THEOREM Let $\Pi \in \hat{G} - {}^0\hat{G}$ -- then there exists a proper cuspidal $P = LU$ and a $\Theta \in {}^0\hat{L}$ such that Π occurs as a subrepresentation of $\mathrm{Ind}_{P,\Theta}^G$:

$$\langle \chi_\Pi, \chi_\pi \rangle_G \neq 0 \quad (\pi = \mathrm{Ind}_{P,\Theta}^G) \quad \text{(cf. §5, \#5).}$$

PROOF Proceed by induction on the semisimple k-rank s of $\underline{G}$, there being nothing to prove if $s = 0$, so assume that $s > 0$ -- then there exists a proper cuspidal $P = LU$ such that $V(\Pi) \neq V(\Pi)_U$. Claim: $V(\Pi)_U$ is P-invariant: $\forall\ \ell_0 \in L$, $\forall\ u_0 \in U$,

$\forall\ v \in V(\Pi): \sum_{u\in U} \Pi(u)v = 0,$

$$\sum_{u\in U} \Pi(u)\Pi(\ell_0 u_0)v$$

$$= \sum_{u\in U} \Pi(u\ell_0 u_0)v$$

$$= \sum_{u\in U} \Pi(\ell_0(\ell_0^{-1}u\ell_0)u_0)v$$

$$= \Pi(\ell_0)\Big(\sum_{u\in U} \Pi(\ell_0^{-1}u\ell_0)\Big)\Pi(u_0)v$$

$$= \Pi(\ell_0)\Big(\sum_{u\in U} \Pi(u)\Big)\Pi(u_0)v$$

$$= \Pi(\ell_0)\sum_{u\in U} \Pi(uu_0)v$$

$$= \Pi(\ell_0)\sum_{u\in U} \Pi(u)v$$

$$= \Pi(\ell_0)0 = 0.$$

Consequently, P operates on the quotient $V(\Pi)/V(\Pi)_U$. Moreover, its restriction to U is trivial: $\forall\ u_0 \in U,\ \forall\ v \in V(\Pi),$

$$\sum_{u\in U} \Pi(u)(\Pi(u_0)v-v)$$

$$= \sum_{u\in U} \Pi(uu_0)v - \sum_{u\in U} \Pi(u)v$$

$$= \sum_{u\in U} \Pi(u)v - \sum_{u\in U} \Pi(u)v$$

$$= 0$$

=>

$$\Pi(u_0)v \equiv v \bmod V(\Pi)_U.$$

On the other hand, while its restriction to L need not be irreducible, there is in any event an L-invariant subspace $\mathcal{V}$ of $V(\Pi)$ containing $V(\Pi)_U$ such that the quotient representation Θ of L on

$$V(\Pi)/V(\Pi)_U/\mathcal{V}/V(\Pi)_U$$

$$\approx V(\Pi)/\mathcal{V}$$

is irreducible. Pass now to $\mathrm{Ind}_{P,\Theta}^G$ and note that Π occurs as a subrepresentation of $\mathrm{Ind}_{P,\Theta}^G$ (see below). Accordingly, if $\Theta \in {}^0\hat{L}$, then we are done. If, however, $\Theta \notin {}^0\hat{L}$, then, thanks to the induction hypothesis, there exists a proper cuspidal subgroup $*P = *L*U$ of L and a discrete series representation $*\Theta$ of $*L$ such that Θ occurs as a subrepresentation of $\mathrm{Ind}_{*P,*\Theta}^L$. Form $P' = L'U' = *L*UU$, view $*\Theta$ as a representation of P' trivial on $U' = *UU$, and utilize the induction in stages rule (cf. §9, #12)

$$\mathrm{Ind}_{P',*\Theta}^G \approx \mathrm{Ind}_{P,}^G\ \mathrm{Ind}_{*P,*\Theta}^L$$

to conclude that Π, which occurs as a subrepresentation of $\mathrm{Ind}_{P,\Theta}^G$, must actually occur as a subrepresentation of $\mathrm{Ind}_{P',*\Theta}^G$:

$$\Theta \subset \mathrm{Ind}_{*P,*\Theta}^L \Rightarrow \Pi \subset \mathrm{Ind}_{P,\Theta}^G$$

$$\subset \mathrm{Ind}_{P,}^G\mathrm{Ind}_{*P,*\Theta}^L.$$

[Note: To confirm that

$$I_G(\Pi,\mathrm{Ind}_{P,\Theta}^G) \neq 0,$$

define an intertwining operator

$$T:V(\Pi) \to E^G_{P,\Theta}$$

by assigning to each $v \in V$ the function

$$f_v:G \to V(\Pi)/V$$

given by the prescription

$$f_v(x) = \Pi(x^{-1})v + V.]$$

This result reduces the problem of describing the elements of $\hat{G}$ into two parts.

- Isolate the discrete series (Deligne-Lusztig theory).
- Explicate the decomposition of $\mathrm{Ind}^G_{P,\Theta}$ and determine its irreducibility (Howlett-Lehrer theory).

We shall pass in silence on the first of these points (for a recent survey, consult arXiv:1404.0861) and settle for a summary on the second (cf. §15).

<u>27:</u> LEMMA The canonical representation of G on C(G;P) is equivalent to

$$\bigoplus_{\Theta} \mathrm{Ind}^G_{P,\Theta},$$

where Θ runs through the elements of ${}^0\hat{L}$.

<u>28:</u> NOTATION Given a parabolic k-subgroup $\underline{\underline{P}}$ of $\underline{\underline{G}}$, let ${}^0C(P)$ be the subspace of C(P) consisting of those f which are invariant to the right under U and have the property that the function on P/U thereby defined belongs to ${}^0C(P/U)$.

<u>29:</u> LEMMA Let $\underline{\underline{P}}_1, \underline{\underline{P}}_2$ be parabolic k-subgroups of $\underline{\underline{G}}$ and let

$$f_1 \in {}^0C(P_1),\ f_2 \in {}^0C(P_2).$$

Then

$$<r_{P_1 \to P_1 \cap P_2} f_1, r_{P_2 \to P_2 \cap P_1} f_2>_{P_1 \cap P_2} = 0$$

unless $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ have a common Levi subgroup $\underline{\underline{L}}$.

PROOF Ignoring constant factors (signified by $\doteq$), we have

$$<r_{P_1 \to P_1 \cap P_2} f_1, r_{P_2 \to P_2 \cap P_1} f_2>_{P_1 \cap P_2}$$

$$\doteq \Sigma_{P_1 \cap P_2} f_1(x)\overline{f_2(x)}$$

$$\doteq \Sigma_{P_1 \cap P_2/U_1 \cap U_2} f_1(x)\overline{f_2(x)}$$

$$\doteq \Sigma_{P_1 \cap P_2/P_1 \cap U_2} \overline{f_2(x)} \ \Sigma_{P_1 \cap U_2/U_1 \cap U_2} f_1(xu).$$

Let $\pi_1 : P_1 \to P_1/U_1 \approx L_1$ be the canonical projection -- then $*P = \pi_1(P_1 \cap P_2)$ is a cuspidal subgroup of L_1 with unipotent radical $*U = P_1 \cap U_2/U_1 \cap U_2$. Given $x \in P_1 \cap P_2$, write $x = \ell_1 u_1$ $(\ell_1 \in L_1, u_1 \in U_1)$, thus

$$f_1(xu) = f_1(\ell_1 u_1 u)$$

$$= f_1(\ell_1 u_1 u u_1^{-1}),$$

so

$$\Sigma_{P_1 \cap U_2/U_1 \cap U_2} f_1(xu)$$

$$\doteq \Sigma_{*U} f_1(\ell_1 u) = 0$$

unless $*U = \{e\}$, i.e., unless

$$P_1 \cap U_2 = U_1 \cap U_2 \subset U_1.$$

Switching roles leads to

$$P_2 \cap U_1 = U_2 \cap U_1 \subset U_2.$$

Therefore the relevant integrals vanish unless $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ have a common Levi subgroup (cf. §13, #20).]

<u>30:</u> APPLICATION Assume: $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ are not associate -- then

$$\langle r_{P_1 \to P_1 \cap P_2} f_1, r_{P_2 \to P_2 \cap P_1} f_2 \rangle_{P_1 \cap P_2} = 0.$$

<u>31:</u> THEOREM Let $P_1 = L_1U_1$, $P_2 = L_2U_2$ be cuspidal subgroups of G. Suppose that $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ are not associate -- then $\forall\ \Theta_1 \in {}^0\hat{L}_1$, $\forall\ \Theta_2 \in {}^0\hat{L}_2$,

$$\pi_1 = \mathrm{Ind}_{P_1,\Theta_1}^G \text{ and } \pi_2 = \mathrm{Ind}_{P_2,\Theta_2}^G$$

are disjoint:

$$\langle i_{P_1 \to G}\chi_{\Theta_1}, i_{P_2 \to G}\chi_{\Theta_2}\rangle_G = 0 \quad \text{(cf. §10, \#2)}.$$

PROOF In the notation of §8, #4,

$$\langle i_{P_1 \to G}\chi_{\Theta_1}, i_{P_2 \to G}\chi_{\Theta_2}\rangle_G$$

$$= \sum_{s\in S} \langle r_{P_1 \to P_2(s)}\chi_{\Theta_1}, (\chi_{\Theta_2})_s\rangle_{P_2(s)}$$

$$= \sum_{s\in S} \langle r_{P_1 \to P_2(s)}\chi_{\Theta_1}, r_{P_2^s \to P_2(s)} (\chi_{\Theta_2})^s\rangle_{P_2(s)},$$

where

$$P_2(s) = P_2^s \cap P_1 \quad (= sP_2s^{-1} \cap P_1).$$

But $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ are not associate, hence $\underline{\underline{P}}_1$ and $s\underline{\underline{P}}_2s^{-1}$ are not associate (cf. §13, #24). Therefore each of the terms in the sum $\sum_{s\in S}$ must vanish (cf. #30).

32: NOTATION Given a parabolic k-subgroup $\underline{\underline{P}}$ of $\underline{\underline{G}}$ and a Levi subgroup $\underline{\underline{L}} \subset \underline{\underline{P}}$, put

$$\underline{\underline{W}}_{\underline{\underline{L}}} = N_{\underline{\underline{G}}}(\underline{\underline{L}})/\underline{\underline{L}}.$$

33: N.B. If $\underline{\underline{L}}'$ is another Levi subgroup of $\underline{\underline{P}}$, then there is a unique $u \in U$ such that $\underline{\underline{L}}' = u\underline{\underline{L}}u^{-1}$, hence there is a canonical isomorphism

$$\underline{\underline{W}}_{\underline{\underline{L}}} \rightarrow \underline{\underline{W}}_{\underline{\underline{L}}'}.$$

Set

$$W_L = \underline{\underline{W}}_{\underline{\underline{L}}}(k) \quad (= N_G(\underline{\underline{L}})/L).$$

Then each $w \in W_L$ can be represented by an element $n_w \in N_G(\underline{\underline{L}})$.

34: LEMMA The arrow

$$W_L \rightarrow P\backslash G/P$$

given by

$$w \rightarrow Pn_wP$$

is injective.

35: LEMMA W_L operates on ${}^0C(P)$.

36: REDUCTION PRINCIPLE Let $\underline{\underline{P}}_1, \underline{\underline{P}}_2$ be parabolic k-subgroups of $\underline{\underline{G}}$ and let

$$f_1 \in {}^0\mathrm{CL}(P_1),\ f_2 \in {}^0\mathrm{CL}(P_2).$$

Assume: $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ have a common Levi subgroup $\underline{\underline{L}}$ — then

$$\langle i_{P_1 \to G} f_1, i_{P_2 \to G} f_2 \rangle_G$$

$$= \sum_{w \in W_L} \langle r_{P_1 \to L} f_1, r_{P_2 \to L} (w \cdot f_2) \rangle_L.$$

PROOF In the notation of §8, #4,

$$\langle i_{P_1 \to G} f_1, i_{P_2 \to G} f_2 \rangle_G$$

$$= \sum_{s \in S} \langle r_{P_1 \to P_2(s)} f_1, (f_2)_s \rangle_{P_2(s)}$$

$$= \sum_{s \in S} \langle r_{P_1 \to P_2(s)} f_1, r_{P_2^s \to P_2(s)} (f_2)^s \rangle_{P_2(s)},$$

where

$$P_2(s) = P_2^s \cap P_1 \ (= sP_2 s^{-1} \cap P_1).$$

The only nonzero terms in the sum are those for which $\underline{\underline{P}}_1$ and $s\underline{\underline{P}}_2 s^{-1}$ have a common Levi subgroup $\underline{\underline{L}}'$ (cf. #31). Choose $u_1 \in U_1$ such that $u_1 \underline{\underline{L}}' u_1^{-1} = \underline{\underline{L}}$. Next

$$\underline{\underline{L}}' \subset s\underline{\underline{P}}_2 s^{-1} \Rightarrow s^{-1}\underline{\underline{L}}' s \subset \underline{\underline{P}}_2.$$

Choose $u_2 \in U_2$ such that $u_2 \underline{\underline{L}} u_2^{-1} = s^{-1}\underline{\underline{L}}' s$, thus

$$\underline{\underline{L}}' = s u_2 \underline{\underline{L}} u_2^{-1} s^{-1}$$

=>

$$\underline{\underline{L}} = u_1\underline{\underline{L}}'u_1^{-1} = u_1 s u_2 \underline{\underline{L}} u_2^{-1} s^{-1} u_1^{-1}$$

=>

$$u_1 s u_2 \in N_G(\underline{\underline{L}}).$$

On the other hand,

$$u_1 s u_2 \in P_1\backslash G/P_2.$$

Therefore the double cosets $P_1\backslash G/P_2$ that intervene are those containing an element of $N_G(\underline{\underline{L}})$, so

$$<i_{P_1 \to G} f_1, i_{P_2 \to G} f_2>_G$$

$$= \sum_{w \in W_L} <r_{P_1 \to P_2(w)} f_1, r_{P_2^w \to P_2(w)} (w \cdot f_2)>_{P_2(w)}.$$

Noting that $\underline{\underline{L}} = w\underline{\underline{L}}w^{-1} \subset w\underline{\underline{P}}_2 w^{-1}$ is a Levi subgroup of $w\underline{\underline{P}}_2 w^{-1}$, write

$$P_2(w) = P_2^w \cap P_1 = P_1 \cap wP_2 w^{-1}$$

$$= L \cdot (L \cap wU_2 w^{-1}) \cdot (U_1 \cap L) \cdot (U_1 \cap wU_2 w^{-1})$$

with uniqueness of expression -- then

$$L \cap wU_2 w^{-1} = \{e\}, \ U_1 \cap L = \{e\}$$

and

$$<r_{P_1 \to P_2(w)} f_1, r_{P_2^w \to P_2(w)} (w \cdot f_2)>_{P_2(w)}$$

$$= \frac{1}{|P_2(w)|} \sum_{x,u} f_1(xu) \overline{(w \cdot f_2)(xu)},$$

where the sum runs over all $x \in L$ and all $u \in U_1 \cap wU_2w^{-1}$. Since f_1 and $w \cdot f_2$ are invariant to the right under $U_1 \cap wU_2w^{-1}$, the above expression equals

$$\frac{|U_1 \cap wU_2w^{-1}|}{|P_2(w)|} \sum_x f_1(x) \overline{(w \cdot f_2)(x)}$$

$$= \frac{|U_1 \cap wU_2w^{-1}|}{|P_2(w)|} |L| \langle r_{P_1 \to L} f_1, r_{P_2 \to L}(w \cdot f_2)\rangle_L.$$

And

$$\frac{|U_1 \cap wU_2w^{-1}|}{|P_2(w)|} |L|$$

$$= \frac{|U_1 \cap wU_2w^{-1}| \cdot |L|}{|L| \cdot |U_1 \cap wU_2w^{-1}|} = 1.$$

37: SUBLEMMA Let H be a Hilbert space and let $x,y \in H$. Assume:

$$\langle x,x\rangle = \langle x,y\rangle = \langle y,y\rangle.$$

Then $x = y$.

PROOF In fact,

$$\langle x - y, x - y\rangle = \langle x,x\rangle + \langle y,y\rangle - \langle x,y\rangle - \langle y,x\rangle$$

$$= \langle x,y\rangle + \langle x,y\rangle - \langle x,y\rangle - \overline{\langle x,y\rangle}$$

$$= \langle x,y\rangle - \overline{\langle x,y\rangle} = \langle x,y\rangle - \langle x,y\rangle = 0.$$

38: APPLICATION If

$$r_{P_1 \to L} f_1 = r_{P_2 \to L} f_2,$$

then

$$i_{P_1 \to G} f_1 = i_{P_2 \to G} f_2.$$

[It follows from #36 that

$$\langle i_{P_1 \to G} f_1, i_{P_1 \to G} f_1 \rangle_G$$

$$= \langle i_{P_1 \to G} f_1, i_{P_2 \to G} f_2 \rangle_G$$

$$= \langle i_{P_2 \to G} f_2, i_{P_2 \to G} f_2 \rangle_G.]$$

<u>39:</u> NOTATION Given a cuspidal subgroup P = LU of G and a $\Theta \in {}^0\hat{L}$, let

$$W_L(\Theta) = \{w \in W_L : w \cdot \chi_\Theta = \chi_\Theta\}.$$

<u>40:</u> THEOREM

$$\langle i_{P \to G} \chi_\Theta, i_{P \to G} \chi_\Theta \rangle_G = |W_L(\Theta)|.$$

[In #36, take $\underline{\underline{P}}_1 = \underline{\underline{P}}_2 = \underline{\underline{P}}$ and note that

$$\langle r_{P \to L} \chi_\Theta, r_{P \to L} (w \cdot \chi_\Theta) \rangle_L$$

equals 1 if $w \cdot \chi_\Theta = \chi_\Theta$ and equals 0 if $w \cdot \chi_\Theta \neq \chi_\Theta$.]

Let $\mathcal{P}$ be the set of parabolic k-subgroups of $\underline{\underline{G}}$. Decompose $\mathcal{P}$ into association classes: $\mathcal{P} = \coprod \mathcal{C}$. Given $\mathcal{C}$, take a $\underline{\underline{P}} \in \mathcal{C}$ and denote by $\hat{G}(\mathcal{C})$ the subset of $\hat{G}$ comprised of those Π which occur as a subrepresentation of

$$\mathrm{Ind}_{P,\Theta}^G$$

for some $\Theta \in {}^0\hat{L}$.

41: LEMMA $\hat{G}(C)$ is independent of the choice of $\underline{\underline{P}} \in C$.

PROOF The theory does not change if $\underline{\underline{P}}$ is replaced by $x\underline{\underline{P}}x^{-1}$ $(x \in G)$, so if $\underline{\underline{P}}_1$, $\underline{\underline{P}}_2$ are associate, then there is no loss of generality in assuming that $\underline{\underline{P}}_1$ and $\underline{\underline{P}}_2$ have a common Levi subgroup $\underline{\underline{L}}$, thus $\left[\begin{array}{l} L \subset P_1 \\ L \subset P_2 \end{array}\right.$. Given $\Theta \in {}^0\hat{L}$, lift

$$\left[\begin{array}{l} \Theta \text{ to } P_1, \text{ call it } \Theta_1 \\ \Theta \text{ to } P_2, \text{ call it } \Theta_2. \end{array}\right.$$

Then

$$\left[\begin{array}{l} i_{P_1 \to G}\chi_{\Theta_1} \text{ is the character of } \operatorname{Ind}_{P_1,\Theta_1}^G \\ i_{P_2 \to G}\chi_{\Theta_2} \text{ is the character of } \operatorname{Ind}_{P_2,\Theta_2}^G. \end{array}\right.$$

But

$$\left[\begin{array}{l} r_{P_1 \to L}\chi_{\Theta_1} = \chi_\Theta \\ r_{P_2 \to L}\chi_{\Theta_2} = \chi_\Theta \end{array}\right.$$

$\Rightarrow$

$$i_{P_1 \to G}\chi_{\Theta_1} = i_{P_2 \to G}\chi_{\Theta_2} \quad \text{(cf. \#38).}$$

Therefore

$$\operatorname{Ind}_{P_1,\Theta_1}^G \approx \operatorname{Ind}_{P_2,\Theta_2}^G.$$

42: LEMMA If $C_1 \neq C_2$, then

$$\hat{G}(C_1) \cap \hat{G}(C_2) = \emptyset \quad \text{(cf. \#30).}$$

Accordingly:

43: THEOREM There is a disjoint decomposition

$$\hat{G} = \coprod_{C} \hat{G}(C).$$

44: NOTATION Given $\underline{\underline{P}} \in \mathcal{P}$, let $[\underline{\underline{P}}]$ be the association class to which $\underline{\underline{P}}$ belongs.

45: EXAMPLE Take $\underline{\underline{P}} = \underline{\underline{G}}$ -- then the elements of $\hat{G}([G])$ comprise the discrete series for G, i.e., $\hat{G}([G]) = {}^0\hat{G}$.

46: EXAMPLE Take $\underline{\underline{P}} = \underline{\underline{B}}$ -- then the elements of $\hat{G}([B])$ comprise the principal series for G.

47: REMARK W_L operates on ${}^0\hat{L}$, hence ${}^0\hat{L}$ breaks up into W_L-orbits. Let $\Theta_1, \Theta_2 \in {}^0\hat{L}$ -- then there are two possibilities.

- If Θ_1, Θ_2 are on the same W_L-orbit, then

$$\text{Ind}_{P,\Theta_1}^G \approx \text{Ind}_{P,\Theta_2}^G.$$

- If Θ_1, Θ_2 are not on the same W_L-orbit, then

$$\text{Ind}_{P,\Theta_1}^G \text{ and } \text{Ind}_{P,\Theta_2}^G$$

are disjoint.

§15. *HOWLETT-LEHRER THEORY*

In view of §14, #40,

$$\langle i_{P \to G}\chi_\Theta, i_{P \to G}\chi_\Theta\rangle_G = |W_L(\Theta)|.$$

And on general grounds (cf. §5, #11), $\mathrm{Ind}_{P,\Theta}^G$ is irreducible iff

$$\langle i_{P \to G}\chi_\Theta, i_{P \to G}\chi_\Theta\rangle_G = 1.$$

<u>1:</u> DEFINITION Θ is <u>unramified</u> if $|W_L(\Theta)| = 1$.

<u>2:</u> THEOREM $\mathrm{Ind}_{G,\Theta}^P$ is irreducible iff Θ is unramified.

To discuss the decomposability of $\mathrm{Ind}_{P,\Theta}^G$, note that $\Pi \in \hat{G}$ occurs as a subrepresentation of $\mathrm{Ind}_{P,\Theta}^G$ iff

$$\langle \chi_\Pi, i_{P \to G}\chi_\Theta\rangle_G \neq 0.$$

<u>3:</u> LEMMA There is a one-to-one correspondence between the $\Pi \in \hat{G}$ such that

$$\langle \chi_\Pi, i_{P \to G}\chi_\Theta\rangle_G \neq 0$$

and the irreducible representations ρ of

$$I_G(\mathrm{Ind}_{P,\Theta}^G, \mathrm{Ind}_{P,\Theta}^G).$$

and if $\Pi \longleftrightarrow \rho$, then

$$\chi_\rho(1) = \langle \chi_\Pi, i_{P \to G}\chi_\Theta\rangle_G,$$

the positive integer on the right being the multiplicity

$$m(\Pi, \mathrm{Ind}_{P,\Theta}^G)$$

of Π in $\mathrm{Ind}_{P,\Theta}^G$.

4: THEOREM The semisimple algebra

$$I_G(\mathrm{Ind}_{P,\Theta}^G, \mathrm{Ind}_{P,\Theta}^G)$$

is isomorphic to the semisimple algebra

$$C(W_L(\Theta)).$$

The irreducible components of $\mathrm{Ind}_{P,\Theta}^G$ are therefore parameterized by the elements of $W_L(\Theta)$: If $\omega \in W_L(\Theta)$ and if $\Pi(\omega) \in \hat{G}$ is the irreducible component of $\mathrm{Ind}_{P,\Theta}^G$ corresponding to ω, then

$$\langle \chi_{\Pi(\omega)}, i_{P \to G}\chi_\Theta \rangle_G = \chi_\omega(1),$$

the dimension of the representation space of ω.

§16. MODULE LANGUAGE

Let G be a finite group, $\Gamma \subset G$ a subgroup. View C(G) as a left C(G)-module and as a right C(Γ)-module.

1: CONSTRUCTION Let $\theta:\Gamma \to GL(E)$ be a representation of Γ -- then the tensor product

$$C(G) \otimes_{C(\Gamma)} E$$

is a left C(G)-module or, equivalently, a representation, the representation $\text{Ind}_{\Gamma,\theta}^{G}$ of G induced by θ.

2: N.B. The left action is given by

$$\left(\sum_{x\in G} f(x)\delta_x\right)(\delta_y \otimes X)$$

$$= \sum_{x\in G} \delta_{xy} \otimes f(x)X \quad (X \in E)$$

and from the definitions, $\forall\ \gamma \in \Gamma$,

$$\delta_x\delta_\gamma \otimes X = \delta_x \otimes \theta(\gamma)X \quad (X \in E).$$

3: LEMMA Write

$$G = \coprod_{k=1}^{n} x_k\Gamma.$$

Then as a vector space

$$\text{Ind}_{\Gamma,\theta}^{G} = \bigoplus_{k=1}^{n} (\delta_{x_k} \otimes E).$$

[Note: The summand

$$\delta_{x_k} \otimes E = \{\delta_{x_k} \otimes X : X \in E\}$$

$$\approx E(\delta_{x_k} \otimes X \longleftrightarrow X)$$

is the transform of $\delta_e \otimes E \approx E$ under the action of δ_{x_k}:

$$\delta_{x_k}(\delta_e \otimes X) = \delta_{x_k} \otimes X.]$$

The following result justifies the notation and the terminology.

<u>4:</u> THEOREM Set $\pi = \text{Ind}_{\Gamma,\theta}^{G}$ -- then

$$i_{\Gamma \to G}\chi_\theta = \chi_\pi \quad \text{(cf. §9, \#10).}$$

PROOF Let $X_1,\ldots,X_d$ be a basis for E and define $\theta_{ij}(\gamma)$ by

$$\theta(\gamma)X_j = \sum_i \theta_{ij}(\gamma)X_i.$$

Equip $C(G) \otimes_{C(\Gamma)} E$ with the basis $\{\delta_{x_1} \otimes X_1,\ldots,\delta_{x_1} \otimes X_d, \delta_{x_2} \otimes X_1,\ldots,\delta_{x_2} \otimes X_d,$

$$\ldots,\delta_{x_n} \otimes X_1,\ldots,\delta_{x_n} \otimes X_d\}$$

and write $xx_k = x_\ell\gamma$ -- then

$$\begin{aligned}\pi(x)(\delta_{x_k} \otimes X_j) &= \delta_{xx_k} \otimes X_j \\ &= \delta_{x_\ell\gamma} \otimes X_j \\ &= \delta_{x_\ell}\delta_\gamma \otimes X_j \\ &= \delta_{x_\ell} \otimes \theta(\gamma)X_j \\ &= \delta_{x_\ell} \otimes \sum_i \theta_{ij}(\gamma)X_i\end{aligned}$$

$$= \sum_i \theta_{ij}(\gamma)\,\delta_{x_\ell} \otimes X_i$$

$$= \sum_i \theta_{ij}(x_\ell^{-1}xx_k)\,\delta_{x_\ell} \otimes X_i.$$

Define $\mathring{\theta}$ on G by $\mathring{\theta}(\gamma) = [\theta_{ij}(\gamma)]$ $(\gamma \in \Gamma)$ and $\mathring{\theta}(x) = 0_d$ if $x \notin \Gamma$ (0_d the zero d-by-d matrix), thus the block matrix representing $\pi(x)$ is

$$\begin{bmatrix} \mathring{\theta}(x_1^{-1}xx_1) & \mathring{\theta}(x_1^{-1}xx_2) & \cdots\cdots & \mathring{\theta}(x_1^{-1}xx_n) \\ \mathring{\theta}(x_2^{-1}xx_1) & \mathring{\theta}(x_2^{-1}xx_2) & \cdots\cdots & \mathring{\theta}(x_2^{-1}xx_n) \\ \vdots & \vdots & & \vdots \\ \mathring{\theta}(x_n^{-1}xx_1) & \mathring{\theta}(x_n^{-1}xx_2) & \cdots\cdots & \mathring{\theta}(x_n^{-1}xx_n) \end{bmatrix}$$

Taking the trace

$$\chi_\pi(x) = \mathrm{tr}(\pi(x))$$

$$= \sum_{k=1}^{n} \mathrm{tr}(\mathring{\theta}(x_k^{-1}xx_k))$$

$$= \sum_{k=1}^{n} \mathring{\chi}_\theta(x_k^{-1}xx_k)$$

$$= (i_{\Gamma \to G}\chi_\theta)(x)$$

finishes the proof.

<u>5:</u> NOTATION MOD(Γ) is the category of left $C(\Gamma)$-modules and MOD(G) is the category of left C(G)-modules.

[Note: All data is over C and finite dimensional.]

6: N.B. Morphisms are intertwining operators.

7: SCHOLIUM The assignment

$$(\theta,E) \rightarrow \mathrm{Ind}_{\Gamma,\theta}^{G}$$

defines a functor

$$\mathrm{MOD}(\Gamma) \rightarrow \mathrm{MOD}(G).$$

8: NOTATION Given a representation (π,V) of G, denote its restriction to Γ by $\mathrm{Res}_{\Gamma,\pi}^{G}$.

9: SCHOLIUM The assignment

$$(\pi,V) \rightarrow \mathrm{Res}_{\Gamma,\pi}^{G}$$

defines a functor

$$\mathrm{MOD}(G) \rightarrow \mathrm{MOD}(\Gamma).$$

Here now are the fundamental formalities.

10: LEMMA

$$I_G(\mathrm{Ind}_{\Gamma,\theta}^{G}, (\pi,V)) \approx I_\Gamma((\theta,E), \mathrm{Res}_{\Gamma,\theta}^{G}).$$

11: SLOGAN The restriction functor is a right adjoint for the induction functor.

12: LEMMA

$$I_G((\pi,V), \mathrm{Ind}_{\Gamma,\theta}^{G}) \approx I_\Gamma(\mathrm{Res}_{\Gamma,\pi}^{G}, (\theta,E)).$$

<u>13:</u> SLOGAN The restriction functor is a left adjoint for the induction functor.

Moving on:

<u>14:</u> DEFINITION Let $\theta:\Gamma \to GL(E)$ be a representation of Γ -- then

$$\mathrm{Inv}_\Gamma(E) = \{X \in E:\theta(\gamma)X = X \ \forall \ \gamma \in \Gamma\}$$

is the set of <u>Γ-invariants</u> per E.

<u>15:</u> DEFINITION Let $\theta:\Gamma \to GL(E)$ be a representation of Γ -- then

$$\mathrm{CoInv}_\Gamma(E) = E/I_\Gamma E$$

is the set of <u>coinvariants</u> per E.

[Note: $I_\Gamma \subset C(\Gamma)$ is the augmentation ideal, thus $I_\Gamma E$ stands for the set of all finite sums $\sum_i \theta(\gamma_i)X_i$ ($\delta_{\gamma_i} \in I_\Gamma$, $X_i \in E$).]

Specialize and assume that G is a group of Lie type (cf. §13, #3).

<u>16:</u> NOTATION Given a cuspidal subgroup $P = LU$ of G,

$$\mathrm{Inf}_{L,P}:\mathrm{MOD}(L) \to \mathrm{MOD}(P)$$

is the inflation functor.

[In other words, given a representation (θ,E) of L, $\mathrm{Inf}_{L,P}\theta$ is the lift of θ to P, i.e., E viewed as a left $C(P)$-module with trivial U-action.]

<u>17:</u> DEFINITION The composite

$$\mathrm{Ind}_{P,-}^G \circ \mathrm{Inf}_{L,P}$$

defines a functor

$$R^G_{L,P}:\mathrm{MOD}(L) \to \mathrm{MOD}(G)$$

termed <u>Harish-Chandra induction</u>.

<u>18:</u> THEOREM If $P_1 = LU_1$, $P_2 = LU_2$ are cuspidal subgroups of G, then the functors

$$\begin{bmatrix} R^G_{L,P_1} \\ R^G_{L,P_2} \end{bmatrix}$$

are naturally isomorphic.

[Note: Accordingly, the left C(G)-module isomorphism class of $R^G_{L,P}(\theta,E)$ depends only on θ (it being independent of the particular cuspidal subgroup $P = LU$).]

<u>19:</u> LEMMA

$$I_G(R^G_{L,P}(\theta,E),\ (\pi,V))$$

$$\approx I_P(\mathrm{Inf}_{L,P}\theta,\ \mathrm{Res}^G_{P,\pi})$$

$$\approx I_L((\theta,E),\ \mathrm{Inv}_U(\mathrm{Res}^G_{P,\pi})).$$

[Note: For any left C(P)-module M, the set $\mathrm{Inv}_U(M)$ is canonically a left C(L)-module.]

<u>20:</u> SLOGAN The composite of restriction followed by the taking of invariants is a right adjoint for Harish-Chandra induction.

21: LEMMA

$$I_G((\pi,V),\ R^G_{L,P}(\theta,E))$$

$$\approx I_P(\mathrm{Res}^G_{P,\pi},\ \mathrm{Inf}_{L,P}(\theta))$$

$$\approx I_L(\mathrm{CoInv}_U(\mathrm{Res}^G_{P,\pi}),\ (\theta,E)).$$

[Note: For any left $C(P)$-module M, the set $\mathrm{CoInv}_U(M)$ is canonically a left $C(L)$-module.]

22: SLOGAN The composite of restriction followed by the taking of coinvariants is a left adjoint for Harish-Chandra induction.

23: SUBLEMMA For any left $C(P)$-module M,

$$\mathrm{Inv}_U(M) \approx \mathrm{CoInv}_U(M).$$

24: SCHOLIUM The left and right adjoint of Harish-Chandra induction are naturally isomorphic.

25: DEFINITION Harish-Chandra restriction $*R^G_{L,P}$ is the left and right adjoint of Harish-Chandra induction.

26: LEMMA

$$*R^G_{L,P}((\pi,V)) = e_U V,$$

where

$$e_U = \frac{1}{|U|}\sum_{u\in U}\pi(u).$$

<u>27:</u> THEOREM If $P_1 = LU_1$, $P_2 = LU_2$ are cuspidal subgroups of G, then the functors

$$\begin{cases} {}^*R^G_{L,P_1} \\ {}^*R^G_{L,P_2} \end{cases}$$

are naturally isomorphic.

[Note: Accordingly, the left C(L)-module class of ${}^*R^G_{L,P}(\pi,V)$ depends only on π (it being independent of the particular cuspidal parabolic subgroup P = LU).]

APPENDIX

Let P = LU be a cuspidal subgroup of G.

DEFINITION Given $\phi \in CL(L)$, define $\tilde{\phi} \in C(P)$ by the rule

$$\tilde{\phi}(\ell u) = \phi(\ell).$$

LEMMA $\tilde{\phi}$ is a class function, i.e.,

$$\tilde{\phi} \in CL(P).$$

PROOF The claim is that $\forall\ p \in P,\ \forall\ p_1 \in P$,

$$\tilde{\phi}(pp_1p^{-1}) = \tilde{\phi}(p_1).$$

Write $p = \ell u$, $p_1 = \ell_1 u_1$ -- then

$$\begin{aligned} \tilde{\phi}(pp_1p^{-1}) &= \tilde{\phi}(\ell u \ell_1 u_1 u^{-1} \ell^{-1}) \\ &= \tilde{\phi}(\ell u \ell_1 u_1 \ell^{-1} \ell u^{-1} \ell^{-1}) \\ &= \tilde{\phi}(\ell u \ell_1 u_1 \ell^{-1} v) \quad (v = \ell u^{-1} \ell^{-1} \in U) \end{aligned}$$

$$= \tilde{\phi}(\ell u \ell_1 \ell^{-1} \ell u_1 \ell^{-1} v)$$

$$= \tilde{\phi}(\ell u \ell_1 \ell^{-1} v_1 v) \quad (v_1 = \ell u_1 \ell^{-1} \in U)$$

$$= \tilde{\phi}(\ell (\ell_1 \ell^{-1}) (\ell_1 \ell^{-1})^{-1} u (\ell_1 \ell^{-1}) v_1 v)$$

$$= \tilde{\phi}(\ell \ell_1 \ell^{-1} v_2 v_1 v) \ (v_2 = (\ell_1 \ell^{-1})^{-1} u (\ell_1 \ell^{-1}) \in U)$$

$$= \phi(\ell \ell_1 \ell^{-1})$$

$$= \phi(\ell_1) = \tilde{\phi}(p_1).$$

Thus there is an arrow

$$CL(L) \to CL(P) \to CL(G),$$

namely

$$\phi \to \tilde{\phi} \to i_P \to G^{\tilde{\phi}}.$$

On the other hand, there is an arrow

$$CL(G) \to CL(L),$$

namely

$$\psi \to \psi_P | L \equiv r_{G \to L} \psi_P.$$

[Note: $\forall \ \ell \in L, \ \forall \ \ell_1 \in L,$

$$\sum_{u \in U} \psi(\ell \ell_1 \ell^{-1} u) = \sum_{u \in U} \psi(\ell \ell_1 \ell^{-1} u \ell \ell^{-1})$$

$$= \sum_{u \in U} \psi(\ell_1 \ell^{-1} u \ell)$$

$$= \sum_{u \in U} \psi(\ell_1 u).]$$

LEMMA Let $\phi \in CL(L)$, $\psi \in CL(G)$ -- then

$$<i_{P \to G}\tilde{\phi},\psi>_G = <\phi, r_{G \to L}\psi_P>_L.$$

PROOF

$$<i_{P \to G}\tilde{\phi},\psi>_G = \frac{1}{|G|}\sum_{x\in G}(i_{P \to G}\tilde{\phi})(x)\overline{\psi(x)}$$

$$= \frac{1}{|G|}\sum_{x\in G}\frac{1}{|P|}\sum_{y\in G}\overset{\circ}{\tilde{\phi}}(yxy^{-1})\overline{\psi(x)}$$

$$= \frac{1}{|G|}\sum_{y\in G}\frac{1}{|P|}\sum_{x\in G}\overset{\circ}{\tilde{\phi}}(yxy^{-1})\overline{\psi(x)}$$

$$= \frac{1}{|G|}\sum_{y\in G}\frac{1}{|P|}\sum_{x\in G}\overset{\circ}{\tilde{\phi}}(x)\overline{\psi(y^{-1}xy)}$$

$$= \frac{1}{|G|}\sum_{y\in G}\frac{1}{|P|}\sum_{p\in P}\tilde{\phi}(p)\overline{\psi(ypy^{-1})}$$

$$= \frac{1}{|G|}\sum_{y\in G}\frac{1}{|P|}\sum_{\ell,u}\tilde{\phi}(\ell u)\overline{\psi(y\ell uy^{-1})}$$

$$= \frac{1}{|G|}\sum_{y\in G}\frac{1}{|P|}\sum_{\ell,u}\phi(\ell)\overline{\psi(y\ell uy^{-1})}$$

$$= \frac{1}{|G|}\sum_{y\in G}\frac{1}{|P|}\sum_{\ell,u}\phi(\ell)\overline{\psi(\ell u)}$$

$$= \frac{1}{|P|}\sum_{\ell,u}\phi(\ell)\overline{\psi(\ell u)}$$

$$= \frac{1}{|L||U|}\sum_{\ell,u}\phi(\ell)\overline{\psi(\ell u)}$$

$$= \frac{1}{|L|}\sum_{\ell}\phi(\ell)\frac{1}{|U|}\sum_{u\in U}\overline{\psi(\ell u)}$$

$$= \frac{1}{|L|} \sum_{\ell} \phi(\ell)\overline{\psi_P(\ell)}$$

$$= \langle\phi, r_{G \to L}\psi_P\rangle_L.$$

§1. *ORBITAL SUMS*

Let G be a finite group.

1: DEFINITION Given $f \in C(G)$ and $\gamma \in G$, put

$$\mathcal{O}(f,\gamma) = \sum_{x\in G} f(x\gamma x^{-1}),$$

the orbital sum of f at γ.

2: LEMMA The function $\mathcal{O}(f)$ defined by the assignment

$$\gamma \to \mathcal{O}(f,\gamma)$$

is a class function on G, i.e., is an element of CL(G).

3: LEMMA There is an expansion

$$\mathcal{O}(f,\gamma) = \sum_{\Pi\in\hat{G}} \operatorname{tr}(\Pi^*(f))\chi_\Pi(\gamma),$$

where

$$\Pi^*(f) = \sum_{x\in G} f(x)\Pi^*(x).$$

PROOF Since $\mathcal{O}(f)$ is a class function, $\forall\ \gamma \in G$,

$$\mathcal{O}(f,\gamma) = \sum_{\Pi\in\hat{G}} <\mathcal{O}(f),\chi_\Pi>_G\chi_\Pi(\gamma) \qquad \text{(cf. II, §4, \#17).}$$

But

$$<\mathcal{O}(f),\chi_\Pi>_G = \frac{1}{|G|}\sum_{\gamma\in G} \mathcal{O}(f,\gamma)\overline{\chi_\Pi(\gamma)}$$

$$= \frac{1}{|G|}\sum_{\gamma\in G}\sum_{x\in G} f(x\gamma x^{-1})\overline{\chi_\Pi(\gamma)}$$

$$= \frac{1}{|G|}\sum_{x\in G}\sum_{\gamma\in G} f(x\gamma x^{-1})\overline{\chi_\Pi(\gamma)}$$

$$= \frac{1}{|G|} \sum_{x \in G} \sum_{\gamma \in G} f(\gamma) \overline{\chi_\Pi(x^{-1} \gamma x)}$$

$$= \frac{1}{|G|} \sum_{x \in G} \sum_{\gamma \in G} f(\gamma) \overline{\chi_\Pi(\gamma)}$$

$$= \frac{1}{|G|} |G| \sum_{\gamma \in G} f(\gamma) \overline{\chi_\Pi(\gamma)}$$

$$= \sum_{\gamma \in G} f(\gamma) \overline{\chi_\Pi(\gamma)}$$

$$= \sum_{x \in G} f(x) \overline{\chi_\Pi(x)}$$

$$= \text{tr}(\Pi^*(f)).$$

[Note: Recall that

$$\chi_{\Pi^*} = \overline{\chi_\Pi} \quad \text{(cf. II, §4, #4).]}$$

4: N.B. In terms of the Fourier transform,

$$\Pi^*(f) = \hat{f}(\Pi^*) \Rightarrow \text{tr}(\Pi^*(f)) = \text{tr}(\hat{f}(\Pi^*)).]$$

§2. *THE LOCAL TRACE FORMULA*

Let G be a finite group.

1: NOTATION Denote by $\pi_{L,R}$ the representation of $G \times G$ on $C(G)$ given by

$$(\pi_{L,R}(x_1,x_2)f)(x) = f(x_1^{-1}xx_2) \quad \text{(cf. II, §1, \#14).}$$

Define a linear bijection

$$T:C(G) \to C(G \times G/G)$$

via the prescription

$$Tf(x_1,x_2) = f(x_1x_2^{-1}).$$

2: N.B. Embed G diagonally into $G \times G$ -- then $\forall\ x \in G$,

$$Tf((x_1,x_2)(x,x))$$

$$= Tf(x_1x,x_2x) = f(x_1xx^{-1}x_2^{-1})$$

$$= f(x_1x_2^{-1}) = Tf(x_1,x_2).$$

3: NOTATION Set

$$L_{G \times G/G} = \text{Ind}_{G,\theta}^{G \times G},$$

where θ is the trivial representation of G on $E = \mathbf{C}$.

4: LEMMA

$$T \in I_{G \times G}(\pi_{L,R}, L_{G \times G/G}).$$

PROOF $\forall\ x_1,x_2 \in G$, $\forall\ f \in C(G)$,

$$(T\pi_{L,R}(x_1,x_2)f)(y_1,y_2)$$

$$= (\pi_{L,R}(x_1,x_2)f)(y_1y_2^{-1})$$

$$= f(x_1^{-1}y_1y_2^{-1}x_2).$$

And

$$(L_{G \times G/G}(x_1,x_2)Tf)(y_1,y_2)$$

$$= Tf((x_1,x_2)^{-1}(y_1,y_2))$$

$$= Tf((x_1^{-1},x_2^{-1})(y_1,y_2))$$

$$= Tf(x_1^{-1}y_1,x_2^{-1}y_2)$$

$$= f(x_1^{-1}y_1y_2^{-1}x_2).$$

5: N.B. T is unitary: $\forall\ f,g \in C(G)$,

$$\langle Tf,Tg\rangle_{G \times G} = \langle f,g\rangle_G.$$

[By definition,

$$\langle Tf,Tg\rangle_{G \times G}$$

$$= \frac{1}{|G \times G|} \sum_{(x_1,x_2)\in G \times G} Tf(x_1,x_2)\overline{Tg(x_1,x_2)}$$

$$= \frac{1}{|G \times G|} \sum_{(x_1,x_2)\in G \times G} f(x_1x_2^{-1})\overline{g(x_1x_2^{-1})}$$

$$= \frac{1}{|G \times G|} \sum_{x_1\in G} \sum_{x_2\in G} f(x_1x_2^{-1})\overline{g(x_1x_2^{-1})}$$

$$= \frac{1}{|G \times G|} \sum_{x_1 \in G} \sum_{x_2 \in G} f(x_1 x_2)\overline{g(x_1 x_2)}$$

$$= \frac{1}{|G \times G|} \sum_{x_1 \in G} \sum_{x_2 \in G} f(x_2)\overline{g(x_2)}$$

$$= \frac{|G|}{|G \times G|} \sum_{x \in G} f(x)\overline{g(x)}$$

$$= \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)} = \langle f,g\rangle_G.]$$

<u>6:</u> NOTATION Given an $x \in G$, write $C(x)$ for its conjugacy class and G_x for its centralizer (cf. II, §4, #10).

<u>7:</u> EXAMPLE $\forall\, f \in C(G)$ and $\forall\, \gamma \in G$,

$$\mathcal{O}(f,\gamma) = \sum_{x \in G} f(x\gamma x^{-1}) = |G_\gamma| \sum_{x \in C(\gamma)} f(x).$$

<u>8:</u> LEMMA Abbreviate $\chi_{\pi_{L,R}}$ to $\chi_{L,R}$ -- then

$$\chi_{L,R}(x_1,x_2) = |\{x \in G : \delta_x(x_1^{-1} x x_2) \neq 0\}|$$

$$= \begin{cases} |G_x| & (x = x_1 = x_2) \\ 0 & (C(x_1) \neq C(x_2)). \end{cases}$$

[Work instead with the character of $L_{G \times G/G}$ and apply II, §7, #11.]

Given $f_1, f_2 \in C(G)$, define $f \in C(G \times G)$ by

$$f(x_1,x_2) = f_1(x_1) f_2(x_2),$$

and let

$$\pi_{L,R}(f) = \sum_{x_1 \in G} \sum_{x_2 \in G} f_1(x_1) f_2(x_2) \pi_{L,R}(x_1, x_2).$$

Then $\forall \ \phi \in C(G)$,

$$(\pi_{L,R}(f)\phi)(x) = \sum_{x_1 \in G} \sum_{x_2 \in G} f_1(x_1) f_2(x_2) \phi(x_1^{-1} x x_2)$$

$$= \sum_{y \in G} K_f(x,y)\phi(y),$$

where

$$K_f(x,y) = \sum_{z \in G} f_1(xz) f_2(zy).$$

Therefore $\pi_{L,R}(f)$ is an integral operator on $C(G)$ (a.k.a. $L^2(G)$...) with kernel $K_f(x,y)$.

<u>9:</u> LEMMA The $\sqrt{|G|}\ \delta_x$ $(x \in G)$ constitute an orthonormal basis for $C(G)$.

<u>10:</u> LEMMA $\forall \ f = f_1 f_2$,

$$\text{tr}(\pi_{L,R}(f)) = \sum_{x \in G} K_f(x,x).$$

PROOF In fact,

$$\text{tr}(\pi_{L,R}(f)) = \sum_{x \in G} \langle \pi_{L,R}(f) \sqrt{|G|}\delta_x, \sqrt{|G|}\delta_x \rangle_G$$

$$= |G| \sum_{x \in G} \frac{1}{|G|} \sum_{y \in G} (\pi_{L,R}(f)\delta_x)(y)\delta_x(y)$$

$$= \sum_{x \in G} (\pi_{L,R}(f)\delta_x)(x)$$

$$= \sum_{x \in G} \sum_{x_1 \in G} \sum_{x_2 \in G} f_1(x_1) f_2(x_2) \delta_x(x_1^{-1} x x_2)$$

$$= \sum_{x \in G} \sum_{z \in G} f_1(xz) f_2(zy)$$

$$= \sum_{x \in G} K_f(x,x).$$

Enumerate the elements of CON(G), say

$$\text{CON}(G) = \{C_1, \ldots, C_n\}.$$

For each i, fix a $\gamma_i \in C_i$ $(1 \le i \le n)$.

<u>11:</u> LEMMA $\forall\ f = f_1 f_2$,

$$\sum_{x \in G} K_f(x,x) = \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} \mathcal{O}(f_1,\gamma_i)\mathcal{O}(f_2,\gamma_i).$$

PROOF Start with the LHS:

$$\sum_{x \in G} K_f(x,x) = \sum_{x \in G} \sum_{z \in G} f_1(xz) f_2(zx)$$

$$= \sum_{x \in G} \sum_{y \in G} f_1(y) f_2(x^{-1}yx)$$

$$= \sum_{y \in G} F(y),$$

where

$$F(y) = \sum_{x \in G} f_1(y) f_2(x^{-1}yx).$$

Using now §4, #2 below, we have

$$\sum_{y \in G} F(y) = \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} \mathcal{O}(F,\gamma_i).$$

And

$$\mathcal{O}(F,\gamma_i) = \sum_{x \in G} F(x\gamma_i x^{-1})$$

$$= \sum_{x \in G} \sum_{y \in G} f_1(x\gamma_i x^{-1}) f_2(y^{-1} x\gamma_i x^{-1} y)$$

$$= \sum_{x \in G} f_1(x\gamma_i x^{-1}) \sum_{y \in G} f_2(y^{-1} x\gamma_i x^{-1} y)$$

$$= \sum_{x \in G} f_1(x\gamma_i x^{-1}) \sum_{y \in G} f_2(y^{-1} \gamma_i y)$$

$$= \sum_{x \in G} f_1(x\gamma_i x^{-1}) \sum_{y \in G} f_2(y\gamma_i y^{-1})$$

$$= O(f_1,\gamma_i) O(f_2,\gamma_i).$$

12: LEMMA $\forall\ f = f_1 f_2$,

$$\sum_{x \in G} K_f(x,x) = \sum_{\Pi \in \hat{G}} \mathrm{tr}(\hat{f}_1(\Pi))\,\mathrm{tr}(\hat{f}_2(\Pi^*)).$$

PROOF Write

$$\sum_{x \in G} K_f(x,x) = \sum_{x \in G} \sum_{y \in G} f_1(y) f_2(x^{-1} y x)$$

$$= \sum_{y \in G} f_1(y) \sum_{x \in G} f_2(xyx^{-1})$$

$$= \sum_{y \in G} f_1(y) O(f_2,y)$$

$$= \sum_{y \in G} f_1(y) \sum_{\Pi \in \hat{G}} \mathrm{tr}(\Pi^*(f_2))\chi_\Pi(y) \qquad \text{(cf. §1, \#3)}$$

$$= \sum_{\Pi \in \hat{G}} \Big(\sum_{y \in G} f_1(y)\chi_\Pi(y)\Big)\mathrm{tr}(\Pi^*(f_2))$$

$$= \sum_{\Pi \in \hat{G}} \mathrm{tr}(\Pi(f_1))\,\mathrm{tr}(\Pi^*(f_2))$$

$$= \sum_{\Pi \in \hat{G}} \mathrm{tr}(\hat{f}_1(\Pi))\,\mathrm{tr}(\hat{f}_2(\Pi^*)).$$

13: DEFINITION Given $f = f_1 f_2$, the local trace formula is the relation

$$\sum_{\Pi \in \hat{G}} \operatorname{tr}(\hat{f}_1(\Pi)) \operatorname{tr}(\hat{f}_2(\Pi^*))$$

$$= \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} \mathcal{O}(f_1,\gamma_i)\mathcal{O}(f_2,\gamma_i).$$

14: EXAMPLE Suppose that $f_1 = f_2$ is real valued, call it ϕ -- then

$$\operatorname{tr}(\hat{\phi}(\Pi^*)) = \sum_{x \in G} \phi(x)\chi_{\Pi^*}(x)$$

$$= \sum_{x \in G} \phi(x)\overline{\chi_{\Pi}(x)}$$

$$= \sum_{x \in G} \overline{\phi(x)}\,\overline{\chi_{\Pi}(x)}$$

$$= \overline{\sum_{x \in G} \phi(x)\chi_{\Pi}(x)}$$

$$= \overline{\operatorname{tr}(\hat{\phi}(\Pi))}.$$

Therefore

$$\sum_{\Pi \in \hat{G}} \operatorname{tr}(\hat{\phi}(\Pi))\overline{\operatorname{tr}(\hat{\phi}(\Pi))}$$

$$= \sum_{\Pi \in \hat{G}} |\operatorname{tr}(\hat{\phi}(\Pi))|^2$$

$$= \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} \mathcal{O}(\phi,\gamma_i)^2.$$

[Note: Specialize and take $f_1 = f_2 = \delta_e$ -- then

$$\operatorname{tr}(\hat{\delta}_e(\Pi)) = \chi_{\Pi}(e) = d_{\Pi}$$

and

$$O(\delta_e, \gamma_i) = 0 \quad (\gamma_i \neq e)$$

while

$$O(\delta_e, e) = \sum_{x \in G} \delta_e(xex^{-1})$$

$$= \sum_{x \in G} \delta_e(e) = |G|.$$

Consequently,

$$\sum_{\Pi \in \hat{G}} d_\Pi^2 = \frac{1}{|G|} |G|^2 = |G| \quad \text{(cf. II, §3, #5 and II, §5, #9).]}$$

From the definitions,

$$\begin{cases} \operatorname{tr}(\hat{f}_1(\Pi)) = \sum_{x \in G} f_1(x) \chi_\Pi(x) = |G| \langle f_1, \chi_{\Pi^*} \rangle_G \\ \operatorname{tr}(\hat{f}_2(\Pi^*)) = \sum_{x \in G} f_2(x) \chi_{\Pi^*}(x) = |G| \langle f_2, \chi_\Pi \rangle_G. \end{cases}$$

Therefore

$$\sum_{\Pi \in \hat{G}} \operatorname{tr}(\hat{f}_1(\Pi)) \operatorname{tr}(\hat{f}_2(\Pi^*))$$

$$= |G|^2 \sum_{\Pi \in \hat{G}} \langle f_1, \chi_{\Pi^*} \rangle_G \langle f_2, \chi_\Pi \rangle_G$$

$$= |G|^2 \sum_{\Pi \in \hat{G}} \langle f_1, \chi_\Pi \rangle_G \langle f_2, \chi_{\Pi^*} \rangle_G.$$

<u>15:</u> <u>N.B.</u> Assume in addition that f_1 and f_2 are class functions. Write

$$\begin{cases} f_1(x) = \sum_{\Pi \in \hat{G}} \langle f_1, \chi_\Pi \rangle_G \chi_\Pi(x) \\ \overline{f_2(x)} = \sum_{\Pi \in \hat{G}} \langle \bar{f}_2, \chi_\Pi \rangle_G \chi_\Pi(x) \end{cases} \quad \text{(cf. II, §4, #17).}$$

Then

$$\langle f_1, \bar{f}_2 \rangle_G = \sum_{\Pi \in \hat{G}} \langle f_1, \chi_\Pi \rangle_G \overline{\langle \bar{f}_2, \chi_\Pi \rangle_G}$$

(first orthogonality relations)

$$= \sum_{\Pi \in \hat{G}} \langle f_1, \chi_\Pi \rangle_G \langle f_2, \chi_{\Pi^*} \rangle_G .$$

On the other hand,

$$\langle f_1, \overline{f_2} \rangle_G = \frac{1}{|G|} \sum_{x \in G} f_1(x) \overline{\overline{f_2(x)}}$$

$$= \frac{1}{|G|} \sum_{i=1}^{n} \sum_{x \in C_i} f_1(x) f_2(x)$$

$$= \frac{1}{|G|} \sum_{i=1}^{n} |C_i| f_1(\gamma_i) f_2(\gamma_i)$$

$$= \sum_{i=1}^{n} \frac{|C_i|}{|G|} f_1(\gamma_i) f_2(\gamma_i)$$

$$= \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} f_1(\gamma_i) f_2(\gamma_i)$$

=>

$$|G|^2 \langle f_1, \overline{f_2} \rangle_G$$

$$= \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} |G| f_1(\gamma_i) \, |G| f_2(\gamma_i)$$

$$= \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} \mathcal{O}(f_1, \gamma_i) \mathcal{O}(f_2, \gamma_i) .$$

The irreducible representations of $G \times G$ are the outer tensor products

$$\Pi_1 \underline{\otimes} \Pi_2 \quad (\Pi_1, \Pi_2 \in \hat{G}) \quad \text{(cf. II, §5, #13)}.$$

Moreover,

$$\chi_{\Pi_1 \underline{\otimes} \Pi_2} = \chi_{\Pi_1} \chi_{\Pi_2}.$$

Consider now the direct sum decomposition

$$L_{G \times G/G} = \bigoplus_{\Pi_1, \Pi_2 \in \hat{G}} m(\Pi_1 \underline{\otimes} \Pi_2, L_{G \times G/G}) \Pi_1 \underline{\otimes} \Pi_2.$$

Then

$$\mathrm{tr}(L_{G \times G/G}(f))$$

$$= \sum_{\Pi_1, \Pi_2 \in \hat{G}} m(\Pi_1 \underline{\otimes} \Pi_2, L_{G \times G/G}) \mathrm{tr}(\Pi_1(f_1)) \mathrm{tr}(\Pi_2(f_2)).$$

I.e. (cf. #4):

$$\mathrm{tr}(\Pi_{L,R}(f))$$

$$= \sum_{\Pi_1, \Pi_2 \in \hat{G}} m(\Pi_1 \underline{\otimes} \Pi_2, L_{G \times G/G}) \mathrm{tr}(\Pi_1(f_1)) \mathrm{tr}(\Pi_2(f_2)).$$

I.e. (cf. #12):

$$\sum_{\Pi \in \hat{G}} \mathrm{tr}(\Pi(f_1)) \mathrm{tr}(\Pi^*(f_2))$$

$$= \sum_{\Pi_1, \Pi_2 \in \hat{G}} m(\Pi_1 \underline{\otimes} \Pi_2, L_{G \times G/G}) \mathrm{tr}(\Pi_1(f_1)) \mathrm{tr}(\Pi_2(f_2)).$$

Therefore, thanks to I, §3, #9,

$$m(\Pi_1 \underline{\otimes} \Pi_2, L_{G \times G/G})$$

must vanish unless $\Pi_1 = \Pi$, $\Pi_2 = \Pi^*$, in which case the coefficient is equal to 1.

<u>16:</u> SCHOLIUM

$$\Pi_{L,R} \approx \bigoplus_{\Pi \in \hat{G}} \Pi \underline{\otimes} \Pi^*.$$

§3. *THE GLOBAL PRE-TRACE FORMULA*

Let G be a finite group, $\Gamma \subset G$ a subgroup.

<u>1:</u> NOTATION Set

$$L_{G/\Gamma} = \operatorname{Ind}_{\Gamma,\theta}^{G},$$

where θ is the trivial representation of Γ on $E = \mathbf{C}$.

[Note: Accordingly, $\chi_\theta = 1_\Gamma$ and $E_{\Gamma,\theta}^{G} = C(G/\Gamma)$.]

<u>2:</u> EXAMPLE In the special case when $\Gamma = \{e\}$, $L_{G/\Gamma} = L$, the left translation representation of G on C(G) (cf. II, §1, #12).

<u>3:</u> <u>N.B.</u> The pair $(G \times G, G)$ figuring in §2 is an instance of the overall setup.

Given $f \in C(G)$, $\phi \in C(G/\Gamma)$, we have

$$
\begin{aligned}
&(L_{G/\Gamma}(f)\phi)(x) \\
&\quad = \sum_{y \in G} f(y)\,(L_{G/\Gamma}(y)\phi)(x) \\
&\quad = \sum_{y \in G} f(y)\,\phi(y^{-1}x) \\
&\quad = \sum_{y \in G} f(xy^{-1})\,\phi(y) \\
&\quad = \sum_{y \in G} f(xy^{-1})\,\frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} \phi(y\gamma) \\
&\quad = \sum_{y \in G} \frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} f(xy^{-1})\,\phi(y\gamma)
\end{aligned}
$$

$$= \sum_{\gamma \in \Gamma} \frac{1}{|\Gamma|} \sum_{y \in G} f(xy^{-1}) \phi(y\gamma)$$

$$= \sum_{\gamma \in \Gamma} \frac{1}{|\Gamma|} \sum_{y \in G} f(x\gamma y^{-1}) \phi(y)$$

$$= \sum_{y \in G} \left(\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma y^{-1})\right) \phi(y)$$

$$= \sum_{y \in G} K_f(x,y) \phi(y),$$

where

$$K_f(x,y) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma y^{-1}).$$

To summarize:

<u>4:</u> LEMMA $\forall\ f \in C(G)$, $\forall\ \phi \in C(G/\Gamma)$, $\forall\ x \in G$,

$$(L_{G/\Gamma}(f)\phi)(x) = \sum_{y \in G} K_f(x,y)\phi(y),$$

where

$$K_f(x,y) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma y^{-1}).$$

Write

$$G = \coprod_{k=1}^{n} x_k \Gamma.$$

Then for any $f \in C(G)$,

$$\sum_{x \in G} f(x) = \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} f(x_k \gamma),$$

thus for any $\phi \in C(G/\Gamma)$,

$$\sum_{x \in G} \phi(x) = |\Gamma| \sum_{k=1}^{n} \phi(x_k).$$

5: RAPPEL (cf. II, §9, #2) The Hilbert space structure on $C(G/\Gamma)$ is defined by the inner product

$$<\phi,\psi>_\theta = \frac{1}{|G|} \sum_{x\in G} \phi(x)\overline{\psi(x)}$$

$$= \frac{|\Gamma|}{|G|} \sum_{k=1}^{n} \phi(x_k)\overline{\psi(x_k)}.$$

6: NOTATION Define functions $\delta_k \in C(G/\Gamma)$ by the rule

$$\delta_k(x_\ell\gamma) = \delta_{k\ell} \quad (1 \le k, \ell \le n).$$

7: LEMMA The

$$\Delta_k = \left(\frac{|G|}{|\Gamma|}\right)^{1/2}\delta_k$$

constitute an orthonormal basis for $C(G/\Gamma)$.

PROOF A given $\phi \in C(G/\Gamma)$ admits the decomposition

$$\phi = \sum_{k=1}^{n} \phi(x_k)\delta_k.$$

In addition,

$$<\Delta_k,\Delta_\ell>_\theta = \frac{|\Gamma|}{|G|} \sum_{j=1}^{n} \Delta_k(x_j)\Delta_\ell(x_j)$$

$$= \frac{|\Gamma|}{|G|}\,\frac{|G|}{|\Gamma|} = 1$$

if $k = \ell$ and is 0 otherwise.

8: LEMMA $\forall\, f \in C(G)$,

$$\operatorname{tr}(L_{G/\Gamma}(f)) = \sum_{x\in G} K_f(x,x).$$

PROOF In fact,

$$\mathrm{tr}(L_{G/\Gamma}(f)) = \sum_{k=1}^{n} \langle L_{G/\Gamma}(f)\Delta_k, \Delta_k\rangle_\theta$$

$$= \sum_{k=1}^{n} \frac{|\Gamma|}{|G|}\frac{|G|}{|\Gamma|} \sum_{\ell=1}^{n} (L_{G/\Gamma}(f)\delta_k)(x_\ell)\delta_k(x_\ell)$$

$$= \sum_{k=1}^{n} (L_{G/\Gamma}(f)\delta_k)(x_k)$$

$$= \sum_{k=1}^{n} \sum_{y\in G} f(x_k y^{-1})\delta_k(y)$$

$$= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{\gamma\in\Gamma} f(x_k\gamma^{-1}x_\ell^{-1})\delta_k(x_\ell\gamma)$$

$$= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{\gamma\in\Gamma} f(x_k\gamma^{-1}x_\ell^{-1})\delta_{k\ell}$$

$$= \sum_{k=1}^{n} \sum_{\gamma\in\Gamma} f(x_k\gamma^{-1}x_k^{-1})$$

$$= \sum_{k=1}^{n} \sum_{\gamma\in\Gamma} f(x_k\gamma x_k^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{k=1}^{n} |\Gamma| f(x_k\gamma x_k^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{k=1}^{n} \sum_{\eta\in\Gamma} \sum_{\gamma\in\Gamma} f(x_k\eta\gamma\eta^{-1}x_k^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{x\in G} \sum_{\gamma\in\Gamma} f(x\gamma x^{-1})$$

$$= \sum_{x\in G} \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} f(x\gamma x^{-1})$$

$$= \sum_{x\in G} K_f(x,x).$$

9: EXAMPLE Take $\Gamma = G$ -- then $\forall\ f \in C(G)$,

$$\mathrm{tr}(L_{G/G}(f)) = \sum_{x\in G} \frac{1}{|G|} \sum_{y\in G} f(xyx^{-1})$$

$$= \frac{1}{|G|} \sum_{x\in G} \sum_{y\in G} f(xyx^{-1})$$

$$= \frac{1}{|G|} \sum_{x\in G} \sum_{y\in G} f(y)$$

$$= \frac{|G|}{|G|} \sum_{y\in G} f(y) = \sum_{x\in G} f(x).$$

10: EXAMPLE Fix $C \in \mathrm{CON}(G)$ and $x \in C$ -- then

$$|C|\chi_{L_{G/\Gamma}}(x) = \frac{|G|}{|\Gamma|} |C \cap \Gamma| \quad \text{(cf. II, §9, #6).}$$

[Work with $f = \chi_C$, thus

$$\mathrm{tr}(L_{G/\Gamma}(\chi_C)) = \sum_{y\in G} \chi_C(y)\chi_{L_{G/\Gamma}}(y)$$

$$= \sum_{y\in C} \chi_C(y)\chi_{L_{G/\Gamma}}(y)$$

$$= |C|\chi_{L_{G/\Gamma}}(x).$$

Meanwhile

$$\mathrm{tr}(L_{G/\Gamma}(\chi_C)) = \sum_{y\in G} K_{\chi_C}(y,y)$$

$$= \sum_{y \in G} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_C(y\gamma y^{-1})$$

$$= \sum_{y \in G} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_C(\gamma)$$

$$= \frac{|G|}{|\Gamma|} |C \cap \Gamma|.$$

On general grounds, there is a direct sum decomposition

$$L_{G/\Gamma} = \bigoplus_{\Pi \in \hat{G}} m(\Pi, L_{G/\Gamma})\Pi.$$

[Note:

$$m(\Pi, L_{G/\Gamma}) \neq 0$$

iff the restriction of Π to Γ contains the trivial representation θ of Γ on $E = \mathbf{C}$ (cf. II, §9, #9) (but see below (cf. #14)).]

<u>11:</u> SCHOLIUM $\forall f \in C(G)$,

$$\mathrm{tr}(L_{G/\Gamma}(f)) = \sum_{\Pi \in \hat{G}} m(\Pi, L_{G/\Gamma}) \mathrm{tr}(\hat{f}(\Pi)).$$

[Note: Explicated,

$$\mathrm{tr}(\hat{f}(\Pi)) = \sum_{x \in G} f(x)\chi_\Pi(x) = \mathrm{tr}(\Pi(f)).]$$

<u>12:</u> DEFINITION Given $f \in C(G)$, the <u>global pre-trace formula</u> is the relation

$$\sum_{\Pi \in \hat{G}} m(\Pi, L_{G/\Gamma}) \mathrm{tr}(\hat{f}(\Pi))$$

$$= \sum_{x \in G} K_f(x,x) = \sum_{x \in G} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma x^{-1}).$$

<u>13:</u> APPLICATION Take $\Gamma = \{e\}$ -- then

$$\sum_{\Pi\in\hat{G}} m(\Pi, L_{G/\Gamma}) \operatorname{tr}(\hat{f}(\Pi))$$

becomes

$$\sum_{\Pi\in\hat{G}} d_\Pi \operatorname{tr}(\hat{f}(\Pi))$$

while

$$\sum_{x\in G} \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} f(x\gamma x^{-1})$$

becomes

$$|G| f(e).$$

I.e.:

$$f(e) = \frac{1}{|G|} \sum_{\Pi\in\hat{G}} d_\Pi \operatorname{tr}(\hat{f}(\Pi)),$$

the so-called "Plancherel theorem" for G.

<u>14:</u> APPLICATION Fix $\Pi_0 \in \hat{G}$ and take $f = \overline{\chi_{\Pi_0}}$.

- $\Pi \neq \Pi_0$

$$\Rightarrow \operatorname{tr}(\Pi(\overline{\chi_{\Pi_0}})) = \sum_{x\in G} \overline{\chi_{\Pi_0}(x)}\, \chi_\Pi(x)$$

$$= 0.$$

- $\Pi = \Pi_0$

$$\Rightarrow \operatorname{tr}(\Pi_0(\overline{\chi_{\Pi_0}})) = \sum_{x\in G} \overline{\chi_{\Pi_0}(x)}\, \chi_{\Pi_0}(x)$$

$$= |G|.$$

Therefore

$$\sum_{\Pi\in\hat{G}} m(\Pi, L_{G/\Gamma}) \operatorname{tr}(\Pi(\overline{\chi_{\Pi_0}}))$$

reduces to

$$|G|m(\Pi_0,L_{G/\Gamma}).$$

On the other hand,

$$\sum_{x\in G} K_f(x,x) = \sum_{x\in G} \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \overline{\chi_{\Pi_0}(x\gamma x^{-1})}$$

$$= \sum_{x\in G} \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \overline{\chi_{\Pi_0}(\gamma)}$$

$$= \frac{|G|}{|\Gamma|} \sum_{\gamma\in\Gamma} \overline{\chi_{\Pi_0}(\gamma)}$$

$$= |G|\langle 1_\Gamma,\chi_{\Pi_0}|\Gamma\rangle_\Gamma$$

$$= |G|m(\theta,\Pi_0|\Gamma).$$

So

$$|G|m(\Pi_0,L_{G/\Gamma}) = |G|m(\theta,\Pi_0|\Gamma)$$

$\Rightarrow$

$$m(\Pi_0,L_{G/\Gamma}) = m(\theta,\Pi_0|\Gamma).$$

[Note: As above, θ is the trivial representation of Γ on $E = \mathbf{C}$.]

<u>15:</u> <u>N.B.</u> Take $\Gamma = \{e\}$ -- then

$$m(\theta,\Pi_0|\Gamma) = d_{\Pi_0},$$

hence

$$m(\Pi_0,L_{G/\Gamma}) = d_{\Pi_0} \quad \text{(cf. II, §5, \#8)}.$$

§4. *THE GLOBAL TRACE FORMULA*

Let G be a finite group, $\Gamma \subset G$ a subgroup.

1: NOTATION For any $\gamma \in \Gamma$,

$$\left[\begin{array}{l} G_\gamma = \text{centralizer of } \gamma \text{ in } G \\ \Gamma_\gamma = \text{centralizer of } \gamma \text{ in } \Gamma. \end{array}\right.$$

Given an $f \in C(G)$, we have

$$\operatorname{tr}(L_{G/\Gamma}(f)) = \sum_{x \in G} K_f(x,x),$$

where

$$K_f(x,x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma x^{-1}) \qquad (\text{cf. §3, \#8}).$$

Enumerate the elements of $\operatorname{CON}(\Gamma)$, say

$$\operatorname{CON}(\Gamma) = \{C_1, \ldots, C_n\}.$$

For each i, fix a $\gamma_i \in C_i$ $(1 \leq i \leq n)$.

2: LEMMA $\forall f \in C(G)$,

$$\sum_{x \in G} K_f(x,x) = \sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} O(f,\gamma_i).$$

PROOF Write

$$\Gamma = \coprod_k \gamma_{i,k} \Gamma_{\gamma_i}.$$

Then $\forall x \in G$,

$$K_f(x,x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma x^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{i=1}^{n} \sum_{\gamma \in C_i} f(x\gamma x^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{i=1}^{n} \sum_{k} f(x\gamma_{i,k}\gamma_i\gamma_{i,k}^{-1}x^{-1}).$$

Therefore

$$\sum_{x \in G} K_f(x,x) = \frac{1}{|\Gamma|} \sum_{i=1}^{n} \sum_{k} \sum_{x \in G} f(x\gamma_{i,k}\gamma_i\gamma_{i,k}^{-1}x^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{i=1}^{n} [\Gamma:\Gamma_{\gamma_i}] \sum_{x \in G} f(x\gamma_i x^{-1}).$$

Write

$$G = \coprod_{k} x_{i,k}\Gamma_{\gamma_i}.$$

Then

$$\sum_{x \in G} K_f(x,x)$$

$$= \frac{1}{|\Gamma|} \sum_{i=1}^{n} [\Gamma:\Gamma_{\gamma_i}] \sum_{k} \sum_{\eta_i \in \Gamma_{\gamma_i}} f(x_{i,k}\eta_i\gamma_i\eta_i^{-1}x_{i,k}^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{i=1}^{n} [\Gamma:\Gamma_{\gamma_i}][\Gamma_{\gamma_i}] \sum_{k} f(x_{i,k}\gamma_i x_{i,k}^{-1})$$

$$= \sum_{i=1}^{n} \frac{[\Gamma:\Gamma_{\gamma_i}][\Gamma_{\gamma_i}]}{|\Gamma|} \sum_{k} f(x_{i,k}\gamma_i x_{i,k}^{-1})$$

$$= \sum_{i=1}^{n} \sum_{k} f(x_{i,k}\gamma_i x_{i,k}^{-1}).$$

Write

$$\left[\begin{array}{l} G = \coprod_{\ell} y_{i,\ell} G_{\gamma_i} \\ G_{\gamma_i} = \coprod_{m} z_{i,m} \Gamma_{\gamma_i} \end{array}\right.$$

=>

$$G = \coprod_{\ell} \coprod_{m} y_{i,\ell} z_{i,m} \Gamma_{\gamma_i}.$$

Then

$$\sum_{x \in G} K_f(x,x)$$

$$= \sum_{i=1}^{n} \sum_{\ell} \sum_{m} f(y_{i,\ell} z_{i,m} \gamma_i z_{i,m}^{-1} y_{i,\ell}^{-1})$$

$$= \sum_{i=1}^{n} [G_{\gamma_i} : \Gamma_{\gamma_i}] \sum_{\ell} f(y_{i,\ell} \gamma_i y_{i,\ell}^{-1})$$

$$= \sum_{i=1}^{n} \frac{|G_{\gamma_i}|}{|\Gamma_{\gamma_i}|} \sum_{\ell} f(y_{i,\ell} \gamma_i y_{i,\ell}^{-1})$$

$$= \sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} \mathcal{O}(f, \gamma_i).$$

<u>3:</u> <u>N.B.</u> $\forall\ \gamma$,

$$\mathcal{O}(f,\gamma) = |G_{\gamma}| \sum_{x \in G/G_{\gamma}} f(x\gamma x^{-1}),$$

the sum on the right being taken over a set of representatives for the left cosets of G_{γ} in G.

4: EXAMPLE Take $\Gamma = G$ -- then $\forall f \in C(G)$,

$$\sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} O(f,\gamma_i)$$

$$= \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} \sum_{x\in G} f(x\gamma_i x^{-1})$$

$$= \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} |G_{\gamma_i}| \sum_{y_i\in G/G_{\gamma_i}} f(y_i\gamma_i y_i^{-1})$$

$$= \sum_{i=1}^{n} \sum_{y_i\in G/G_{\gamma_i}} f(y_i\gamma_i y_i^{-1})$$

$$= \sum_{i=1}^{n} \sum_{y\in C_i} f(y)$$

$$= \sum_{x\in G} f(x) \quad \text{(cf. §3, \#9).}$$

5: EXAMPLE Suppose that $f \in CL(G)$ -- then

$$\sum_{x\in G} K_f(x,x) = \sum_{x\in G} \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} f(x\gamma x^{-1})$$

$$= \sum_{x\in G} \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} f(\gamma)$$

$$= \frac{|G|}{|\Gamma|} \sum_{\gamma\in\Gamma} f(\gamma).$$

In the other direction,

$$\sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} O(f,\gamma_i) = \sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} \sum_{x\in G} f(x\gamma_i x^{-1})$$

$$= |G| \sum_{i=1}^{n} \frac{f(\gamma_i)}{|\Gamma_{\gamma_i}|}.$$

Therefore

$$\frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} f(\gamma) = \sum_{i=1}^{n} \frac{f(\gamma_i)}{|\Gamma_{\gamma_i}|}.$$

6: DEFINITION Given $f \in C(G)$, the global trace formula is the relation

$$\sum_{\Pi\in\hat{G}} m(\Pi, L_{G/\Gamma}) \operatorname{tr}(\hat{f}(\Pi))$$

$$= \sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} O(f,\gamma_i) \qquad \text{(cf. §3, \#12)}.$$

7: EXAMPLE (POISSON SUMMATION) Take G abelian and identify $\hat{G}$ with the character group of G: $\Pi \longleftrightarrow \chi$, hence

$$\hat{f}(\chi) = \sum_{x\in G} f(x)\chi(x).$$

Consider now the sum

$$\sum_{\chi\in\hat{G}} m(\chi, L_{G/\Gamma})\hat{f}(\chi).$$

Let $\Gamma^{\perp} = \{\chi \in \hat{G} : \chi(\gamma) = 1 \ \forall \ \gamma \in \Gamma\}$ -- then

$$\left[\begin{array}{l} \chi \in \Gamma^{\perp} \Rightarrow m(\chi, L_{G/\Gamma}) = 1 \\ \\ \chi \notin \Gamma^{\perp} \Rightarrow m(\chi, L_{G/\Gamma}) = 0 \end{array}\right. \qquad \text{(cf. §3, \#14)}.$$

Therefore matters on the "spectral side" reduce to

$$\sum_{\chi\in\Gamma^{\perp}} \hat{f}(\chi).$$

And on the "geometric side",

$$\sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} O(f,\gamma_i) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} O(f,\gamma)$$

$$= \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \sum_{x\in G} f(x\gamma x^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} |G| f(\gamma)$$

$$= \frac{|G|}{|\Gamma|} \sum_{\gamma\in\Gamma} f(\gamma).$$

Therefore

$$\frac{1}{|G|} \sum_{\chi\in\Gamma^{\perp}} \hat{f}(\chi) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} f(\gamma).$$

Each element ζ in the center $Z(\Gamma)$ of Γ determines a one-element conjugacy class $\{\zeta\}$.

8: DEFINITION The central contribution to the global trace formula is the subsum

$$\sum_{\zeta\in Z(\Gamma)} \frac{1}{|\Gamma_\zeta|} O(f,\zeta)$$

of

$$\sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} O(f,\gamma_i).$$

Accordingly,

$$\sum_{\zeta\in Z(\Gamma)} \frac{1}{|\Gamma_\zeta|} O(f,\zeta) = \frac{1}{|\Gamma|} \sum_{\zeta\in Z(\Gamma)} O(f,\zeta)$$

$$= \frac{1}{|\Gamma|} \sum_{\zeta \in Z(\Gamma)} \sum_{x \in G} f(x\zeta x^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{\zeta \in Z(\Gamma)} |G| f(\zeta)$$

$$= \frac{|G|}{|\Gamma|} \sum_{\zeta \in Z(\Gamma)} f(\zeta).$$

§1. *UNITARY REPRESENTATIONS*

Let G be a compact group.

<u>1:</u> NOTATION d_G is normalized Haar measure on G:

$$\int_G 1 d_G(x) = 1.$$

<u>2:</u> LEMMA

$$L^1(G) \supset L^2(G) \supset C(G)$$

and

- $\forall f \in L^2(G), \ ||f||_2 \geq ||f||_1.$
- $\forall f \in C(G), \ ||f||_2 \leq ||f||_\infty.$

<u>3:</u> <u>N.B.</u> The convolution operator

$$*:L^2(G) \times L^2(G) \to C(G)$$

is given by

$$(f * g)(x) = \int_G f(xy^{-1})g(y)d_G(y)$$

$$= \int_G f(y)g(y^{-1}x)d_G(y).$$

<u>4:</u> DEFINITION A <u>unitary representation</u> of G on a Hilbert space H is a homomorphism $\pi:G \to UN(H)$ from G to the unitary group $UN(H)$ of H such that $\forall a \in H$, the map

$$x \to \pi(x)a$$

of G into H is continuous.

5: DEFINITION

- The left translation representation of G on $L^2(G)$ is the prescription

$$L(x)f(y) = f(x^{-1}y).$$

- The right translation representation of G on $L^2(G)$ is the prescription

$$R(x)f(y) = f(yx).$$

[Note: Both L and R are unitary.]

6: N.B. There is also a unitary representation $\pi_{L,R}$ of $G \times G$ on $L^2(G)$, namely

$$(\pi_{L,R}(x_1,x_2)f)(x) = f(x_1^{-1}xx_2).$$

7: DEFINITION A unitary representation π of G on a Hilbert space $\mathcal{H} \neq \{0\}$ is irreducible if the only closed subspaces of $\mathcal{H}$ which are invariant under π are $\{0\}$ and $\mathcal{H}$.

8: THEOREM Let π be a unitary representation of G -- then π is the Hilbert space direct sum of finite dimensional irreducible unitary representations.

9: APPLICATION Every irreducible unitary representation of G is finite dimensional.

10: NOTATION $\hat{G}$ is the set of unitary equivalence classes of irreducible unitary representations of G.

[Note: Generically, $\Pi \in \hat{G}$ with representation space $V(\Pi)$ and $d_\Pi = \dim V(\Pi)$ is its dimension.]

11: N.B. Let π be a unitary representation of G -- then there exist cardinal numbers $n_\Pi (\Pi \in \hat{G})$ such that

$$\pi = \hat{\bigoplus_{\Pi \in \hat{G}}} n_\Pi \Pi.$$

12: EXAMPLE Take $\pi = L$ -- then

$$L = \hat{\bigoplus_{\Pi \in \hat{G}}} d_\Pi \Pi.$$

[Note: There is also an analog of A, III, §2, #16.]

13: THEOREM $\forall\, x \in G\ (x \neq e)$, $\exists$ an irreducible unitary representation Π such that $\Pi(x) \neq \text{id}$ (Gelfand-Raikov).

14: APPLICATION

$$\bigcap_{\Pi \in \hat{G}} \text{Ker } \Pi = \{e\}.$$

15: LEMMA Given $\Pi \in \hat{G}$, suppose that $A \in \text{Hom}(V(\Pi), V(\Pi))$ has the property that $\forall\, x \in G$,

$$A\Pi(x) = \Pi(x)A.$$

Then A is a scalar multiple of the identity (Schur), call it λ_A.

§2. *EXPANSION THEORY*

Let G be a compact group.

1: DEFINITION Let π be a finite dimensional unitary representation of G -- then its character is the function

$$\chi_\pi : G \to C$$

given by the prescription

$$\chi_\pi(x) = \mathrm{tr}(\pi(x)) \quad (x \in G).$$

2: DEFINITION The character of an irreducible unitary representation is called an irreducible character.

3: LEMMA Let $\Pi_1, \Pi_2 \in \hat{G}$ and suppose that $\Pi_1 \neq \Pi_2$ -- then

$$\langle \chi_{\Pi_1}, \chi_{\Pi_2} \rangle = 0.$$

4: LEMMA Let $\Pi \in \hat{G}$ -- then

$$\langle \chi_\Pi, \chi_\Pi \rangle = 1.$$

5: DEFINITION A continuous complex valued function φ on G is of positive type if for all $x_1, \ldots, x_n \in G$ and $\lambda_1, \ldots, \lambda_n \in C$,

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} \varphi(x_i^{-1} x_j) \geq 0.$$

6: N.B. The sum of two functions of positive type is of positive type and a positive scalar multiple of a function of positive type is of positive type.

7: LEMMA If $\pi:G \to UN(H)$ is a unitary representation and if $a \in H$, then

$$\varphi(x) = \langle\pi(x)a,a\rangle \quad (x \in G)$$

is of positive type.

[Note:

$$||\varphi||_\infty = \langle a,a\rangle.]$$

8: EXAMPLE $\forall\ \Pi \in \hat{G}$, χ_Π is of positive type.

[Fix an orthonormal basis $v_1,\ldots,v_n$ in $V(\Pi)$ -- then

$$\chi_\Pi(x) = \langle\Pi(x)v_1,v_1\rangle + \cdots + \langle\Pi(x)v_n,v_n\rangle,$$

from which the assertion.]

9: NOTATION Given $\Pi \in \hat{G}$ and $f \in L^2(G)$, put

$$\Pi(f) = \int_G f(x)\Pi(x)d_G(x).$$

10: LEMMA $\forall\ f_1,f_2 \in L^2(G)$,

$$\Pi(f_1 * f_2) = \Pi(f_1) \circ \Pi(f_2).$$

11: NOTATION Given $f \in L^2(G)$, define $f^* \in L^2(G)$ by

$$f^*(x) = \overline{f(x^{-1})} \quad (= \overline{\check{f}(x)}).$$

12: LEMMA $\forall\ f \in L^2(G)$, $\forall\ v_1,v_2 \in V(\Pi)$,

$$\langle\Pi(f)v_1,v_2\rangle = \langle v_1,\Pi(f^*)v_2\rangle,$$

i.e.,

$$\Pi(f)^* = \Pi(f^*).$$

PROOF

$$
\begin{aligned}
<\Pi(f)v_1,v_2> &= \int_G f(x)<\Pi(x)v_1,v_2>d_G(x) \\
&= \int_G <v_1,\overline{f(x)}\,\Pi(x^{-1})v_2>d_G(x) \\
&= \int_G <v_1,\overline{f(x^{-1})}\,\Pi(x)v_2>d_G(x) \\
&= \int_G <v_1,f^*(x)\Pi(x)v_2>d_G(x) \\
&= <v_1,\int_G f^*(x)\Pi(x)v_2d_G(x)> \\
&= <v_1,\Pi(f^*)v_2>.
\end{aligned}
$$

<u>13:</u> THEOREM Let $f \in L^2(G)$ -- then

$$\int_G |f(x)|^2 d_G(x) = \sum_{\Pi\in\hat{G}} d_\Pi \mathrm{tr}(\Pi(f)\Pi(f)^*).$$

<u>14:</u> THEOREM Let $f \in L^2(G)$ -- then

$$f = \sum_{\Pi\in\hat{G}} d_\Pi (f * \chi_\Pi),$$

the series converging in $L^2(G)$.

<u>15:</u> THEOREM Let

$$f \in \mathrm{span}_C(L^2(G) * L^2(G)) \subset C(G).$$

Then

$$f(e) = \sum_{\Pi\in\hat{G}} d_\Pi \mathrm{tr}(\Pi(f)).$$

PROOF Put $f = f_1 * f_2$:

$$f(e) = \int_G f_1(x^{-1}) f_2(x) d_G(x)$$

$$= \int_G \overline{\overline{f_1(x^{-1})}} f_2(x) d_G(x)$$

$$= \int_G f_2(x) \overline{\overline{f_1(x^{-1})}} d_G(x)$$

$$= \int_G f_2(x) \overline{f_1^*(x)} d_G(x)$$

$$= \langle f_2, f_1^* \rangle$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \operatorname{tr}(\Pi(f_1 * f_2))$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \operatorname{tr}(\Pi(f)).$$

[Note: This is the so-called "Plancherel theorem" for G (cf. A, III, §3, #13).

<u>16:</u> <u>N.B.</u> The foregoing may fail if f is only assumed to be continuous (e.g., take $G = S^1 \ldots$).

<u>17:</u> DEFINITION A function $f \in L^2(G)$ is said to be an <u>L^2 class function</u> if

$$f(x) = f(yxy^{-1})$$

for almost all x and all y.

18: $\forall\ \Pi \in \hat{G}$, χ_Π is an L^2 class function.

19: THEOREM Suppose that $f \in L^2(G)$ is an L^2 class function -- then

$$f = \sum_{\Pi \in \hat{G}} \langle f, \chi_\Pi \rangle \chi_\Pi,$$

the series converging in $L^2(G)$, and

$$||f||^2 = \sum_{\Pi \in \hat{G}} |\langle f, \chi_\Pi \rangle|^2.$$

20: SCHOLIUM The $\{\chi_\Pi : \Pi \in \hat{G}\}$ constitute an orthonormal basis for the set of L^2 class functions.

21: NOTATION Write $C(G)_{fin}(L)$ for the set of G-finite functions in $C(G)$ per L:

$$f \in C(G)_{fin}(L) \iff \dim\{L(x)f : x \in G\}_{\ell in} < \infty.$$

22: NOTATION Write $C(G)_{fin}(R)$ for the set of G-finite functions in $C(G)$ per R:

$$f \in C(G)_{fin}(R) \iff \dim\{R(x)f : x \in G\}_{\ell in} < \infty.$$

23: LEMMA

$$C(G)_{fin}(L) = C(G)_{fin}(R).$$

24: NOTATION Write $C(G)_{fin}$ unambiguously for the G-finite functions per either action.

Recalling §1, #6, $\pi_{L,R}$ operates on $C(G)_{fin}$ and it turns out that

$$C(G)_{fin} \approx \bigoplus_{\Pi \in \hat{G}} V(\Pi^*) \otimes V(\Pi).$$

Here the identification sends an element

$$v^* \otimes v \in V(\Pi^*) \otimes V(\Pi)$$

to

$$f_{v^* \otimes v} \in C(G)_{fin},$$

where

$$f_{v^* \otimes v}(x) = v^*(\Pi(x^{-1})v).$$

[Note:

$$L^2(G) \approx \hat{\bigoplus_{\Pi \in \hat{G}}} V(\Pi^*) \otimes V(\Pi).]$$

<u>25:</u> THEOREM $C(G)_{fin}$ is dense in $C(G)$.

<u>26:</u> THEOREM $C(G)_{fin}$ is dense in $L^2(G)$.

<u>27:</u> DEFINITION A function $f \in C(G)$ is said to be a <u>continuous class function</u> if $f(x) = f(yxy^{-1})$ for all $x,y \in G$ (written $f \in CL(G)$).

<u>28:</u> EXAMPLE $\forall\ \Pi \in \hat{G}$, χ_Π is a continuous class function: $\chi_\Pi \in CL(G)$.

<u>29:</u> THEOREM The span of the χ_Π $(\Pi \in \hat{G})$ equals the set of continuous class functions in $C(G)_{fin}$.

<u>30:</u> THEOREM The span of the χ_Π $(\Pi \in \hat{G})$ is dense in the set of continuous class functions.

§3. *STRUCTURE THEORY*

Let G be a compact group.

1: NOTATION $G^0 \subset G$ is the connected component of the identity of G.

2: LEMMA G^0 is a closed normal subgroup of G.

3: LEMMA The quotient G/G^0 is compact and totally disconnected.

4: DEFINITION A topological group possessing a neighborhood of the identity which does not contain a nontrivial subgroup is said to be a group with no small subgroups.

5: RAPPEL A Lie group has no small subgroups.

6: THEOREM The following conditions on a compact group G are equivalent.

- G is a Lie group.
- G has no small subgroups.
- G has a faithful finite dimensional representation.

7: REMARK Every compact group is the projective limit of compact Lie groups.

Let G be a compact Lie group.

8: N.B. Every finite group (discrete topology) is a compact Lie group.

9: EXAMPLE The product $\prod_{n=1}^{\infty} SU(n)$ is a compact group but it is not a Lie group.

10: EXAMPLE The p-adic integers

$$Z_p = \lim_{\substack{\leftarrow \\ n \geq 1}} (Z/p^n Z)$$

are a compact group but they are not a Lie group.

11: DEFINITION A torus is a compact Lie group which is isomorphic to $R^n/Z^n \approx (R/Z)^n$ for some $n \geq 0$.

[Note: The nonnegative integer n is called the rank of the torus.]

12: THEOREM Every compact abelian Lie group is isomorphic to the product of a torus and a finite abelian group.

13: DEFINITION A compact Lie group is topologically cyclic if it contains an element whose powers are dense.

14: LEMMA Every torus T is topologically cyclic.

[Note: There are infinitely many topologically cyclic elements in T and their totality has full measure in any Haar measure on T.]

15: THEOREM A compact Lie group is topologically cyclic iff it is isomorphic to the product of a torus and a finite cyclic group.

Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra.

16: LEMMA G^0 is an open normal subgroup of G.

Therefore the compact quotient G/G^0 is discrete, hence is a finite group, the group of components of G.

17: NOTATION $Z(G)$ is the center of G, $Z(G)^0 \subset Z(G)$ is the connected component of the identity element of $Z(G)$.

18: N.B. In general, $Z(G)$ is not connected (consider $SU(3)$).

19: THEOREM Assume that G is connected -- then $Z(G)^0$ is a compact abelian Lie subgroup of G and its Lie algebra is the center of $\mathfrak{g}$, i.e., the ideal

$$\{X \in \mathfrak{g}: [X,Y] = 0 \ \forall \ Y \in \mathfrak{g}\}.$$

20: DEFINITION

- A Lie algebra is simple if it is noncommutative and has no proper nontrivial ideals.
- A Lie algebra is semisimple if it is noncommutative and has no proper nontrivial commutative ideals.
- A Lie algebra is reductive if it is the direct sum of an abelian Lie algebra and a semisimple Lie algebra.

21: N.B. A Lie group is simple, semisimple, or reductive if this is the case of its Lie algebra.

22: LEMMA A semisimple Lie algebra has a trivial center (it being a commutative ideal).

23: LEMMA A semisimple Lie algebra can be decomposed as a finite direct sum of simple ideals.

24: DEFINITION If G and H are Lie groups and if H is a subgroup of G, then H is a Lie subgroup of G if the arrow $H \rightarrow G$ of inclusion is continuous.

[Note: If G is a Lie group and if H is a closed subgroup of G, then H is a Lie group.]

25: N.B. A Lie subgroup of a compact Lie group needn't be compact nor carry the relative topology.

26: THEOREM Let G be a compact Lie group and let H be a semisimple connected Lie subgroup of G -- then as a subset of G, H is closed, and as a Lie subgroup of G, H carries the relative topology.

27: NOTATION

- $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$.
- $\mathfrak{g}_{ss}$ is the ideal in $\mathfrak{g}$ spanned by $[\mathfrak{g},\mathfrak{g}]$.

28: LEMMA $\mathfrak{g}_{ss}$ is a semisimple Lie algebra.

29: THEOREM Let G be a compact Lie group -- then

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{ss},$$

thus $\mathfrak{g}$ is reductive or still, G is reductive.

30: NOTATION G_{ss} is the analytic subgroup of G corresponding to $\mathfrak{g}_{ss}$.

31: NOTATION G* is the commutator subgroup of G, i.e., the subgroup of G generated by the

$$xyx^{-1}y^{-1} \quad (x,y \in G).$$

[Note: G* is necessarily normal.]

32: THEOREM Assume that G is connected -- then G* is a compact connected Lie subgroup of G with Lie algebra $\mathfrak{g}_{ss}$, so $G^* = G_{ss}$, hence is semisimple.

33: THEOREM Assume that G is connected -- then G is the commuting product $Z(G)^0 G_{ss}$.

34: THEOREM Assume that G is connected -- then

$$G \approx (Z(G)^0 \times G_{ss})/\Delta,$$

where

$$\Delta \approx Z(G)^0 \cap G_{ss}$$

is embedded in $Z(G)^0 \times G_{ss}$ via the arrow $z \to (z^{-1}, z)$.

[Note: Spelled out, there is an exact sequence

$$\{1\} \to Z(G)^0 \cap G_{ss} \xrightarrow{\iota} Z(G)^0 \times G_{ss} \xrightarrow{\mu} G \to \{1\},$$

where

$$\iota(z) = (z^{-1}, z), \quad \mu(z,x) = zx.]$$

35: N.B. Structurally, $Z(G)^0$ is a torus and

$$Z(G_{ss}) = Z(G) \cap G_{ss}$$

is a finite abelian group.

36: SCHOLIUM Assume that G is connected -- then G is semisimple iff Z(G) is finite.

[Note: Here is another way to put it: G is semisimple iff $G = G_{ss}$ or still, iff $G = G^*$. To see that connectedness is essential, consider the 8 element quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ -- then its commutator group is $\{\pm 1\}$.]

37: EXAMPLE The center of $G/Z(G)$ is trivial, so $G/Z(G)$ (which is connected) is semisimple.

There are simple ideals $\mathfrak{h}_i \subset \mathfrak{g}_{ss}$ such that

$$\mathfrak{g}_{ss} = \bigoplus_{i=1}^{r} \mathfrak{h}_i$$

with $[\mathfrak{h}_i, \mathfrak{h}_j] = 0$ for $i \neq j$ and such that the span of $[\mathfrak{h}_i, \mathfrak{h}_i] = \mathfrak{h}_i$.

Put $H_i = \exp \mathfrak{h}_i$.

38: LEMMA H_i is a compact connected normal Lie subgroup of G_{ss} and its Lie algebra is $\mathfrak{h}_i$ (hence H_i is simple).

39: LEMMA A proper compact normal Lie subgroup of H_i is necessarily discrete, finite, and central.

40: LEMMA There is a decomposition

$$G_{ss} = H_1 \cdots H_r,$$

where H_i and H_j commute $(i \neq j)$.

41: N.B. The differential of the arrow

$$H_1 \times \cdots \times H_r \to G_{ss}$$

defined by the rule

$$(x_1, \ldots, x_r) \to x_1 \cdots x_r$$

is the identity map, thus its kernel Δ is discrete and normal, thus finite and central as well, so

$$G_{ss} \approx (H_1 \times \cdots \times H_r)/\Delta.$$

APPENDIX

Let G be a compact connected Lie group.

DEFINITION G is tall if for each positive integer n, there are but finitely many elements of $\hat{G}$ of degree n.

THEOREM G is semisimple iff G is tall.

REMARK If G is not semisimple, then G possesses infinitely many nonisomorphic irreducible representations of degree 1.

§4. *MAXIMAL TORI*

Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra.

1: LEMMA Every connected abelian subgroup $A \subset G$ is contained in a maximal connected abelian subgroup $T \subset G$.

2: N.B. T is compact.

[In fact, $\bar{T}$ is connected and abelian.]

3: DEFINITION A maximal torus $T \subset G$ is a maximal connected abelian subgroup of G.

[Note: T is a torus... .]

4: THEOREM Assume that G is connected and let $T_1 \subset G$, $T_2 \subset G$ be maximal tori -- then $\exists\ x \in G$ such that $xT_1x^{-1} = T_2$.

5: THEOREM Assume that G is connected and let $T \subset G$ be a maximal torus -- then

$$G = \bigcup_{x \in G} xTx^{-1}.$$

6: APPLICATION The exponential map $\exp: \mathfrak{g} \to G$ is surjective.

[Every element of G belongs to a maximal torus and the exponential map of a torus is surjective.]

7: LEMMA Assume that G is connected and let $T \subset G$ be a maximal torus -- then the centralizer of T in G is T itself.

<u>8:</u> APPLICATION The center of G is contained in T, i.e., $Z(G) \subset T$.

[Note: More is true, viz.

$$Z(G) = \bigcap_{T},$$

the intersection being taken over all maximal tori in G.]

<u>9:</u> LEMMA Assume that G is connected and let $T \subset G$ be a maximal torus -- then T is a maximal abelian subgroup.

<u>10:</u> REMARK A maximal abelian subgroup need not be a maximal torus.

[In SO(3), there is a maximal abelian subgroup which is isomorphic to $(Z/2Z)^2$, hence is not a maximal torus.]

<u>11:</u> NOTATION Given a torus $T \subset G$, let N(T) be its normalizer in G.

<u>12:</u> LEMMA The quotient N(T)/T is finite iff T is a maximal torus.

Let G be a compact connected Lie group, $T \subset G$ a maximal torus.

<u>13:</u> DEFINITION The <u>Weyl group</u> of T in G is the quotient

$$W = N(T)/T.$$

<u>14:</u> <u>N.B.</u> Different choices of T give rise to isomorphic Weyl groups.

Fix a maximal torus $T \subset G$ -- then N(T) operates on T by conjugation:

$$\begin{cases} N(T) \times T \to T \\ \quad (n,t) \to nTn^{-1}. \end{cases}$$

Since T operates trivially on itself, there is an induced operation of the Weyl group:

$$W \times T \to T.$$

[Note: The action of W is on the left, thus the orbit space is denoted by $W\backslash T$.]

15: LEMMA The canonical homomorphism $W \to \text{Aut}\, T$ is injective.

16: LEMMA Two elements of T are conjugate in G iff they lie on the same orbit under the action of W.

17: RAPPEL Let G be a compact group and let X be a Hausdorff topological space on which G operates to the left -- then the action arrow

$$G \times X \to X$$

is a closed map. Equip the orbit space $G\backslash X$ with the quotient topology and let $\pi: X \to G\backslash X$ be the projection. Then:

- $G\backslash X$ is a Hausdorff space.
- X is compact iff $G\backslash X$ is compact.
- $\pi: X \to G\backslash X$ is open, closed, and proper.

18: EXAMPLE $W\backslash T$ is a compact Hausdorff space.

19: NOTATION CON(G) is the set of conjugacy classes of G.

Geometrically, CON(G) is the orbit space under the action of G on itself via inner automorphisms:

$$\begin{cases} G \times G \to G \\ (x,y) \to xyx^{-1}. \end{cases}$$

It carries the quotient topology per the projection $G \to CON(G)$ under which it is a compact Hausdorff space.

20: RAPPEL A one-to-one continuous map from a compact Hausdorff space X onto a Hausdorff space Y is a homeomorphism.

21: THEOREM The arrow

$$W\backslash T \to CON(G)$$

which sends the W-orbit Wt of $t \in T$ to the conjugacy class of $t \in T$ in G is a (well defined) homeomorphism.

[The map is injective (cf. #16), continuous (see below), and surjective (cf. #5), so #20 is applicable.]

[Note: To check the continuity of the arrow

$$W\backslash T \to CON(G),$$

bear in mind that $W\backslash T$ has the quotient topology, thus it suffices to check the continuity of the composition

$$T \to W\backslash T \to CON(G).$$

But this map is just the restriction to T of the arrow

$$G \to CON(G).]$$

22: NOTATION

- Given $f \in C(T)$ and $w \in W$, $w \cdot f$ is the function in $C(T)$ defined by the rule

$$(w \cdot f)(t) = f(n^{-1}tn) \qquad (w = nT).$$

- Given $f \in C(G)$ and $x \in G$, $x \cdot f$ is the function in $C(G)$ defined by the rule

$$(x \cdot f)(y) = f(x^{-1}yx).$$

23: N.B. These rules define operations

$$\begin{cases} W \times C(T) \rightarrow C(T) \\ G \times C(G) \rightarrow C(G) \end{cases}$$

with associated invariants

$$\begin{cases} C(W\backslash T) = C(T)^W \\ CL(G) = C(G)^G. \end{cases}$$

[Note: CL(G) is the subspace of C(G) comprised of the continuous class functions (cf. §2, #27) or still, the space C(CON(G)).]

24: LEMMA The arrow

$$f \rightarrow f|T$$

of restriction defines an isomorphism

$$CL(G) \rightarrow C(T)^W.$$

§5. *REGULARITY*

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Consider the polynomial

$$\det((t + 1) - \mathrm{Ad}(x)) = \sum_{i=0}^{n} D_i(x)t^i \quad (x \in G),$$

where t is an indeterminate and n = dim G. The D_i are real analytic functions on G and $D_n = 1$. Let ℓ be the smallest positive integer such that $D_\ell \neq 0$ -- then ℓ is called the rank of G and an element $x \in G$ is said to be singular or regular according to whether $D_\ell(x) = 0$ or not.

1: NOTATION G^{reg} is the set of regular elements in G.

2: LEMMA G^{reg} is an open, dense subset of G while its complement, the set of singular elements, is a set of Haar measure zero (right or left).

3: N.B. G^{reg} is inner automorphism invariant and stable under multiplication by elements from the center of G.

From this point forward, assume that G is a compact connected Lie group.

4: LEMMA The set of singular elements in G is a finite union of submanifolds of G, each of dimension $\leq$ dim G - 3.

[Note: Therefore G^{reg} is path connected.]

5: RAPPEL The fundamental group of a connected Lie group is abelian.

Fix a maximal torus T.

6: LEMMA The quotient G/T is simply connected.

7: LEMMA The induced map $\pi_1(T) \to \pi_1(G)$ is surjective.

PROOF Consider the exact sequence

$$\pi_1(T) \to \pi_1(G) \to \pi_1(G/T)$$

arising from the fibration $T \to G \to G/T$.

8: THEOREM $\pi_1(G)$ is a finitely generated abelian group.

[Note: If G is semisimple, then $\pi_1(G)$ is finite, thus its universal covering group $\tilde{G}$ is compact.]

9: LEMMA An element $x \in G$ is regular iff x lies in a unique maximal torus.

Put

$$T^{reg} = T \cap G^{reg}.$$

10: THEOREM

$$G^{reg} = \bigcup_{x \in G} xT^{reg}x^{-1}.$$

11: THEOREM The map

$$\mu: G/T \times T^{reg} \to G^{reg}$$

that sends

$$(xT,t) \text{ to } xtx^{-1}$$

is a surjective, $|W|$-to-one local diffeomorphism.

[To verify the "$|W|$-to-one" claim, observe first that $\forall\ w \in W$ $(w = nT)$,

$$\kappa(xn^{-1}T, ntn^{-1}) = xn^{-1} \cdot ntn^{-1} \cdot nx^{-1}$$

$$= xtx^{-1} = \kappa(xT, t),$$

hence

$$|\kappa^{-1}(xtx^{-1})| \geq |W|.$$

In the opposite direction, suppose that

$$xtx^{-1} = ysy^{-1} \quad (t, s \in T^{reg}).$$

Then there is a $w \in W$ such that

$$s = ntn^{-1} \quad (w = nT) \quad (\text{cf. §4, \#16})$$

from which

$$xtx^{-1} = yntn^{-1}y^{-1},$$

so $x^{-1}yn \in G_t$, the centralizer of t in G. But

$$t \in T^{reg} \Rightarrow G_t^0 = T$$

which implies that conjugation by $x^{-1}yn$ preserves T (G_t^0 being the identity component of G_t), i.e.,

$$n' \equiv x^{-1}yn \in N$$

$\Rightarrow$

$$(yT, s) = (x(x^{-1}yn)n^{-1}T, ntn^{-1})$$

$$= (xn'n^{-1}T, ntn^{-1})$$

$$= (xn'n^{-1}T, n(n^{-1}y^{-1}x)t(x^{-1}yn)n^{-1})$$

$$= (x(n'n^{-1})T, (n'n^{-1})^{-1}t(n'n^{-1}))$$

$$\in \kappa^{-1}(xtx^{-1}).]$$

[Note: G_t^0 is a compact connected Lie group and $T \subset G_t^0$ is a maximal torus. If $T \neq G_t^0$, $\exists\, z \in G_t^0$: $zTz^{-1} \neq T$ (cf. §4, #5 (applied to G_t^0)). But then

$$t = ztz^{-1} \in zTz^{-1},$$

contradicting the regularity of t (cf. #9).]

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{t}$ the Lie algebra of T. Since G is compact, there is a positive definite symmetric bilinear form on $\mathfrak{g}$ which is invariant under the adjoint representation:

$$\mathrm{Ad}:G \to \mathrm{Aut}\ \mathfrak{g}.$$

Denote by $\mathfrak{g}/\mathfrak{t}$ the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ -- then $\mathfrak{g}/\mathfrak{t}$ is stable under Ad T, which gives rise to an induced action

$$\mathrm{Ad}_{G/T}:T \to \mathrm{Aut}\ \mathfrak{g}/\mathfrak{t}.$$

Denoting by $I_{G/T}$ the identity map $\mathfrak{g}/\mathfrak{t} \to \mathfrak{g}/\mathfrak{t}$, one may then attach to each $t \in T$ the endomorphism

$$\mathrm{Ad}_{G/T}(t^{-1}) - I_{G/T}$$

of $\mathfrak{g}/\mathfrak{t}$.

<u>12:</u> LEMMA The determinant of

$$\mathrm{Ad}_{G/T}(t^{-1}) - I_{G/T}$$

is positive on the subset of T comprised of the topologically cyclic elements.

<u>13:</u> INTEGRATION FORMULA For any continuous function $f \in C(G)$,

$$\int_G f(x)d_G(x)$$

$$= \frac{1}{|W|} \int_T \left[\det(\mathrm{Ad}_{G/T}(t^{-1}) - I_{G/T}) \int_G f(xtx^{-1}) d_G(x) \right] d_T(t).$$

[Note: $d_G(x)$ is normalized Haar measure on G and $d_T(t)$ is normalized Haar measure on T.]

14: SCHOLIUM For any continuous class function $f \in CL(G)$,

$$\int_G f(x) d_G(x)$$

$$= \frac{1}{|W|} \int_T \det(\mathrm{Ad}_{G/T}(t^{-1}) - I_{G/T}) f(t) d_T(t).$$

APPENDIX

Consider the polynomial

$$\det(t - \mathrm{ad}(X)) = \sum_{i=0}^{n} d_i(X) t^i \quad (X \in \mathfrak{g}),$$

where t is an indeterminate and $n = \dim \mathfrak{g}$. The d_i are polynomial functions on $\mathfrak{g}$ and $d_n = 1$. Let ℓ be the smallest positive integer such that $d_\ell \neq 0$ -- then ℓ is called the rank of $\mathfrak{g}$ and an element $X \in \mathfrak{g}$ is said to be singular or regular according to whether $d_\ell(X) = 0$ or not.

N.B. The rank of $\mathfrak{g}$ equals the rank of G, both being equal to the dimension of $\mathfrak{t}$.

NOTATION $\mathfrak{g}^{reg}$ is the set of regular elements in $\mathfrak{g}$.

LEMMA $\mathfrak{g}^{reg}$ is an open, dense subset of $\mathfrak{g}$.

NOTATION $\mathfrak{G} \equiv \text{Int}\,\mathfrak{g}$ is the adjoint group of $\mathfrak{g}$.

[Note: Recall that the arrow

$$\text{Ad}:G \rightarrow \mathfrak{G}$$

is surjective with kernel $Z(G)$, so

$$G/Z(G) \approx \mathfrak{G}.]$$

Put

$$\mathfrak{t}^{\text{reg}} = \mathfrak{t} \cap \mathfrak{g}^{\text{reg}}.$$

THEOREM

$$\mathfrak{g}^{\text{reg}} = \bigcup_{x\in\mathfrak{G}} x(\mathfrak{t}^{\text{reg}}).$$

§6. WEIGHTS AND ROOTS

Let G be a compact connected semisimple Lie group, $T \subset G$ a maximal torus. Denote their respective Lie algebras by $\mathfrak{g},\mathfrak{t}$ and let $\mathfrak{g}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}}$ stand for their complexifications.

Suppose that (π,V) is a representation of G -- then V can be equipped with a G-invariant inner product, thus rendering matters unitary.

<u>1:</u> LEMMA $d\pi$ is skew-adjoint on $\mathfrak{g}$ (hence self-adjoint on $\sqrt{-1}\,\mathfrak{g}$).

[Given $X \in \mathfrak{g}$, apply $\frac{d}{dt}\Big|_{t=0}$ to

$$<\pi(\exp t)v_1, \pi(\exp tX)v_2> = <v_1,v_2>$$

to get

$$<d\pi(X)v_1,v_2> + <v_1,d\pi(X)v_2> = 0.]$$

<u>2:</u> <u>N.B.</u> $\forall\ X \in \mathfrak{g}$,

$$\pi(\exp X) = e^{d\pi(X)}.$$

<u>3:</u> LEMMA V is simultaneously diagonalizable under the action of $\mathfrak{t}_{\mathbf{C}}$.

[This is because

$$\{d\pi(H):H \in \mathfrak{t}_{\mathbf{C}}\}$$

is a commuting family of normal operators.]

Consequently, there is a finite set $\Phi(V) \subset \mathfrak{t}_{\mathbf{C}}^* - \{0\}$, the elements of which being the <u>weights</u> of V, such that

$$V = V^0 \bigoplus_{\lambda\in\Phi(V)} V^{\lambda},$$

where

$$V^0 = \{v \in V : d\pi(H)v = 0\} \quad (H \in \mathfrak{t}_C)$$

and

$$V^\lambda = \{v \in V : d\pi(H)v = \lambda(H)v\} \quad (H \in \mathfrak{t}_C).$$

4: LEMMA Fix a $\lambda \in \Phi(V)$ -- then $\lambda|\mathfrak{t}$ is purely imaginary and $\lambda|\sqrt{-1}\,\mathfrak{t}$ is purely real.

5: N.B. Given $t \in T$, choose $H \in \mathfrak{t}$ such that $t = \exp H$ -- then $\forall\, v \in V^\lambda$,

$$\pi(t)v = \pi(\exp H)v = e^{d\pi(H)}v = e^{\lambda(H)}v.$$

6: RAPPEL Denote by I_x the inner automorphism $y \to xyx^{-1}$ attached to $x \in G$ -- then the adjoint representation of G is the homomorphism $\mathrm{Ad}:G \to \mathrm{Aut}\,\mathfrak{g}$ defined by the rule

$$\mathrm{Ad}(x) = (dI_x)_e$$

and the adjoint representation of $\mathfrak{g}$ is the homomorphism $\mathrm{ad}:\mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$ defined by the rule

$$\mathrm{ad}(X) = (d\mathrm{Ad})_e(X).$$

7: N.B. $\forall\, X,Y \in \mathfrak{g}$,

$$\mathrm{ad}(X)Y = [X,Y].$$

- For each $x \in G$, extend the domain of $\mathrm{Ad}(x)$ from $\mathfrak{g}$ to $\mathfrak{g}_C$ by complex linearity.
- For each $X \in \mathfrak{g}$, extend the domain of $\mathrm{ad}(X)$ from $\mathfrak{g}$ to $\mathfrak{g}_C$ by complex linearity.

<u>8:</u> LEMMA $(\mathrm{Ad},\mathfrak{g}_C)$ is a representation of G with differential $(\mathrm{ad},\mathfrak{g}_C)$.

Take now $V = \mathfrak{g}_C$, let $\pi = \mathrm{Ad}$, and abbreviate $(\mathfrak{g}_C)^\alpha$ to $\mathfrak{g}^\alpha$ $(\alpha \in \Phi(\mathfrak{g}_C))$ -- then $\mathfrak{g}^0 = \mathfrak{t}_C$ and there is a weight space decomposition

$$\mathfrak{g}_C = \mathfrak{g}^0 \underset{\alpha\in\Phi(\mathfrak{g}_C)}{\oplus} \mathfrak{g}^\alpha.$$

<u>9:</u> TERMINOLOGY The elements $\alpha \in \Phi(\mathfrak{g}_C)$ are called the <u>roots</u> of the pair $(\mathfrak{g}_C,\mathfrak{t}_C)$.

<u>10:</u> <u>N.B.</u>

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g}_C : [H,X] = \alpha(H)X\ (H \in \mathfrak{t}_C)\}.$$

<u>11:</u> LEMMA $\forall\ \alpha \in \Phi(\mathfrak{g}_C)$, $\forall\ \lambda \in \Phi(V) \cup \{0\}$,

$$d\pi(\mathfrak{g}^\alpha)V^\lambda \subset V^{\alpha+\lambda}.$$

PROOF Let $H \in \mathfrak{t}_C$, $X_\alpha \in \mathfrak{g}^\alpha, v_\lambda \in V^\lambda$ -- then

$$d\pi(H)d\pi(X_\alpha)v_\lambda$$

$$= (d\pi(X_\alpha)d\pi(H) + [d\pi(H),\ d\pi(X_\alpha)])v_\lambda$$

$$= (d\pi(X_\alpha)d\pi(H) + d\pi([H,X_\alpha])v_\lambda$$

$$= (d\pi(X_\alpha)d\pi(H) + \alpha(H)d\pi(X_\alpha))v_\lambda$$

$$= (\lambda(H) + \alpha(H))d\pi(X_\alpha)v_\lambda$$

$$\Rightarrow$$

$$d\pi(X_\alpha)v_\lambda \in V^{\alpha+\lambda}.$$

[Note: Take $\lambda = 0$ to see that

$$d\pi(\mathfrak{g}^{\alpha})V^0 \subset V^{\alpha}.$$

<u>12:</u> APPLICATION $\forall\ \alpha,\beta \in \Phi(\mathfrak{g}_C) \cup \{0\}$,

$$[\mathfrak{g}^{\alpha},\mathfrak{g}^{\beta}] \subset \mathfrak{g}^{\alpha+\beta}.$$

<u>13:</u> LEMMA Let $<\ ,\ >$ be an Ad G invariant inner product on $\mathfrak{g}_C$ -- then for all $\alpha,\beta \in \Phi(\mathfrak{g}_C) \cup \{0\}$,

$$<\mathfrak{g}^{\alpha},\mathfrak{g}^{\beta}> = 0 \text{ if } \alpha + \beta \neq 0.$$

<u>14:</u> LEMMA $\forall\ \alpha \in \Phi(\mathfrak{g}_C)$, $\dim \mathfrak{g}^{\alpha} = 1$ and the only multiples of α in $\Phi(\mathfrak{g}_C)$ are $\pm\ \alpha$.

<u>15:</u> NOTATION $\sigma:\mathfrak{g}_C \to \mathfrak{g}_C$ is the map that sends $Z = X + \sqrt{-1}\ Y$ to $\bar{Z} = X - \sqrt{-1}\ Y$ $(X,Y \in \mathfrak{g})$.

<u>16:</u> LEMMA σ is an R-linear involution which preserves the bracket, i.e.,

$$\sigma([Z_1,Z_2]) = [\sigma Z_1,\sigma Z_2] \qquad (Z_1,Z_2 \in \mathfrak{g}_C).$$

<u>17:</u> <u>N.B.</u> $\forall\ \alpha \in \Phi(\mathfrak{g}_C)$,

$$\sigma\mathfrak{g}^{\alpha} = \mathfrak{g}^{-\alpha}.$$

<u>18:</u> RAPPEL The <u>Killing form</u> of $\mathfrak{g}_C$ is the bilinear form $B:\mathfrak{g}_C \times \mathfrak{g}_C \to C$ given by

$$B(Z_1,Z_2) = \mathrm{tr}(\mathrm{ad}(Z_1) \circ \mathrm{ad}(Z_2)).$$

19: PROPERTIES

- $\forall\, x \in G$, $\forall\, Z_1, Z_2 \in \mathfrak{g}_c$,

$$B(\mathrm{Ad}(x)Z_1,\ \mathrm{Ad}(x)Z_2) = B(Z_1, Z_2).$$

- $\forall\, Z, Z_1, Z_2 \in \mathfrak{g}_c$,

$$B(\mathrm{ad}(Z)Z_1, Z_2) = -\ B(Z_1,\ \mathrm{ad}(Z)Z_2).$$

20: N.B. The prescription

$$\langle Z_1, Z_2 \rangle_\sigma = -\ B(Z_1, \sigma Z_2)$$

is an Ad G invariant inner product on $\mathfrak{g}_c$.

Every $\alpha \in \Phi(\mathfrak{g}_c)$ is determined by its restriction to either $\mathfrak{t}$ or $\sqrt{-1}\ \mathfrak{t}$, so α can be viewed as an element of $(\sqrt{-1}\ \mathfrak{t})^*$ (purely real) or of $\mathfrak{t}^*$ (purely imaginary).

21: CONSTRUCTION B induces an isomorphism between $\sqrt{-1}\ \mathfrak{t}$ and $(\sqrt{-1}\ \mathfrak{t})^*$ as follows: Given $\lambda \in (\sqrt{-1}\ \mathfrak{t})^*$, define $H_\lambda \in \sqrt{-1}\ \mathfrak{t}$ by the relation

$$\lambda(H) = B(H, H_\lambda) \qquad (H \in \sqrt{-1}\ \mathfrak{t}).$$

[Note: B is negative definite on $\mathfrak{t} \times \mathfrak{t}$, hence B is a real inner product on the real vector space $\sqrt{-1}\ \mathfrak{t}$ and for $\lambda_1, \lambda_2 \in (\sqrt{-1}\ \mathfrak{t})^*$, one writes

$$B(\lambda_1, \lambda_2) = B(H_{\lambda_1}, H_{\lambda_2}).]$$

22: DEFINITION The vector $H_\alpha \in \sqrt{-1}\ \mathfrak{t}$ is called the root vector associated with α.

23: LEMMA The roots span $(\sqrt{-1}\ \mathfrak{t})^*$ and the root vectors span $\sqrt{-1}\ \mathfrak{t}$.

<u>24:</u> LEMMA Let $X_\alpha \in \mathfrak{g}^\alpha$, $X_{-\alpha} \in \mathfrak{g}^{-\alpha}$ -- then

$$[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha}) H_\alpha.$$

PROOF First of all,

$$[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \subset \mathfrak{g}^{\alpha-\alpha} = \mathfrak{g}^0 = \mathfrak{t}_{\mathbf{C}} \quad \text{(cf. \#12)},$$

thus

$$[X_\alpha, X_{-\alpha}] \in \mathfrak{t}_{\mathbf{C}}.$$

Proceeding, $\forall\ H \in \mathfrak{t}_{\mathbf{C}}$,

$$\begin{aligned}
&B([X_\alpha, X_{-\alpha}], H) \\
&\quad = - B([X_{-\alpha}, X_\alpha], H) \\
&\quad = - B(\text{ad}(X_{-\alpha}) X_\alpha, H) \\
&\quad = B(X_{-\alpha}, \text{ad}(X_{-\alpha}) H) \\
&\quad = B(X_\alpha, [X_{-\alpha}, H]) \\
&\quad = - B(X_\alpha, [H, X_{-\alpha}]) \\
&\quad = - B(X_\alpha, -\alpha(H) X_{-\alpha}) \\
&\quad = \alpha(H) B(X_\alpha, X_{-\alpha}) \\
&\quad = B(H, H_\alpha) B(X_\alpha, X_{-\alpha}) \\
&\quad = B(H_\alpha, H) B(X_\alpha, X_{-\alpha}) \\
&\quad = B(B(X_\alpha, X_{-\alpha}) H_\alpha, H)
\end{aligned}$$

$\Rightarrow$

$$[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha}) H_\alpha.$$

25: NOTATION Put

$$h_\alpha = 2 \frac{H_\alpha}{B(H_\alpha, H_\alpha)} .$$

Then $\alpha(h_\alpha) = 2$.

26: N.B. $\forall\ \lambda \in (\sqrt{-1}\ t)^*$,

$$\lambda(h_\alpha) = \lambda(2 \frac{H_\alpha}{B(H_\alpha, H_\alpha)})$$

$$= \lambda(2 \frac{H_\alpha}{B(\alpha,\alpha)}$$

$$= 2 \frac{\lambda(H_\alpha)}{B(\alpha,\alpha)}$$

$$= 2 \frac{B(H_\alpha, H_\lambda)}{B(\alpha,\alpha)}$$

$$= 2 \frac{B(H_\lambda, H_\alpha)}{B(\alpha,\alpha)}$$

$$= 2 \frac{B(\lambda,\alpha)}{B(\alpha,\alpha)}$$

and analogously, $\forall\ H \in \sqrt{-1}\ \mathfrak{t}$,

$$\alpha(H) = 2 \frac{B(H, h_\alpha)}{B(h_\alpha, h_\alpha)} .$$

27: NORMALIZATION Scale the data and choose $e_\alpha \in \mathfrak{g}^\alpha$, $f_\alpha \in \mathfrak{g}^{-\alpha}$ such that

$$[e_\alpha, f_\alpha] = h_\alpha,$$

hence

$$\left[\begin{array}{l} [h_\alpha, e_\alpha] = 2e_\alpha \\ [h_\alpha, f_\alpha] = -2f_\alpha. \end{array}\right.$$

Consequently,

$$\text{span}_C\{h_\alpha, e_\alpha, f_\alpha\} \approx \mathfrak{sl}(2,C),$$

where

$$h_\alpha \longleftrightarrow h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_\alpha \longleftrightarrow e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f_\alpha \longleftrightarrow f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

<u>28:</u> <u>N.B.</u> Under this correspondence,

$$\begin{aligned} \mathfrak{su}(2) &\approx \text{span}_R\{\sqrt{-1}\, h_\alpha,\ e_\alpha - f_\alpha,\ \sqrt{-1}\,(e_\alpha + f_\alpha)\} \\ &\equiv \mathfrak{s}_\alpha, \end{aligned}$$

where

$$\sqrt{-1}\, h_\alpha \longleftrightarrow \sqrt{-1}\, h = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}$$

and

$$e_\alpha - f_\alpha \longleftrightarrow e - f = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \sqrt{-1}(e_\alpha + f_\alpha) \longleftrightarrow \sqrt{-1}(e + f) = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}.$$

<u>29:</u> LEMMA The analytic subgroup S_α of G with Lie algebra $\mathfrak{s}_\alpha$ is compact and isomorphic to SU(2) or $SU(2)/Z_2$.

<u>30:</u> LEMMA Let (π,V) be a unitary representation of G -- then $\forall\ \lambda \in \Phi(V)$, $\lambda(h_\alpha) \in Z$.

PROOF In SU(2), $e^{2\pi\sqrt{-1}\,h} = I$. This said, let $\phi_\alpha:SU(2) \to G$ be the arrow realizing the preceding setup and consider $\pi \circ \phi_\alpha$:

$$\begin{aligned} I &= \pi(\phi_\alpha(e^{2\pi\sqrt{-1}\,h})) \\ &= \pi(e^{2\pi d\phi_\alpha(\sqrt{-1}\,h)}) \\ &= \pi(e^{2\pi\sqrt{-1}\,h_\alpha}) = e^{2\pi\sqrt{-1}\,d\pi(h_\alpha)}. \end{aligned}$$

On the other hand, $\forall\ v \in V^\lambda$,

$$\begin{aligned} v &= e^{2\pi\sqrt{-1}\,d\pi(h_\alpha)}v \\ &= e^{2\pi\sqrt{-1}\lambda(h_\alpha)}v \quad \text{(cf. \#5)}. \end{aligned}$$

Therefore $\lambda(h_\alpha) \in Z$.

§7. *LATTICES*

Let V be a finite dimensional vector space over R.

1: DEFINITION A lattice in V is an additive subgroup $L \subset V$ such that

- L is closed;
- L is discrete;
- L spans V.

2: EXAMPLE Z^n is a lattice in R^n.

3: DEFINITION A basis for a lattice $L \subset V$ is a set $\{e_1,\ldots,e_n\} \subset L$ $(n = \dim V)$ such that

$$L = \{\sum_{i=1}^{n} k_ie_i : k_i \in Z\}.$$

4: LEMMA Every lattice has a basis.

5: DEFINITION If L,K are lattices in V, then L is a sublattice of K if L is a subset of K.

6: LEMMA If L is a sublattice of K, then K/L is a finite group G. Moreover, there is a one-to-one correspondence between the subgroups $H \subset G$ and the lattices $L \subset M \subset K$, viz.

$$\pi(M) = H \text{ and } M = \pi^{-1}(H),$$

where $\pi:K \to K/L$ is the projection.

7: NOTATION Given a lattice $L \subset V$, let

$$L^* = \{v^* \in V^*: v^*(x) \in Z \ \forall\ x \in L\}.$$

8: LEMMA L^* is a lattice in V^*, the dual of L.

Let $\{e_1,\ldots,e_n\}$ be a basis for a lattice $L \subset V$. Define $\{f_1,\ldots,f_n\}$ by

$$f_j(e_i) = \delta_{ij}.$$

9: LEMMA $\{f_1,\ldots,f_n\}$ is a basis for L^*.

10: APPLICATION

$$L^{**} \approx L.$$

[In fact, the condition

$$f_j(e_i) = \delta_{ij}$$

is symmetric in f and e.]

11: LEMMA Suppose that L is a sublattice of K -- then $K^* \subset L^*$ and

$$L^*/K^* \approx \widehat{K/L}.$$

PROOF The first point is obvious. As for the second, define a homomorphism $\rho: L^* \to \widehat{K/L}$ by stipulating that

$$\rho(\ell^*)(x + L) = \exp(2\pi\sqrt{-1}\ \ell^*(x)).$$

Then the kernel of ρ is K^*, so ρ induces an injection $L^*/K^* \to \widehat{K/L}$, thus

$$|L^*/K^*| \leq |\widehat{K/L}| = |K/L|.$$

But then by duality,

$$|L^*/K^*| \geq |K^{**}/L^{**}| = |K/L|.$$

Let G be a compact connected semisimple Lie group, $T \subset G$ a maximal torus.

12: CONVENTION Identify $(\sqrt{-1}\ \mathfrak{t})^{**}$ with $\sqrt{-1}\ \mathfrak{t}$ and let L be a lattice in $(\sqrt{-1}\ \mathfrak{t})^*$ -- then its dual is the lattice $L^* \subset \sqrt{-1}\ \mathfrak{t}$ specified via the prescription

$$\{H \in \sqrt{-1}\ \mathfrak{t}:\lambda(H) \in Z\ \forall\ \lambda \in L\}.$$

13: DEFINITION The root lattice is the lattice L_{rt} in $(\sqrt{-1}\ \mathfrak{t})^*$ generated by the $\alpha \in \Phi(\mathfrak{g}_C)$.

14: DEFINITION The weight lattice is the lattice L_{wt} in $(\sqrt{-1}\ \mathfrak{t})^*$ given by

$$\{\lambda \in (\sqrt{-1}\ \mathfrak{t})^*:\lambda(h_\alpha) \in Z\ \forall\ \alpha \in \Phi(\mathfrak{g}_C)\}.$$

15: LEMMA L_{rt} is a sublattice of L_{wt}.

Given a character $\chi:T \rightarrow S^1$, there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{d\chi} & \sqrt{-1}\ R \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ T & \xrightarrow[\chi]{} & S^1 \end{array}$$

and the arrow $\chi \rightarrow d\chi$ implements an identification of $\hat{T}$ with the lattice

$$d\hat{T} \equiv \{\lambda \in (\sqrt{-1}\ \mathfrak{t})^*:\lambda|\exp^{-1}(e) \subset 2\pi\sqrt{-1}\ Z\}.$$

Here

$$d\chi \in \text{Hom}_R(\mathfrak{t},\sqrt{-1}\ R)$$

which we shall view as an element of

$$\text{Hom}_R(\sqrt{-1}\ \mathfrak{t},R)$$

by writing

$$d\lambda(\sqrt{-1}\,H) = \sqrt{-1}\,d\lambda(H) \qquad (H \in \mathfrak{t}).$$

[Note: $\sqrt{-1}\,\mathbf{R}$ is the Lie algebra of S^1, the exponential map $\exp: \sqrt{-1}\,\mathbf{R} \to S^1$ being the usual exponential function $\sqrt{-1}\,\theta \to e^{\sqrt{-1}\,\theta}$.]

16: LEMMA L_{rt} is a sublattice of $d\hat{T}$ and $d\hat{T}$ is a sublattice of L_{wt}.

17: THEOREM

- $Z(G) \approx d\hat{T}/L_{rt}$
- $\pi_1(G) \approx L_{wt}/d\hat{T}$.

§8. *WEYL CHAMBERS AND WEYL GROUPS*

Let G be a compact connected semisimple Lie group, $T \subset G$ a maximal torus, $\Phi(\mathfrak{g}_{\mathbb{C}})$ the roots of the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

<u>1:</u> DEFINITION A subset Ψ of $\Phi(\mathfrak{g}_{\mathbb{C}})$ is a <u>simple system</u> of roots if it is a vector space basis for $(\sqrt{-1}\,\mathfrak{t})^*$ and has the property that every root can be written as a linear combination

$$\sum_{\alpha \in \Psi} n_\alpha \alpha,$$

where the n_α are integers all of the same sign.

<u>2:</u> DEFINITION The elements in a simple system of roots are said to be <u>simple</u>.

<u>3:</u> <u>N.B.</u> Simple systems exist (cf. infra).

<u>4:</u> CONSTRUCTION Let Ψ be a simple system of roots.

- The <u>positive roots</u> per Ψ is the set

$$\Phi^+ = \{\beta \in \Phi(\mathfrak{g}_{\mathbb{C}}) : \beta = \sum_{\alpha \in \Psi} n_\alpha \alpha \quad (n_\alpha \in \mathbb{Z}_{\geq 0})\}.$$

- The <u>negative roots</u> per Ψ is the set

$$\Psi^- = \{\beta \in \Phi(\mathfrak{g}_{\mathbb{C}}) : \beta = \sum_{\alpha \in \Psi} n_\alpha \alpha \quad (n_\alpha \in \mathbb{Z}_{\leq 0})\}.$$

Accordingly,

$$\Phi(\mathfrak{g}_{\mathbb{C}}) = \Phi^+ \coprod \Phi^-.$$

5: DEFINITION

- The connected components of

$$(\sqrt{-1}\ \mathfrak{t})^* - \bigcup_{\alpha \in \Phi(\mathfrak{g}_C)} \alpha^\perp$$

are called the Weyl chambers of $(\sqrt{-1}\ \mathfrak{t})^*$.

- The connected components of

$$\sqrt{-1}\ \mathfrak{t} - \bigcup_{\alpha \in \Phi(\mathfrak{g}_C)} h_\alpha^\perp$$

are called the Weyl chambers of $\sqrt{-1}\ \mathfrak{t}$.

6: DEFINITION

- If $C \subset (\sqrt{-1}\ \mathfrak{t})^*$ is a Weyl chamber, then $\alpha \in \Phi(\mathfrak{g}_C)$ is said to be C-positive if $B(C,\alpha) > 0$ and C-negative if $B(C,\alpha) < 0$.

- If $C \subset \sqrt{-1}\ \mathfrak{t}$ is a Weyl chamber, then $\alpha \in \Phi(\mathfrak{g}_C)$ is said to be C-positive if $B(C,h_\alpha) > 0$ and C-negative if $B(C,h_\alpha) < 0$.

7: DEFINITION

- If $C \subset (\sqrt{-1}\ \mathfrak{t})^*$ is a Weyl chamber and if α is C-positive, then α is decomposable w.r.t. C if there exist $\beta,\gamma \in \Phi(\mathfrak{g}_C)$ such that $\alpha = \beta + \gamma$ (otherwise, α is indecomposable w.r.t. C).

- If $C \subset \sqrt{-1}\ \mathfrak{t}$ is a Weyl chamber and if α is C-positive, then α is decomposable w.r.t. C if there exist $\beta,\gamma \in \Phi(\mathfrak{g}_C)$ such that $\alpha = \beta + \gamma$ (otherwise, α is indecomposable w.r.t. C).

8: NOTATION

- Given a Weyl chamber $C \subset (\sqrt{-1}\ \mathfrak{t})^*$, let $\Psi(C)$ be the subset of $\Phi(\mathfrak{g}_C)$

comprised of those α which are C-positive and indecomposable.

- Given a Weyl chamber $C \subset \sqrt{-1}\,\mathfrak{t}$, let $\Psi(C)$ be the subset of $\Phi(\mathfrak{g}_C)$ comprised of those α which are C-positive and indecomposable.

9: LEMMA In either case, $\Psi(C)$ is a simple system of roots.

10: NOTATION

- Given a simple system of roots Ψ, let

$$C(\Psi) = \{\lambda \in (\sqrt{-1}\,\mathfrak{t})^* : B(\lambda,\alpha) > 0 \ \forall\ \alpha \in \Psi\}.$$

- Given a simple system of roots Ψ, let

$$C(\Psi) = \{H \in \sqrt{-1}\,\mathfrak{t} : B(H,h_\alpha) > 0 \ \forall\ \alpha \in \Psi\}.$$

11: LEMMA In either case, $C(\Psi)$ is a Weyl chamber.

12: THEOREM

- There is a one-to-one correspondence between the simple systems of roots and the Weyl chambers of $(\sqrt{-1}\,\mathfrak{t})^*$:

$$\begin{cases} \Psi \rightarrow C(\Psi) \\ C \rightarrow \Psi(C). \end{cases}$$

- There is a one-to-one correspondence between the simple systems of roots and the Weyl chambers of $\sqrt{-1}\,\mathfrak{t}$:

$$\begin{cases} \Psi \rightarrow C(\Psi) \\ C \rightarrow \Psi(C). \end{cases}$$

The Weyl group $W = N(T)/T$ operates via Ad on $\sqrt{-1}\,\mathfrak{t}$ and $(\sqrt{-1}\,\mathfrak{t})^*$.

13: LEMMA The action of W on $\sqrt{-1}\,\mathfrak{t}$ and $(\sqrt{-1}\,\mathfrak{t})^*$ is faithful, i.e., $w \in W$ acts trivially iff w is the identity element.

PROOF Suppose that $\mathrm{Ad}(n)$ $(n \in N)$ is the identity on $\mathfrak{t}$ and consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{\mathrm{Ad}(n)} & \mathfrak{t} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ T & \xrightarrow[I_n]{} & T \end{array} .$$

Then

$$\exp \mathfrak{t} = T$$

and $\forall\ X \in \mathfrak{t}$,

$$I_n(\exp X) = n(\exp X)n^{-1} = \exp(\mathrm{Ad}(n)X) = \exp X.$$

Therefore n centralizes T, hence $n \in T$ (cf. §4, #7), i.e., n represents the identity element of W.

14: LEMMA W preserves $\Phi(\mathfrak{g}_{\mathbb{C}})$ and $wh_\alpha = h_{w\alpha}$ $(w \in W)$.

15: NOTATION

- Given $\alpha \in \Phi(\mathfrak{g}_{\mathbb{C}})$, define

$$r_\alpha : (\sqrt{-1}\,\mathfrak{t})^* \to (\sqrt{-1}\,\mathfrak{t})^*$$

by

$$r_\alpha(\lambda) = \lambda - 2\,\frac{B(\lambda,\alpha)}{B(\alpha,\alpha)}\,\alpha = \lambda - \lambda(h_\alpha)\alpha.$$

- Given $\alpha \in \Phi(\mathfrak{g}_C)$, define

$$r_{h_\alpha}:\sqrt{-1}\,\mathfrak{t} \to \sqrt{-1}\,\mathfrak{t}$$

by

$$r_{h_\alpha}(H) = h - 2\,\frac{B(H,h_\alpha)}{B(h_\alpha,h_\alpha)}\,h_\alpha = H - \alpha(H)h_\alpha.$$

[Note: Geometrically, r_α is the reflection of $(\sqrt{-1}\,\mathfrak{t})^*$ across the hyperplane perpendicular to α and r_{h_α} is the reflection of $\sqrt{-1}\,\mathfrak{t}$ across the hyperplane perpendicular to h_α.]

16: NOTATION Depending on the context, $W(\Phi(\mathfrak{g}_C))$ is the group generated by

$$\{r_\alpha:\alpha \in \Phi(\mathfrak{g}_C)\} \text{ or } \{r_{h_\alpha}:\alpha \in \Phi(\mathfrak{g}_C)\}.$$

17: N.B. $W(\Phi(\mathfrak{g}_C))$ operates on $\mathfrak{t}^*$ and $\mathfrak{t}$ (extension by complex linearity).

18: LEMMA $\forall\ \alpha \in \Phi(\mathfrak{g}_C)$, $\exists\ n_\alpha \in N(T)$ such that the action of n_α on $(\sqrt{-1}\,\mathfrak{t})^*$ is given by r_α and the action of n_α on $\sqrt{-1}\,\mathfrak{t}$ is given by r_{h_α}.

19: THEOREM

- Per $(\sqrt{-1}\,\mathfrak{t})^*$, $W \approx W(\Phi(\mathfrak{g}_C))$.
- Per $\sqrt{-1}\,\mathfrak{t}$, $W \approx W(\Phi(\mathfrak{g}_C))$.

[Note: It follows from #18 that in either case,

$$W(\Phi(\mathfrak{g}_C)) \subset W,$$

so the crux is the reversal of this.]

20: LEMMA W operates simply transitively on the set of Weyl chambers in $(\sqrt{-1}\,\mathfrak{t})^*$ or $\sqrt{-1}\,\mathfrak{t}$.

[Note: In other words, there is exactly one element of the Weyl group mapping a given Weyl chamber onto another one.]

21: N.B. It is a corollary that $|W|$ is the cardinality of the set of Weyl chambers.

22: EXAMPLE Given a Weyl chamber C (be it in $(\sqrt{-1}\,\mathfrak{t})^*$ or $\sqrt{-1}\,\mathfrak{t}$), there exists a unique element $w^{\circ} \in W$ which maps C to its negative $-C$, hence $w^{\circ}\Psi(C) = -\Psi(C)$.

[Note: In general, $-e \notin W$.]

23: THEOREM Let

$$C \subset (\sqrt{-1}\,\mathfrak{t})^* \text{ or } C \subset \sqrt{-1}\,\mathfrak{t}$$

be a Weyl chamber -- then its closure $\bar{C}$ is a fundamental domain for the action of W, i.e., $\bar{C}$ meets each W orbit exactly once.

Fix a Weyl chamber $C \subset (\sqrt{-1}\,\mathfrak{t})^*$ and thereby determine the simple system $\Psi(C)$, hence Φ^+.

24: NOTATION $W(C)$ is the subgroup of $W(\Phi(\mathfrak{g}_c))$ generated by the $r_\alpha (\alpha \in \Psi(C))$.

25: LEMMA

$$W(C) = W(\Phi(\mathfrak{g}_c)).$$

26: NOTATION Given $w \in W(\Phi(\mathfrak{g}_c))$, let $\ell(w)$ be the smallest k such that w can be factored as a product $r_{\alpha_1} \cdots r_{\alpha_k}$, where the $\alpha_i \in \Psi(C)$ (set $\ell(w) = 0$ if $w = e$).

[Note: $\ell(w)$ is referred to as the <u>length</u> of w.]

<u>27:</u> LEMMA $\ell(w)$ is the number of $\alpha \in \Phi^+$ such that $w\alpha \in \Phi^-$.

<u>28:</u> APPLICATION If $w\Phi^+ = \Phi^+$, then $w = e$.

<u>29:</u> <u>N.B.</u> The assignment

$$w \to \det(w) = (-1)^{\ell(w)} \in \{\pm 1\}$$

is a character of W.

<u>30:</u> LEMMA If $\lambda \in L_{wt}$, then $\forall\, w \in W$, $\lambda - w\lambda \in L_{rt}$.

PROOF This is obvious if $w = r_\alpha$ for some $\alpha \in \Psi(C)$. In general, $w = r_{\alpha_1} \cdots r_{\alpha_k}$ $(k = \ell(w))$ and one can write

$$\lambda - w\lambda = (\lambda - r_k(\lambda)) + (r_k(\lambda) - r_{k-1}(r_k(\lambda)) + \cdots .$$

Let $\alpha_1, \ldots, \alpha_\ell$ be an enumeration of the elements of $\Psi(C)$.

[Note: Recall that ℓ is the rank of G or still, the dimension of T or still, the dimension of $\sqrt{-1}\,\mathfrak{t}$ or still, the dimension of $(\sqrt{-1}\,\mathfrak{t})^*$.]

<u>31:</u> DEFINITION The <u>fundamental weights</u> are the $\omega_i \in L_{wt}$ per the prescription

$$2\,\frac{B(\omega_i,\alpha_j)}{B(\alpha_j,\alpha_j)} = \delta_{ij} \quad (1 \le i,j \le \ell).$$

<u>32:</u> LEMMA The set $\{\omega_1, \ldots, \omega_\ell\}$ is a basis for L_{wt}.

33: DEFINITION A weight $\lambda \in L_{wt}$ is said to be dominant if $B(\lambda,\alpha) \geq 0$ for all $\alpha \in \Psi(C)$.

34: N.B. To say that $\lambda \in L_{wt}$ is dominant amounts to saying that $\lambda \in \bar{C}$ (the closure of C).

35: LEMMA A weight $\lambda \in L_{wt}$ is dominant iff it is a linear combination with nonnegative integral coefficients of the ω_i.

36: NOTATION Put

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

37: N.B. Ultimately, ρ depends on the choice of C.

38: LEMMA $\forall\, w \in W$,

$$w\rho = \rho - \sum_{\alpha \in \Phi^+, w^{-1}\alpha \in \Phi^-} \alpha.$$

39: APPLICATION $\forall\, \alpha \in \Psi(C)$,

$$r_\alpha(\rho) = \rho - \alpha.$$

[Note: $\forall\, \alpha \in \Psi(C)$,

$$r_\alpha(\Phi^+ - \{\alpha\}) = \Phi^+ - \{\alpha\}.]$$

40: LEMMA

$$\rho = \omega_1 + \cdots + \omega_\ell.$$

PROOF Given $\alpha_i \in \Psi(C)$,

$$\alpha_i = \rho - r_{\alpha_i}(\rho)$$

$$= \rho - \left(\rho - 2\frac{B(\rho,\alpha_i)}{B(\alpha_i,\alpha_i)}\right)$$

$$= 2\frac{B(\rho,\alpha_i)}{B(\alpha_i,\alpha_i)}\alpha_i$$

=>

$$2\frac{B(\rho,\alpha_i)}{B(\alpha_i,\alpha_i)} = 1 \Rightarrow \rho \in L_{wt} \text{ (see below)}.$$

Now write

$$\rho = n_1\omega_1 + \cdots + n_\ell\omega_\ell.$$

Then

$$1 = 2\frac{B(\rho,\alpha_j)}{B(\alpha_j,\alpha_j)} = 2\frac{B(\sum_i n_i\omega_i,\alpha_j)}{B(\alpha_j,\alpha_j)}$$

$$= \sum_i n_i\, 2\frac{B(\omega_i,\alpha_j)}{B(\alpha_j,\alpha_j)}$$

$$= \sum_i n_i\delta_{ij} = n_i \Rightarrow 1 = n_i.$$

Therefore

$$\rho = \omega_1 + \cdots + \omega_\ell.$$

<u>41:</u> <u>N.B.</u> It follows that ρ is a dominant weight.

APPENDIX

LEMMA Suppose that $\lambda \in (\sqrt{-1}\ \mathfrak{t})^*$ has the property that

$$2\,\frac{B(\lambda,\alpha_i)}{B(\alpha_i,\alpha_i)} \in Z \qquad (i = 1,\ldots,\ell).$$

Then $\lambda \in L_{wt}$.

PROOF It is a question of showing that $\forall\ \alpha \in \Phi^+$,

$$\lambda(h_\alpha) = 2\,\frac{B(\lambda,\alpha)}{B(\alpha,\alpha)} \in Z.$$

To this end, let $\alpha = \sum_{i=1}^{n} n_i\alpha_i \in \Phi^+$ and proceed by induction on $|\alpha| = \sum_{i=1}^{\ell} n_i$, the <u>level</u> of α. The case $|\alpha| = 1$ is the hypothesis, so assume that the assertion is true for all levels $< |\alpha|$. Choose α_i such that $B(\alpha,\alpha_i) > 0$, hence

$$\beta = r_{\alpha_i}(\alpha) = \alpha - 2\,\frac{B(\alpha,\alpha_i)}{B(\alpha_i,\alpha_i)}\,\alpha_i$$

is positive and has level $< |\alpha|$, thus

$$2\,\frac{B(\lambda,\alpha)}{B(\alpha,\alpha)} = 2\,\frac{B(r_{\alpha_i}(\lambda),\beta)}{B(\beta,\beta)}$$

$$= 2\,\frac{B(\lambda,\beta)}{B(\beta,\beta)} - 2\,\frac{B(\lambda,\alpha_i)}{B(\alpha_i,\alpha_i)}\ 2\,\frac{B(\alpha_i,\beta)}{B(\beta,\beta)}$$

is an integer.

§9. *DESCENT*

Let G be a compact connected semisimple Lie group, $T \subset G$ a maximal torus, $\Phi(\mathfrak{g}_C)$ the roots of the pair $(\mathfrak{g}_C, \mathfrak{t}_C)$, $C \subset (\sqrt{-1}\ \mathfrak{t})^*$ a Weyl chamber, $\Psi(=\Psi(C))$ the simple system of roots thereby determined, and Φ^+ (Φ^-) the positive (negative) roots per Ψ.

<u>1:</u> RAPPEL Given a character $\chi: T \to S^1$, there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{d\chi} & \sqrt{-1}\ R \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ T & \xrightarrow[\chi]{} & S^1 \end{array}$$

and the arrow $\chi \to d\chi$ implements an identification of $\hat{T}$ with the lattice

$$d\hat{T} \equiv \{\lambda \in (\sqrt{-1}\ \mathfrak{t})^*: \lambda\,|\exp^{-1}(e) \subset 2\pi\sqrt{-1}\ Z\}.$$

<u>2:</u> <u>N.B.</u> $d\hat{T}$ is a sublattice of L_{wt} and

$$\pi_1(G) \approx L_{wt}/d\hat{T} \quad \text{(cf. §7, #17)}.$$

[Note: Therefore $L_{wt} = d\hat{T}$ iff G is simply connected.]

<u>3:</u> NOTATION Each $\lambda \in d\hat{T}$ determines a character $\xi_\lambda \in \hat{T}$ such that

$$\xi_\lambda(\exp H) = e^{\lambda(H)} \quad (H \in \mathfrak{t}).$$

<u>4:</u> DEFINITION A function $f: \mathfrak{t} \to C$ <u>descends to T</u> if it factors through the exponential map, i.e., if $f(H + Z) = f(H)$ $\forall\ H \in \mathfrak{t}$ and $\forall\ Z \in \mathfrak{t}$ such that $\exp Z = e$.

If $f:\mathfrak{t} \to C$ descends to T, then there is a function $F:T \to C$ such that

$$F(\exp H) = f(H) \quad (H \in \mathfrak{t}).$$

5: EXAMPLE Given $\lambda \in d\hat{T}$, the function $H \to e^{\lambda(H)}$ descends to T $(F = \xi_\lambda)$.

6: EXAMPLE Put

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \quad (\text{cf. §8, \#36}).$$

Then $\forall\ w \in W$,

$$w\rho - \rho \in L_{rt} \subset d\hat{T} \quad (\text{cf. §8, \#38}),$$

thus the function

$$H \to e^{(w\rho-\rho)(H)}$$

descends to T $(F = \xi_{w\rho-\rho})$.

7: N.B. It is not claimed nor is it true in general that the function $H \to e^{\rho(H)}$ descends to T.

8: DEFINITION $\Delta:\mathfrak{t} \to C$ is the function

$$H \to \prod_{\alpha \in \Phi^+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2}) \quad (H \in \mathfrak{t}).$$

[Note: $\alpha/2$ need not belong to L_{wt}.]

9: LEMMA

$$\Delta = e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}).$$

Therefore Δ descends to T iff e^{ρ} descends to T.

<u>10:</u> LEMMA $|\Delta|^2$ descends to T.

PROOF $\forall\, H \in \mathfrak{t}$,

$$|\Delta(H)|^2 = \Delta(H)\overline{\Delta(H)}$$

$$= e^{\rho(H)} \prod_{\alpha\in\Phi^+} (1 - e^{-\alpha(H)})\overline{e^{\rho(H)} \prod_{\alpha\in\Phi^+} (1 - e^{-\alpha(H)})}$$

$$= e^{\rho(H)} \prod_{\alpha\in\Phi^+} (1 - e^{-\alpha(H)})\, e^{-\rho(H)} \overline{\prod_{\alpha\in\Phi^+} (1 - e^{-\alpha(H)})}$$

$$= \prod_{\alpha\in\Phi^+} (1 - e^{-\alpha(H)})\, \overline{(1 - e^{-\alpha(H)})}$$

$$= \prod_{\alpha\in\Phi^+} |1 - e^{-\alpha(H)}|^2,$$

which descends to T.

<u>11:</u> LEMMA $\forall\, t \in T$,

$$\det(\mathrm{Ad}_{G/T}(t^{-1}) - I_{G/T}) = |\Delta(t)|^2.$$

PROOF The complexification of $\mathfrak{g}/\mathfrak{t}$ is the direct sum of the $\mathfrak{g}^\alpha$ on which $t \in T$ acts by $\xi_\alpha(t)$ in the adjoint representation, so

$$\det(\mathrm{Ad}_{G/T}(t^{-1}) - I_{G/T})$$

$$= \prod_{\alpha\in\Phi(\mathfrak{g}_C)} (\xi_\alpha(t^{-1}) - 1)$$

$$= \prod_{\alpha \in \Phi(\mathfrak{g}_c)} (1 - \xi_{-\alpha}(t))$$

$$= \prod_{\alpha \in \Phi^+} |1 - \xi_{-\alpha}(t)|^2.$$

[Note: The number of roots is even.]

12: INTEGRATION FORMULA For any continuous function $f \in C(G)$,

$$\int_G f(x) d_G(x) = \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G f(xtx^{-1}) d_G(x) d_T(t) \qquad \text{(cf. §5, \#13)}.$$

13: SCHOLIUM For any continuous class function $f \in CL(G)$,

$$\int_G f(x) d_G(x) = \frac{1}{|W|} \int_T |\Delta(t)|^2 f(t) d_T(t) \qquad \text{(cf. §5, \#14)}.$$

14: REMARK Let $t \in T$ -- then $t \in T^{reg}$ iff

$$|\Delta(t)|^2 \neq 0$$

or still, iff

$$\prod_{\alpha \in \Phi^+} |1 - \xi_{-\alpha}(t)|^2 \neq 0.$$

15: N.B. Let $H \in \mathfrak{t}$ -- then

$$|\Delta(e^H)|^2 = 2^{|\Phi(\mathfrak{g}_c)|} \prod_{\alpha \in \Phi^+} \sin^2\left(\frac{\alpha(H)}{2\sqrt{-1}}\right).$$

[Note: Bear in mind that $\alpha(H) \in \sqrt{-1}\, R$.]

16: NOTATION Let

$$\Xi = \{H \in \mathfrak{t} : \forall\ \alpha \in \Phi(\mathfrak{g}_c),\ \alpha(H) \notin 2\pi\sqrt{-1}\, Z\}.$$

17: LEMMA Ξ is open and dense in $\mathfrak{t}$. Moreover,

$$\exp \Xi = T^{reg}.$$

18: RAPPEL The inclusion $T \to G$ induces a bijection between the orbits of W in T and the conjugacy classes of G (cf. §4, #16). Consequently, class functions on G are the "same thing" as W-invariant functions on T.

19: NOTATION Given $\lambda \in d\hat{T}$, define $\Psi_\lambda : \Xi \to \mathbf{C}$ by setting

$$\Psi_\lambda(H) = \frac{\sum_{w \in W} \det(w) e^{w(\lambda+\rho)(H)}}{\Delta(H)} \quad (H \in \Xi).$$

20: LEMMA $\forall\, w \in W$,

$$w(\Delta) = (-1)^{\ell(w)} \Delta.$$

Recalling that $\det(w) = (-1)^{\ell(w)}$ (cf. §8, #29), it therefore follows that Ψ_λ is a W-invariant function on Ξ.

Next, $\forall\, H \in \Xi$,

$$\Psi_\lambda(H) = \frac{\sum_{w \in W} \det(w) e^{(w(\lambda+\rho)-\rho)(H)}}{\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha(H)})}.$$

Since

$$e^{(w(\lambda+\rho)-\rho)(H)} = e^{w\lambda(H)} e^{(w\rho-\rho)(H)},$$

the numerator of this fraction descends to T (cf. #5 and #6). The same also goes for the denominator which is nonzero on Ξ. Accordingly, Ψ_λ descends to a W-invariant function on T^{reg}, hence extends to a class function on G^{reg} (cf. §5, #10), denoted still by Ψ_λ.

§10. *CHARACTER THEORY*

Let G be a compact connected semisimple Lie group, $T \subset G$ a maximal torus, and maintain the assumptions/notation of §9.

<u>1:</u> THEOREM Suppose given a $\Pi \in \hat{G}$ -- then there is a $\lambda_\Pi \in d\hat{T}$ subject to $\lambda_\Pi + \rho \in C$ such that $\forall\, x \in G^{reg}$,

$$\chi_\Pi(x) = \Psi_{\lambda_\Pi}(x).$$

The proof proceeds by a series of lemmas.

<u>2:</u> NOTATION Given $\gamma \in C$, define $A_\gamma : \mathfrak{t} \to \mathbf{C}$ by

$$A_\gamma(H) = \sum_{w \in W} \det(w) e^{w\gamma(H)}.$$

Rephrased, the claim becomes the assertion that

$$\chi_\Pi(\exp H)\Delta(H) = A_{\lambda_\Pi+\rho}(H) \qquad (H \in \Xi)$$

for some $\lambda_\Pi \in d\hat{T}$ subject to $\lambda_\Pi + \rho \in C$.

<u>3:</u> NOTATION $d\hat{T}(C)$ is the subset of $d\hat{T}$ consisting of those λ such that $\lambda + \rho \in C$, say $d\hat{T}(C) = \{\lambda_k\}$.

[Note: It turns out that $d\hat{T}(C) = d\hat{T} \cap \bar{C}$ (cf. #9).]

<u>4:</u> LEMMA There exist integers m_k such that $\forall\, H \in \Xi$,

$$\chi_\Pi(\exp H)\Delta(H) = \sum_k m_k A_{\lambda_k+\rho}(H).$$

[Note: The point of departure is the fact that $\chi_\Pi | T$ decomposes as a finite sum

$$\sum_{\lambda \in d\hat{T}} n_\lambda \xi_\lambda \qquad (n_\lambda \in Z_{\geq 0}).]$$

Proceeding,

$$1 = \int_G |\chi_\Pi(x)|^2 d_G(x)$$

$$= \frac{1}{|W|} \int_T |\Delta(t)|^2 |\chi_\Pi(t)|^2 d_T(t) \qquad \text{(cf. §9, \#13).}$$

5: N.B. The function

$$|\sum_k m_k A_{\lambda_k+\rho}|^2$$

descends to T (because $|\Delta|^2$ descends to T (cf. §9, #10)).

Therefore

$$1 = \frac{1}{|W|} \int_T |\sum_k m_k A_{\lambda_k+\rho}|^2 d_T(t).$$

6: LEMMA The function

$$A_{\lambda_k+\rho} \overline{A_{\lambda_{k'}+\rho}}$$

$$= (e^{-\rho} A_{\lambda_k+\rho}) \overline{(e^{-\rho} A_{\lambda_{k'}+\rho})}$$

descends to T (cf. §9, #6).

Therefore

$$\frac{1}{|W|} \int_T A_{\gamma_k+\rho} \overline{A_{\gamma_{k'}+\rho}} d_T(t)$$

$$= \frac{1}{|W|} \int_T (e^{-\rho} A_{\lambda_k+\rho}) \overline{(e^{-\rho} A_{\lambda_{k'}+\rho})} d_T(t)$$

$$= \frac{1}{|W|} \sum_{w,w' \in W} \det(ww') \int_T \xi_{w(\lambda_k+\rho)-\rho} \, \xi_{-w'(\lambda_{k'}+\rho)+\rho} d_T(t).$$

And

$$\int_T \xi_{w(\lambda_k+\rho)-\rho} \, \xi_{-(w'(\lambda_{k'}+\rho)-\rho)} d_T(t)$$

$$= 1 \iff w(\lambda_k+\rho)-\rho = w'(\lambda_{k'}+\rho)-\rho$$

$$\iff w(\lambda_k+\rho) = w'(\lambda_{k'}+\rho)$$

$$\iff w = w' \text{ and } k = k'$$

but is zero otherwise.

Therefore

$$\frac{1}{|W|} \int_T A_{\lambda_k+\rho} \overline{A_{\lambda_{k'}+\rho}} \, d_T(t) = \begin{cases} 1 \text{ if } k = k' \\ 0 \text{ if } k \neq k'. \end{cases}$$

Matters then reduce to the equation

$$1 = \sum_k m_k^2.$$

However, the $m_k \in Z$, hence all but one are zero. Consequently, there is a $\lambda_\Pi \in d\hat{T}$ subject to $\lambda_\Pi + \rho \in C$ such that $\forall\, H \in \Xi$,

$$\chi_\Pi(\exp H)\Delta(H) = \pm A_{\lambda_\Pi+\rho}(H).$$

<u>7:</u> LEMMA The A_γ $(\gamma \in C)$ are linearly independent over Z.

[Given $\gamma,\gamma' \in C$,

$$\langle A_\gamma, A_{\gamma'}\rangle = \begin{cases} 1 \text{ if } \gamma = \gamma' \\ 0 \text{ if } \gamma \neq \gamma', \end{cases}$$

the inner product < , > being by definition the multiplicity of the "zero weight" in

$$\frac{1}{|W|}\Big[\sum_{w\in W} \det(w)e^{w\gamma}\Big]\Big[\sum_{w'\in W} \det(w')e^{-w'\gamma'}\Big]$$

$$= \frac{1}{|W|}\sum_{w,w'\in W} \det(ww')e^{w\gamma-w'\gamma'}.$$

But

$$w\gamma - w'\gamma' = 0 \Rightarrow \gamma = w^{-1}w'\gamma'$$

$$\Rightarrow w = w' \Rightarrow \gamma = \gamma',$$

so the number of solutions is $|W|$ if $\gamma = \gamma'$ and is zero otherwise.]

<u>8:</u> APPLICATION The linear function $\lambda_\Pi + \rho \in C$ is unique.

<u>9:</u> LEMMA Let $\lambda \in d\hat{T}$ -- then

$$\lambda + \rho \in C \Leftrightarrow \lambda \in \bar{C}.$$

PROOF $\forall\ \alpha_i \in \Psi(C)$,

$$2\,\frac{B(\rho,\alpha_i)}{B(\alpha_i,\alpha_i)} = 1 \quad \text{(cf. §8, \#40)}$$

and

$$2\,\frac{B(\lambda,\alpha_i)}{B(\alpha_i,\alpha_i)} \in Z \quad (\lambda \in d\hat{T} \subset L_{wt}).$$

The stated equivalence then follows upon writing

$$2\,\frac{B(\lambda+\rho,\alpha_i)}{B(\alpha_i,\alpha_i)} = 2\,\frac{B(\rho,\alpha_i)}{B(\alpha_i,\alpha_i)} + 2\,\frac{B(\lambda,\alpha_i)}{B(\alpha_i,\alpha_i)}$$

$$= 1 + 2\,\frac{B(\lambda,\alpha_i)}{B(\alpha_i,\alpha_i)}\,.$$

10: APPLICATION

$$\lambda_\Pi + \rho \in C \Rightarrow \lambda_\Pi \in \bar{C}.$$

Return now to the expression

$$\chi_\Pi(\exp H)\,\Delta(H) = \pm A_{\lambda_\Pi+\rho}(H)$$

valid for $H \in \Xi$, the objective then being to establish that it is the plus sign which obtains.

11: LEMMA $\forall\, H \in \mathfrak{t}$,

$$\Delta(H) = \sum_{w\in W} \det(w)\, e^{w\rho(H)}\,.$$

[Note: There is no vicious circle here in that the formula can be derived by direct (albeit somewhat tedious) manipulation, the derivation being independent of the preceding considerations (but consistent with the final outcome).]

From this it follows that $\forall\, H \in \Xi$,

$$\pm\,\chi_\Pi(\exp H) = \frac{\sum_{w\in W} \det(w)\, e^{w(\lambda_\Pi+\rho)(H)}}{\sum_{w\in W} \det(w)\, e^{w\rho(H)}}$$

$$= \Psi_{\lambda_\Pi}(H).$$

12: NOTATION Define $H_\rho \in \sqrt{-1}\,\mathfrak{t}$ by the relation

$$\rho(H) = B(H,H_\rho)\ (H \in \sqrt{-1}\,\mathfrak{t}) \qquad \text{(cf. §6, #21)}.$$

13: LEMMA $\sqrt{-1}\ t\ H_\rho \in \Xi$ for small positive t.

14: LEMMA

$$\lim_{t \downarrow 0} \chi_\Pi(\exp \sqrt{-1}\ t\ H_\rho) = d_\Pi.$$

[For $\chi_\Pi | T$ is continuous and $d_\Pi = \chi_\Pi(e)$.]

15: APPLICATION

$$\pm\, d_\Pi = \lim_{t \downarrow 0} \Psi_{\lambda_\Pi}(\sqrt{-1}\ t\ H_\rho).$$

16: SUBLEMMA $\forall\ w \in W$,

$$w(\lambda_\Pi + \rho)(\sqrt{-1}\ t\ H_\rho)$$

$$= \sqrt{-1}\ t\ (\lambda_\Pi + \rho)(w^{-1}H_\rho)$$

$$= \sqrt{-1}\ t\ B(H_{\lambda_\Pi+\rho}, w^{-1}H_\rho)$$

$$= \sqrt{-1}\ t\ B(wH_{\lambda_\Pi+\rho}, H_\rho)$$

$$= \sqrt{-1}\ t\ \rho(wH_{\lambda_\Pi+\rho})$$

$$= (w^{-1}\rho)(\sqrt{-1}\ t\ H_{\lambda_\Pi+\rho}).$$

<u>17:</u> LEMMA

$$\lim_{t\downarrow 0} \Psi_{\lambda_\Pi}(\sqrt{-1}\ t\ H_\rho) = \frac{\prod_{\alpha\in\Phi^+} B(\lambda_\Pi+\rho,\alpha)}{\prod_{\alpha\in\Phi^+} B(\rho,\alpha)}.$$

PROOF Write

$$\sum_{w\in W} \det(w) e^{w(\lambda_\Pi+\rho)(\sqrt{-1}\ t\ H_\rho)}$$

$$= \sum_{w\in W} \det(w) e^{(w^{-1}\rho)(\sqrt{-1}\ t\ H_{\lambda_\Pi+\rho})}$$

$$= \sum_{w\in W} \det(w^{-1}) e^{(w^{-1}\rho)(\sqrt{-1}\ t\ H_{\lambda_\Pi+\rho})}$$

$$= \sum_{w\in W} \det(w) e^{(w\rho)(\sqrt{-1}\ t\ H_{\lambda_\Pi+\rho})}$$

$$= \Delta(\sqrt{-1}\ t\ H_{\lambda_\Pi+\rho})$$

$$= \prod_{\alpha\in\Phi^+} (e^{\alpha(\sqrt{-1}\ t\ H_{\lambda_\Pi+\rho})/2} - e^{-\alpha(\sqrt{-1}\ t\ H_{\lambda_\Pi+\rho})/2})$$

$$= \prod_{\alpha\in\Phi^+} \sqrt{-1}\ t\ \alpha(H_{\lambda_\Pi+\rho}) + o(1)$$

$$= (\sqrt{-1}\ t)^{|\Phi^+|} \prod_{\alpha\in\Phi^+} B(\lambda_\Pi+\rho,\alpha) + o(1).$$

Analogously,

$$\sum_{w\in W} \det(w) e^{w\rho(\sqrt{-1}\ t\ H_\rho)}$$

$$= (\sqrt{-1}\, t)^{|\Phi^+|} \prod_{\alpha\in\Phi^+} B(\rho,\alpha) + o(1).$$

Taking the limit as $t \downarrow 0$ then finishes the proof.

18: N.B. Both ρ and $\lambda_\Pi + \rho$ belong to C, thus $\forall\ \alpha \in \Phi^+$,

$$B(\rho,\alpha) > 0 \text{ and } B(\lambda_\Pi + \rho,\alpha) > 0,$$

so

$$\lim_{t\downarrow 0} \Psi_{\lambda_\Pi}(\sqrt{-1}\, t\, H_\rho) > 0.$$

19: APPLICATION

$$d_\Pi = \lim_{t\downarrow 0} \Psi_{\lambda_\Pi}(\sqrt{-1}\, t\, H_\rho).$$

I.e.: The plus sign prevails.

20: SCHOLIUM

$$d_\Pi = \frac{\prod_{\alpha\in\Phi^+} B(\lambda_\Pi + \rho,\alpha)}{\prod_{\alpha\in\Phi^+} B(\rho,\alpha)}.$$

21: LEMMA The arrow from $\hat{G}$ to $d\hat{T} \cap \bar{C}$ that sends Π to λ_Π is well-defined (cf. #8) and injective.

PROOF Given $\Pi_1,\Pi_2 \in \hat{G}$, suppose that $\lambda_{\Pi_1} = \lambda_{\Pi_2}$ -- then $\lambda_{\Pi_1} + \rho = \lambda_{\Pi_2} + \rho$, hence

$$\Psi_{\lambda_{\Pi_1}} = \Psi_{\lambda_{\Pi_2}},$$

which implies that $\chi_{\Pi_1} = \chi_{\Pi_2}$ on G^{reg} or still, by continuity, $\chi_{\Pi_1} = \chi_{\Pi_2}$ on G, so $\Pi_1 = \Pi_2$.

<u>22:</u> LEMMA The arrow from $\hat{G}$ to $d\hat{T} \cap \bar{C}$ that sends Π to λ_Π is surjective.

PROOF Fix a $\lambda \in d\hat{T} \cap \bar{C}$ -- then

$$\int_G |\Psi_\lambda(x)|^2 d_G(x) = \frac{1}{|W|} \int_{T^{reg}} |\Delta(t)|^2 |\Psi_\lambda(t)|^2 d_T(t)$$

$$= \frac{1}{|W|} \int_T A_{\lambda+\rho} \overline{A_{\lambda+\rho}} \, d_T(t)$$

$$= 1.$$

Therefore Ψ_λ is an L^2 class function (cf. §2, #17). Now fix a $\Pi_0 \in \hat{G}$:

$$<\Psi_\lambda, \chi_{\Pi_0}>$$

$$= \int_G \Psi_\lambda(x) \overline{\chi_{\Pi_0}(x)} d_G(x)$$

$$= \frac{1}{|W|} \int_{T^{reg}} |\Delta(t)|^2 \Psi_\lambda(t) \overline{\chi_{\Pi_0}(t)} d_T(t)$$

$$= \frac{1}{|W|} \int_{T^{reg}} |\Delta(t)|^2 \Psi_\lambda(t) \overline{\Psi_{\lambda_{\Pi_0}}(t)} d_T(t)$$

$$= \begin{cases} 1 \text{ if } \lambda_{\Pi_0} = \lambda \\ 0 \text{ if } \lambda_{\Pi_0} \neq \lambda \end{cases}$$

=>

$$\Psi_\lambda = \sum_{\Pi \in \hat{G}} <\Psi_\lambda, \chi_\Pi> \chi_\Pi \qquad \text{(cf. §2, #19)}$$

$$= \chi_\Pi$$

for a unique $\Pi \in \hat{G}$ with $\lambda_\Pi = \lambda$.

<u>23:</u> SCHOLIUM

$$\left[\begin{array}{l} \hat{G} \longleftrightarrow d\hat{T} \cap \bar{C} \\ \\ \Pi \longleftrightarrow \lambda_{\Pi}. \end{array}\right.$$

[Note: W operates on $d\hat{T}$ and $d\hat{T} \cap \bar{C}$ is a fundamental domain for this action (cf. §8, #23), hence $\hat{G}$ is parametrized by the orbits of W in $d\hat{T}$.]

<u>24:</u> <u>N.B.</u>

$$\lambda_{\Pi^*} = - w^{\circ} \lambda_{\Pi} \quad \text{(cf. §8, \#22)}.$$

<u>25:</u> REMARK It is clear that if $\lambda_{\Pi} = 1_G$, then $\lambda_{\Pi} = 0$.

<u>26:</u> LEMMA In the restriction of χ_{Π} to T, $\xi_{\lambda_{\Pi}}$ occurs with multiplicity 1.

APPENDIX

There are two directions in which the theory can be extended.

- Drop the assumption that G is semisimple and work with an arbitrary compact connected Lie group.

- Drop the assumption that G is connected and work with an arbitrary compact Lie group.

As regards the first point, no essential difficulties are encountered. As regards the second point, however, there are definitely some subtleties (see Chapter 1 of D. Vogan's book "Unitary Representations of Reductive Lie Groups").

NOTATION Let G be a compact semisimple Lie group, $T \subset G$ a maximal torus,

$C \subset \sqrt{-1}\,\mathfrak{t}$ a Weyl chamber and let

$$N_G(C) = \{x \in G: Ad(x)C \subset C\}.$$

LEMMA

$$N_G(C) \cap G^0 = T,\ N_G(C)G^0 = G.$$

Therefore

$$G/G^0 \approx N_G(C)/T.$$

N.B. Each element of G is conjugate to an element of $N_G(C)$.

§11. *THE INVARIANT INTEGRAL*

Let G be a compact connected semisimple Lie group, $T \subset G$ a maximal torus etc.

1: NOTATION Set

$$\pi = \prod_{\alpha \in \Phi^+} \alpha \equiv \prod_{\alpha > 0} \alpha.$$

2: LEMMA π is a homogeneous polynomial of degree $r(= |\Phi^+|)$ and $\forall\, w \in W$,

$$w\pi = \det(w)\pi.$$

3: LEMMA If p is a homogeneous polynomial such that $\forall\, w \in W$,

$$wp = \det(w)p,$$

then p can be written as πP, where P is a homogeneous W-invariant polynomial.

4: N.B. $P = 0$ if $\deg p < r$ and $P = C$ (a constant) if $\deg p = r$.

5: DEFINITION Given $f \in C^\infty(\mathfrak{g})$ and $H \in \mathfrak{t}$, put

$$\phi_f(H) = \pi(H) \int_G f(\mathrm{Ad}(x)H)\,d_G(x),$$

the invariant integral of f at H.

6: FUNCTIONAL EQUATION $\forall\, w \in W$ $(w = nT)$,

$$\begin{aligned}\phi_f(wH) &= \pi(wH) \int_G f(\mathrm{Ad}(x)wH)\,d_G(x)\\ &= \det(w)\pi(H) \int_G f(\mathrm{Ad}(x)\mathrm{Ad}(n)H)\,d_G(x)\\ &= \det(w)\pi(H) \int_G f(\mathrm{Ad}(xn)H)\,d_G(x)\\ &= \det(w)\pi(H) \int_G f(\mathrm{Ad}(x)H)\,d_G(x)\end{aligned}$$

$$= \det(w)\phi_f(H).$$

7: LEMMA

$$f \in C^\infty(\mathfrak{g}) \Rightarrow \phi_f \in C^\infty(\mathfrak{t}).$$

8: LEMMA

$$f \in C_c^\infty(\mathfrak{g}) \Rightarrow \phi_f \in C_c^\infty(\mathfrak{t}).$$

9: LEMMA

$$f \in \mathcal{C}(\mathfrak{g}) \Rightarrow \phi_f \in \mathcal{C}(\mathfrak{t}).$$

[If $D \in \mathcal{P}(\mathfrak{g})$ is a polynomial differential operator, then there exist a finite number of elements $D_1,\ldots,D_p \in \mathcal{P}(\mathfrak{g})$ and analytic functions $a_1,\ldots,a_p$ on G such that $\forall\ x \in G$,

$$\mathrm{Ad}(x)D = \sum_{i=1}^{p} a_i(x)D_i.$$

[Note: An automorphism of $\mathfrak{g}$ extends to an automorphism of $\mathcal{P}(\mathfrak{g})$.]

10: NOTATION Set

$$\tilde{\pi} = \prod_{\alpha\in\Phi^+} H_\alpha \equiv \prod_{\alpha>0} H_\alpha.$$

11: N.B. $\partial(\tilde{\pi})(\pi)$ is a constant (explicated infra).

[The point is that π is a homogeneous polynomial of degree r and $\partial(\tilde{\pi})$ is a polynomial differential operator of degree r.]

12: RAPPEL For the record,

$$\partial H_\alpha(f)\Big|_H = \frac{d}{dt} f(H + tH_\alpha)\Big|_{t=0}.$$

In particular, if f is linear, then

$$\partial H_\alpha(f)\Big|_H = f(H_\alpha),$$

a constant.]

Put

$$F(H) = \int_G f(\mathrm{Ad}(x)H)d_G(x) \qquad (H \in \mathfrak{t}).$$

Then

$$(\partial(\tilde{\pi}) \circ \pi)F\Big|_{H=0} = F(H;\partial(\tilde{\pi}) \circ \pi)\Big|_{H=0}$$

$$= \partial(\tilde{\pi})(\pi)F(0)$$

$$= \partial(\tilde{\pi})(\pi)f(0).$$

<u>13:</u> THEOREM

$$\partial(\tilde{\pi})(\pi) = |W| \prod_{\alpha>0} B(\rho,\alpha).$$

PROOF The sum

$$\sum_{w\in W} \det(w)(w\rho)^k$$

is a homogeneous polynomial of degree k which transforms according to the determinant per the action of W, hence vanishes if $0 \leq k < r$ but if $k = r$,

$$\frac{1}{r!} \sum_{w\in W} \det(w)(w\rho)^r = C(\rho)\pi$$

for some constant $C(\rho)$ (cf. #4). To calculate $C(\rho)$, note that ρ^r is a homogeneous polynomial of degree r, thus $\partial(\tilde{\pi})(\rho)^r$ is a constant, so

$$\partial(\tilde{\pi})(\det(w)(w\rho)^r)$$

$$= w(\partial(\tilde{\pi})(\rho)^r)$$

$$= \partial(\tilde{\pi})(\rho)^r$$

$$= \prod_{\alpha>0} \partial(H_\alpha)(\rho)^r$$

$$= r! \prod_{\alpha>0} B(\rho,\alpha).$$

Therefore, on the one hand,

$$\partial(\tilde{\pi})\left(\frac{1}{r!}\sum_{w\in W} \det(w)(w\rho)^r\right)$$

$$= \frac{1}{r!}\sum_{w\in W} \partial(\tilde{\pi})(\det(w)(w\rho)^r)$$

$$= \frac{1}{r!}|W|r! \prod_{\alpha>0} B(\rho,\alpha)$$

$$= |W| \prod_{\alpha>0} B(\rho,\alpha),$$

while on the other

$$\partial(\tilde{\pi})\left(\frac{1}{r!}\sum_{w\in W} \det(w)(w\rho)^r\right)$$

$$= C(\rho)\partial(\tilde{\pi})(\pi).$$

Consequently,

$$|W| \prod_{\alpha>0} B(\rho,\alpha) = C(\rho)\partial(\tilde{\pi})(\pi)$$

$\Rightarrow$

$$\frac{1}{r!}\sum_{w\in W} \det(w)(w\rho)^r$$

$$= \frac{|W| \prod_{\alpha>0} B(\rho,\alpha)}{\partial(\tilde{\pi})(\pi)}\pi.$$

Let $H = \sqrt{-1}\ t\ H_\rho$ (cf. §10, #13) and write $\lim\limits_{H \to 0}$ in place of $\lim\limits_{t\downarrow 0}$:

$$1 = \frac{\Delta(H)}{\Delta(H)}$$

$$= \frac{\sum\limits_{w\in W} \det(w)\, e^{w\rho(H)}}{\prod\limits_{\alpha>0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}$$

$$= \lim_{H \to 0} \frac{\sum\limits_{w\in W} \det(w)\, e^{w\rho(H)}}{\prod\limits_{\alpha>0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}$$

$$= \lim_{H \to 0} \frac{\sum\limits_{w\in W} \det(w)\, e^{w\rho(H)}}{e^{-\rho(H)} \prod\limits_{\alpha>0} (e^{\alpha(H)} - 1)}$$

$$= \lim_{H \to 0} \left[\frac{e^{\rho(H)}}{\prod\limits_{\alpha>0} \frac{e^{\alpha(H)} - 1}{\alpha(H)}} \times \frac{\sum\limits_{w\in W} \det(w)\, e^{w\rho(H)}}{\pi(H)} \right]$$

$$= \lim_{H \to 0} \frac{\sum\limits_{w\in W} \det(w)\, e^{w\rho(H)}}{\pi(H)}$$

which upon expansion of the exponentials equals

$$\lim_{H \to 0} (C(\rho) + o(1)) = C(\rho)$$

=>

$$1 = C(\rho) = \frac{|W| \prod_{\alpha>0} B(\rho,\alpha)}{\partial(\tilde{\pi})(\pi)}$$

=>

$$\partial(\tilde{\pi})(\pi) = |W| \prod_{\alpha>0} B(\rho,\alpha).$$

<u>14:</u> APPLICATION Given $f \in C^{\infty}(\mathfrak{g})$,

$$\begin{aligned} \phi_f(0;\partial(\tilde{\pi})) &= (\partial(\tilde{\pi})\phi_f)(0) \\ &= (|W| \prod_{\alpha>0} B(\rho,\alpha))f(0). \end{aligned}$$

§12. *PLANCHEREL*

Keeping to the overall setup of §11, assume in addition that G is simply connected, so

$$L_{wt} = d\hat{T} \quad (\text{cf. } §7, \#17)$$

and e^ρ descends to T, so does Δ, thus

$$\Delta(t) = \xi_\rho(t) \prod_{\alpha>0} (1 - \xi_\alpha(t^{-1})) \quad (t \in T).$$

<u>1:</u> NOTATION Put

$$\mathcal{W} = L_{wt}, \ \mathcal{W}^+ = L_{wt} \cap \bar{C}.$$

<u>2:</u> <u>N.B.</u> The elements of $\mathcal{W}^+$ are the dominant weights (cf. §8, #34).

<u>3:</u> NOTATION Given $\Lambda \in \mathcal{W}^+$, Π_Λ is the irreducible unitary representation of G associated with Λ, χ_Λ its character,

$$d_\Lambda = \frac{\prod_{\alpha>0} B(\Lambda+\rho,\alpha)}{\prod_{\alpha>0} B(\rho,\alpha)}$$

its dimension (cf. §10, #20).

<u>4:</u> <u>N.B.</u> On T^{reg},

$$\chi_\Lambda(t)\Delta(t) = \sum_{w\in W} \det(w)\xi_{w(\Lambda+\rho)}(t).$$

It is wellknown that

$$C^\infty(G) * C^\infty(G) = C^\infty(G),$$

so on the basis of §2, #15, the Plancherel theorem is in force:

$$f(e) = \sum_{\Pi \in \hat{G}} d_\Pi \operatorname{tr}(\Pi(f))$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \int_G f(x) \chi_\Pi(x) d_G(x)$$

or still,

$$f(e) = \sum_{\Lambda \in \mathcal{W}^+} d_\Lambda \int_G f(x) \chi_\Lambda(x) d_G(x).$$

Our objective now will be to give another proof of this relation which is independent of the factorization theory for $C^\infty(G)$ but hinges instead on the result formulated in §11, #13.

5: NOTATION Given $f \in C^\infty(G)$ and $t \in T$, put

$$F_f(t) = \Delta(t) \int_G f(xtx^{-1}) d_G(x),$$

the invariant integral of f at t.

6: LEMMA

$$F_f \in C^\infty(T).$$

Owing to §9, #12,

$$\int_G f(x) d_G(x) = \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G f(xtx^{-1}) d_G(x) d_T(t)$$

which equals

$$\frac{1}{|W|} \int_T \overline{\Delta(t)} \Delta(t) \int_G f(xtx^{-1}) d_G(x) d_T(t)$$

or still,

$$\frac{1}{|W|} \int_T \overline{\Delta(t)} F_f(t) d_T(t)$$

or still,

$$\frac{(-1)^r}{|W|} \int_T \Delta(t) F_f(t) d_T(t) \qquad (r = |\Phi^+|).$$

Therefore

$$\sum_{\Lambda \in \mathcal{W}^+} d_\Lambda \int_G f(x) \chi_\Lambda(x) d_G(x)$$

$$= \frac{(-1)^r}{|W|} \sum_{\Lambda \in \mathcal{W}^+} d_\Lambda \int_T \Delta(t) \chi_\Lambda(t) F_f(t) d_T(t).$$

7: LEMMA $\forall\, w \in W$,

$$\det(w) \prod_{\alpha>0} B(\Lambda+\rho,\alpha) = \prod_{\alpha>0} B(w(\Lambda+\rho),\alpha).$$

Proceeding,

$$\frac{(-1)^r}{|W|} \sum_{\Lambda \in \mathcal{W}^+} d_\Lambda \int_T \Delta(t) \chi_\Lambda(t) F_f(t) d_T(t)$$

$$= \frac{(-1)^r}{|W|} \sum_{\Lambda \in \mathcal{W}^+} d_\Lambda \int_T \sum_{w \in W} \det(w) \xi_{w(\Lambda+\rho)}(t) F_f(t) d_T(t)$$

$$= \frac{(-1)^r}{|W| \prod_{\alpha>0} B(\rho,\alpha)} \sum_{\Lambda \in \mathcal{W}^+} \sum_{w \in W} \int_T \det(w) \prod_{\alpha>0} B(\Lambda+\rho,\alpha) \xi_{w(\Lambda+\rho)}(t) F_f(t) d_T(t)$$

$$= \frac{(-1)^r}{|W| \prod_{\alpha>0} B(\rho,\alpha)} \sum_{\Lambda \in \mathcal{W}^+} \sum_{w \in W} \int_T \prod_{\alpha>0} B(w(\Lambda+\rho),\alpha) \xi_{w(\Lambda+\rho)}(t) F_f(t) d_T(t)$$

$$= \frac{(-1)^r}{|W| \prod_{\alpha>0} B(\rho,\alpha)} \sum_{\lambda \in \mathcal{W}} \int_T \prod_{\alpha>0} B(\lambda,\alpha) \xi_\lambda(t) F_f(t) d_T(t),$$

the $\lambda \in \mathcal{W}$ for which $\prod_{\alpha>0} B(\lambda,\alpha) = 0$ making no contribution.

8: REMARK The elements $\lambda \in \mathcal{W}$ such that $w\lambda \neq \lambda$ when $w \neq e$ $(w \in W)$ are in a one-to-one correspondence with the pairs $(\Lambda,w) \in \mathcal{W}^+ \times W$ via the arrow $(\Lambda,w) \rightarrow w(\Lambda+\rho)$.

To isolate $f(e)$, put $\check{f}(X) = f(\exp X)$ $(X \in \mathfrak{g})$ -- then $\check{f} \in C^\infty(\mathfrak{g})$ and $\forall\ H \in \mathfrak{t}$,

$$F_f(\exp H) = \Delta(\exp H) \int_G f(x(\exp H)x^{-1})d_G(x)$$

$$= \Delta(\exp H) \int_G f(\exp(\mathrm{Ad}(x)H))d_G(x)$$

$$= \Delta(\exp H) \int_G \check{f}(\mathrm{Ad}(x)H)d_G(x).$$

9: LEMMA Let λ be a linear function on $\mathfrak{t}_C$ -- then there exists a unique Ad G invariant analytic function Γ_λ on $\mathfrak{g}$ such that $\forall\ H \in \mathfrak{t}$,

$$\Gamma_\lambda(H)\pi(H) = \sum_{w\in W} \det(w)e^{w\lambda(H)}.$$

10: APPLICATION Take $\lambda = \rho$ -- then there exists a unique Ad G invariant analytic function Γ_ρ on $\mathfrak{g}$ such that $\forall\ H \in \mathfrak{t}$,

$$\Gamma_\rho(H)\pi(H) = \sum_{w\in W} \det(w)e^{w\rho(H)}$$

$$= \Delta(H) \quad (\text{cf. §10, \#11}).$$

Therefore

$$\Delta(\exp H) \int_G \check{f}(\mathrm{Ad}(x)H)d_G(x)$$

$$= \Gamma_\rho(H)\pi(H) \int_G \check{f}(\mathrm{Ad}(x)H)d_G(x)$$

$$= \pi(H)\Gamma_\rho(H) \int_G \check{f}(\mathrm{Ad}(x)H)d_G(x)$$

$$= \pi(H) \int_G \Gamma_\rho(H)\check{f}(\mathrm{Ad}(x)H)d_G(x)$$

$$= \pi(H) \int_G \Gamma_\rho(\mathrm{Ad}(x)H)\check{f}(\mathrm{Ad}(x)H)d_G(x)$$

$$= \phi_{\Gamma_\rho \check{f}}(H).$$

Summary: $\forall\ H \in \mathfrak{t}$,

$$F_f(\exp H) = \phi_{\Gamma_\rho \check{f}}(H).$$

<u>11:</u> SUBLEMMA In $\{H:\pi(H)\pi \neq 0\}$,

$$\Gamma_\rho(0) = \lim_{H \to 0} \frac{\Delta(H)}{\pi(H)}$$

$$= \lim_{H \to 0} \frac{\prod_{\alpha>0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}{\prod_{\alpha>0} \alpha(H)}$$

$$= \lim_{H \to 0} \prod_{\alpha>0} \left[\frac{e^{\alpha(H)/2} - e^{-\alpha(H)/2}}{\alpha(H)} \right]$$

$$= 1.$$

Next

$$F_f(\exp H) = \phi_{\Gamma_\rho \check{f}}(H)$$

=>

$$\partial(\tilde{\pi})F_f(\exp H) = \partial(\tilde{\pi})\phi_{\Gamma_\rho \check{f}}(H)$$

=>

$$(\partial(\tilde{\pi})F_f \circ \exp)(0) = (\partial(\tilde{\pi})\phi_{\Gamma_\rho \check{f}})(0)$$

$$= (|W| \prod_{\alpha>0} B(\rho,\alpha))(\Gamma_\rho \check{f})(0) \qquad \text{(cf. §11, \#14).}$$

And

$$(\Gamma_\rho \check{f})(0) = \Gamma_\rho(0)\check{f}(0) = \check{f}(0) = f(e).$$

Therefore

$$f(e) = \frac{1}{|W| \prod_{\alpha>0} B(\rho,\alpha)} \lim_{H \to 0} F_f(\exp H; \partial(\tilde{\pi})).$$

12: NOTATION Given $\lambda \in \mathcal{W}$, put

$$\hat{F}_f(\lambda) = \int_T F_f(t)\xi_\lambda(t)d_T(t),$$

the Fourier transform of F_f.

13: N.B. Assume that the Haar measure on $\hat{T}$ is normalized so that Fourier inversion is valid (thus each $\lambda \in \mathcal{W}$ is assigned mass 1).

Write

$$\lim_{H \to 0} F_f(\exp H; \partial(\tilde{\pi}))$$

$$= \int_{\hat{T}} \hat{F}_f(\lambda) \lim_{H \to 0} \xi_{-\lambda}(\exp H; \partial(\tilde{\pi}))d_{\hat{T}}(\lambda)$$

$$= \int_{\hat{T}} \hat{F}_f(\lambda) \lim_{H \to 0} \partial(\tilde{\pi})e^{-\lambda(H)}d_{\hat{T}}(\lambda)$$

$$= (-1)^r \int_{\hat{T}} \hat{F}_f(\lambda) \prod_{\alpha>0} B(\lambda,\alpha) d_{\hat{T}}(\lambda)$$

$$= (-1)^r \int_{\hat{T}} \prod_{\alpha>0} B(\lambda,\alpha) \hat{F}_f(\lambda) d_{\hat{T}}(\lambda)$$

$$= (-1)^r \int_{\hat{T}} \prod_{\alpha>0} B(\lambda,\alpha) \left(\int_T F_f(t) \xi_\lambda(t) d_T(t)\right) d_{\hat{T}}(\lambda)$$

$$= (-1)^r \sum_{\lambda\in\mathcal{W}} \int_T B(\lambda,\alpha) \xi_\lambda(t) F_f(t) d_T(t).$$

Therefore

$$f(e) = \frac{(-1)^r}{|W| \prod_{\alpha>0} B(\rho,\alpha)} \sum_{\lambda\in\mathcal{W}} \int_T B(\lambda,\alpha) \xi_\lambda(t) F_f(t) d_T(t)$$

$$= \sum_{\Lambda\in\mathcal{W}^+} d_\Lambda \int_G f(x) \chi_\Lambda(x) d_G(x),$$

the relation at issue.

§13. *DETECTION*

Let G be a compact group.

1: DEFINITION The character ring X(G) is the free abelian group generated by the irreducible characters of G (i.e., by the χ_Π ($\Pi \in \hat{G}$)) under pointwise addition and multiplication with unit 1_G.

2: DEFINITION An element of X(G) is called a virtual character.

3: NOTATION CL(G) is the subspace of C(G) comprised of the continuous class functions (cf. §2, #27).

4: LEMMA A class function $f \in CL(G)$ is a virtual character of G iff

$$<f,\chi_\Pi> = \int_G f(x)\overline{\chi_\Pi(x)}d_G(x) \in Z$$

for all $\Pi \in \hat{G}$.

PROOF The condition is obviously necessary. As for its sufficiency, we have

$$||f||^2 = \sum_{\Pi\in\hat{G}} |<f,\chi_\Pi>|^2 \quad \text{(cf. §2, \#19)},$$

hence

$$<f,\chi_\Pi> = 0$$

for all but finitely many χ_Π, say $\chi_{\Pi_1},\ldots,\chi_{\Pi_n}$ and then

$$f = \sum_{i=1}^{n} <f,\chi_{\Pi_i}>\chi_{\Pi_i} \quad \text{(ibid.)}.$$

[Note: A priori, this is an equality in the L^2-sense, hence is valid almost everywhere. But both sides are continuous, thus the equality is valid everywhere.]

Let G be a compact connected Lie group.

5: NOTATION $CL^{\infty}(G)$ is the set of C^{∞} class functions.

6: RAPPEL The characters of G belong to $CL^{\infty}(G)$.

7: N.B. Therefore X(G) is a subring of the ring of C^{∞} functions on G.

8: REMARK Per #4, suppose that $f \in CL(G)$ has the property that

$$<f,\chi_{\Pi}> \in Z$$

for all $\Pi \in \hat{G}$ -- then it follows after the fact that $f \in CL^{\infty}(G)$.

Let $T \subset G$ be a maximal torus and assign to the symbol X(T) the obvious interpretation.

9: RAPPEL The arrow

$$f \rightarrow f|T$$

of restriction defines an isomorphism

$$CL(G) \rightarrow C(T)^{W} \quad \text{(cf. §4, #24)}.$$

10: APPLICATION Restriction to T induces an injective homomorphism

$$X(G) \rightarrow X(T)^{W}.$$

Take a $\phi \in X(T)^{W}$ and let $f \in CL(G)$ be the class function that restricts to ϕ.

11: LEMMA f is a virtual character of G, i.e., $f \in X(G)$.

PROOF With #4 in view, write

$$<f,\chi_{\Pi}> = \int_G f(x)\overline{\chi_{\Pi}(x)}d_G(x)$$

$$= \frac{1}{|W|} \int_T |\Delta(t)|^2 \phi(t)\overline{\chi_\Pi(t)} d_T(t)$$

$$= \frac{1}{|W|} \int_T \Delta(t)\phi(t)\overline{\Delta(t)\chi_\Pi(t)} d_T(t)$$

$$\in \frac{1}{|W|} (|W|\mathbb{Z}) = \mathbb{Z}.$$

12: SCHOLIUM

$$X(G) \approx X(T)^W.$$

13: N.B. Rephrased, a continuous class function $f:G \to \mathbb{C}$ is a virtual character of G iff $f|T$ is a virtual character of T.

14: THEOREM Let $f \in CL(G)$ -- then $f \in X(G)$ iff its restriction to every finite elementary subgroup of G is a virtual character.

PROOF To establish the nontrivial assertion, let $H \subset G$ be a finite subgroup -- then the assumption on f coupled with A, II, §12, #1 implies that $f|H \in X(H)$. Matters can thus be reinforced, the assumption on f becoming that its restriction to every finite subgroup of G is a virtual character and, thanks to what has been said above, one might just as well work with T rather than G. Choose a sequence $H_1 \subset H_2 \subset \cdots$ of finite subgroups of T whose union is dense in T -- then $\forall\ \chi \in X(T)$,

$$\langle f,\chi\rangle_T = \int_T f\bar{\chi} = \lim_{n \to \infty} \left(\frac{1}{|H_n|} \sum_{h \in H_n} f(h)\overline{\chi(h)}\right)$$

$$= \lim_{n \to \infty} \langle f,\chi\rangle_{H_n}.$$

But

$$f|H_n \in X(H_n)$$

$$\Rightarrow \langle f,\chi\rangle_{H_n} \in \mathbb{Z}$$

$$\Rightarrow \langle f,\chi\rangle_T \in \mathbb{Z}.$$

§14. *INDUCTION*

Let G be a finite group, $\Gamma \subset G$ a subgroup.

<u>1:</u> RAPPEL There is an arrow

$$i_{\Gamma \to G}: CL(\Gamma) \to CL(G)$$

which sends characters of Γ to characters of G (cf. A, II, §9, #10), thus induces an arrow

$$X(\Gamma) \to X(G).$$

<u>2:</u> <u>N.B.</u> If

$$G = \coprod_{k=1}^{n} x_k \Gamma$$

and if $\phi \in CL(\Gamma)$ is a class function, then

$$(i_{\Gamma \to G}\phi)(x) = \sum_{k=1}^{n} \overset{\circ}{\phi}(x_k^{-1} x x_k) \quad \text{(cf. A, II, §7, #10)},$$

i.e.,

$$(i_{\Gamma \to G}\phi)(x) = \sum_{k, x_k^{-1} x x_k \in \Gamma} \phi(x_k^{-1} x x_k).$$

Let G be a compact Lie group, $\Gamma \subset G$ a closed Lie subgroup.

<u>3:</u> NOTATION Given an $x \in G$, write $(G/\Gamma)^x$ for the fixed point set of the action of x on G/Γ.

<u>4:</u> LEMMA A coset $y\Gamma$ in G/Γ lies in $(G/\Gamma)^x$ iff $y^{-1}xy \in \Gamma$.

5: LEMMA If cosets $y_1\Gamma$, $y_2\Gamma$ lie in the same connected component of $(G/\Gamma)^x$, then $y_2\Gamma = yy_1\Gamma$ for some y in the centralizer of x.

6: N.B. If $\phi \in CL(\Gamma)$ is a class function and if $y_2 = yy_1\gamma$, then

$$\begin{aligned}\phi(y_2^{-1}xy_2) &= \phi(\gamma^{-1}y_1^{-1}y^{-1}xyy_1\gamma)\\ &= \phi(y_1^{-1}y^{-1}xyy_1)\\ &= \phi(y_1^{-1}xy_1).\end{aligned}$$

Let $C_1,\ldots,C_m$ be the connected components of $(G/\Gamma)^x$, thus

$$(G/\Gamma)^x = \coprod_{j=1}^{m} C_j,$$

let $\chi(C_j)$ be the Euler characteristic of C_j, and fix elements

$$y_1\Gamma \in C_1,\ldots,y_m\Gamma \in C_m.$$

7: NOTATION Given a class function $\phi \in CL(\Gamma)$, put

$$(i_{\Gamma \to G}\phi)(x) = \sum_{j=1}^{m} \chi(C_j)\phi(y_j^{-1}xy_j).$$

8: LEMMA

$$i_{\Gamma \to G}\phi \in CL(G),$$

the induced class function.

9: N.B. Therefore

$$i_{\Gamma \to G}:CL(\Gamma) \to CL(G).$$

<u>10:</u> REMARK The definition of $i_{\Gamma \to G}\phi$ is independent of the choice of representatives $y_j\Gamma$ for the components of $(G/\Gamma)^x$ but it is not quite obvious that $i_{\Gamma \to G}\phi$ is continuous.

<u>11:</u> RECONCILIATION Take the case when G and $\Gamma \subset G$ are finite. Write

$$G = \coprod_{k=1}^{n} x_k\Gamma.$$

Then, as recalled in #2,

$$(i_{\Gamma \to G}\phi)(x) = \sum_{k,\, x_k^{-1}xx_k \in \Gamma} \phi(x_k^{-1}xx_k)$$

which, in view of #4, is equal to

$$\sum_{\substack{y\Gamma \in G/\Gamma \\ xy\Gamma = y\Gamma}} \phi(y^{-1}xy)$$

or still, is equal to

$$\sum_{y\Gamma \in (G/\Gamma)^x} \phi(y^{-1}xy).$$

But here the C_j are points, say

$$C_j = \{y_j\Gamma\} \quad (\Rightarrow \quad (G/\Gamma)^x = \{\{y_1\Gamma\},\ldots,\{y_m\Gamma\}\},$$

so $\chi(C_j) = 1$, thus

$$\sum_{j=1}^{m} \chi(C_j)\phi(y_j^{-1}xy_j) = \sum_{j=1}^{m} \phi(y_j^{-1}xy_j)$$

$$= \sum_{y\Gamma \in (G/\Gamma)^x} \phi(y^{-1}xy).$$

12: RAPPEL A compact connected Lie group of positive dimension has zero Euler characteristic, so the connected components of a compact Lie group of positive dimension have zero Euler characteristic.

13: EXAMPLE Take $\Gamma = \{e\}$, let $\phi = 1_\Gamma$, and assume that $\dim G > 0$ -- then $(G/\Gamma)^x$ is empty of $x \neq e$, hence for such x,

$$(i_{\Gamma \to G}\phi)(x) = 0,$$

but if $x = e$, then $(G/\Gamma)^e = G$ and

$$(i_{\Gamma \to G}\phi)(e) = \sum_{j=1}^{m} \chi(C_j)\phi(y_j^{-1}ey_j)$$

$$= (\sum_{j=1}^{m} \chi(C_j))\phi(e)$$

$$= 0.$$

Therefore

$$i_{\Gamma \to G}\phi = 0.$$

14: DEFINITION A closed subgroup H of G is generic if it is topologically cyclic and of finite index in its normalizer.

[Note: Let G be a compact connected Lie group, $T \subset G$ a maximal torus -- then T is generic.]

15: DEFINITION An element $x \in G$ is generic if it generates a generic subgroup of G.

[Note: Let G be a compact connected Lie group -- then a generic element is necessarily regular.]

16: LEMMA The generic elements are dense in G.

17: THEOREM Suppose that $x \in G$ is generic -- then

$$|(G/\Gamma)^x| > \infty$$

and

$$(i_{\Gamma \to G}\phi)(x) = \sum_{y\Gamma \in (G/\Gamma)^x} \phi(y^{-1}xy).$$

18: EXAMPLE Take $\Gamma = \{e\}$, let $\phi = 1_\Gamma$, and assume that $\dim G > 0$ -- then at every generic element of G,

$$(i_{\Gamma \to G}\phi)(x) = 0,$$

hence by continuity (in conjunction with #16),

$$i_{\Gamma \to G}\phi = 0 \quad \text{(cf. \#13)}.$$

Let G be a compact Lie group, let $\Gamma_1, \Gamma_2 \subset G$ be closed Lie subgroups, and let

$$G = \bigcup_{s \in S} \Gamma_1 s \Gamma_2$$

be a double coset decomposition of G.

19: N.B.

$$\Gamma_1 \backslash G / \Gamma_2$$

is the orbit space per the action of Γ_1 by left translation on G/Γ_2.

Write

$$\Gamma_1 \backslash G / \Gamma_2 = \coprod_{s \in S} U_s,$$

where each U_s is a connected component of one orbit type for the action of Γ_1 on

G/Γ_2. Fix elements $x_s \in G$ such that $\Gamma_1 x_s \Gamma_2 \in U_s$ and for each s let

$$\Phi_s : CL(\Gamma_2) \to CL(\Gamma_1)$$

denote the following composite: Take a $\phi \in CL(\Gamma_2)$ and form $\phi^s \equiv \phi \circ I_{x_s^{-1}}$ (a class function on $x_s \Gamma_2 x_s^{-1}$), then restrict ϕ^s to $\Gamma_2(s) \equiv x_s \Gamma_2 x_s^{-1} \cap \Gamma_1$, call it ϕ_s, and finally apply $i_{\Gamma_2(s) \to \Gamma_1}$. I.e.:

$$\Phi_s(\phi) = i_{\Gamma_2(s) \to \Gamma_1} \phi_s.$$

<u>20:</u> THEOREM As maps from $CL(\Gamma_2)$ to $CL(\Gamma_1)$,

$$r_{G \to \Gamma_1} \circ i_{\Gamma_2 \to G} = \sum_{s \in S} \chi^{\#}(U_s) \Phi_s,$$

where for each $s \in S$,

$$\chi^{\#}(U_s) = \chi(\bar{U}_s) - \chi(\bar{U}_s - U_s).$$

<u>21:</u> <u>N.B.</u> When G and $\Gamma_1, \Gamma_2 \subset G$ are finite, matters reduce to A, II, §8, #3.

Here is a sketch of the proof.

1. Fix a class function $\phi \in CL(\Gamma_2)$ and a $\gamma_1 \in \Gamma_1$.

2. Let $\tilde{U}_s \subset G/\Gamma_2$ denote the inverse image of U_s under the projection to $\Gamma_1 \backslash G/\Gamma_2$, thus

$$G/\Gamma_2 = \coprod_{s \in S} \tilde{U}_s.$$

3. Let $C_1, \ldots, C_m$ be the connected components of $(G/\Gamma_2)^{\gamma_1}$, thus

$$(G/\Gamma_2)^{\gamma_1} = \coprod_{j=1}^{m} C_j.$$

4. For each pair (s,j), put

$$V_{s,j} = (\Gamma_1 \cdot x_s\Gamma_2) \cap C_j \subset \tilde{U}_s \cap C_j \subset G/\Gamma_2.$$

5. The arrows

$$V_{s,j} \to \tilde{U}_s \cap C_j \to U_s$$

are a fibration sequence, hence by the multiplicativity of the Euler characteristic,

$$\chi(C_j) = \sum_{s \in S} \chi(V_{s,j})\chi^{\#}(U_s).$$

6. Fix elements $\gamma_{s,j} \in \Gamma_1$ such that

$$\gamma_{s,j}x_s\Gamma_2 \in V_{s,j}.$$

Then in particular,

$$\gamma_{s,j}x_s\Gamma_2 \in (G/\Gamma_2)^{\gamma_1}$$

$\Rightarrow$

$$x_s^{-1}\gamma_{s,j}^{-1}\gamma_1\gamma_{s,j}x_s \in \Gamma_2$$

$\Rightarrow$

$$\gamma_{s,j}^{-1}\gamma_1\gamma_{s,j} \in x_s\Gamma_2x_s^{-1},$$

the domain of ϕ^s.

7. From the definitions,

$$(r_{G \to \Gamma_1}(i_{\Gamma_2 \to G}\phi))(\gamma_1)$$

$$= \sum_{j=1}^{m} \chi(C_j)\phi(y_j^{-1}\gamma_1 y_j)$$

$$= \sum_{s\in S} \sum_{j=1}^{m} \chi(V_{s,j})\chi^{\#}(U_s)\phi(x_s^{-1}\gamma_{s,j}^{-1}\gamma_1\gamma_{s,j}x_s)$$

$$= \sum_{s\in S} \chi^{\#}(U_s) \sum_{j=1}^{m} \chi(V_{s,j})\phi^s(\gamma_{s,j}^{-1}\gamma_1\gamma_{s,j}).$$

8. The isotropy subgroup of the action of Γ_1 on $x_s\Gamma_2 \in G/\Gamma_2$ is

$$\Gamma_2(s) = x_s\Gamma_2 x_s^{-1} \cap \Gamma_1.$$

And

$$(\Gamma_1/\Gamma_2(s))^{\gamma_1} \approx (\Gamma_1 \cdot x_s\Gamma_2)^{\gamma_1}$$

$$= \coprod_{j=1}^{m} V_{s,j} \subset G/\Gamma_2.$$

9. Given $s \in S$,

$$\sum_{j=1}^{m} \chi(V_{s,j})\phi^s(\gamma_{s,j}^{-1}\gamma_1\gamma_{s,j})$$

$$= (i_{\Gamma_2(s) \to \Gamma_1}\phi^s)(\gamma_1).$$

10. Therefore

$$(r_{G \to \Gamma_1}(i_{\Gamma_2 \to G}\phi))(\gamma_1)$$

$$= \sum_{s\in S} \chi^{\#}(U_s)(i_{\Gamma_2(s) \to \Gamma_1}\phi^s)(\gamma_1)$$

$$= \sum_{s\in S} \chi^{\#}(U_s)\Phi_s(\phi)(\gamma_1),$$

the contention.

<u>22:</u> THEOREM The arrow

$$i_{\Gamma \to G}:CL(\Gamma) \to CL(G)$$

sends virtual characters to virtual characters, thus induces an arrow

$$X(\Gamma) \to X(G).$$

PROOF Recall first that this is true when G is finite (cf. #1). In general, let $\chi \in X(\Gamma)$ -- then to conclude that

$$i_{\Gamma \to G}\chi \in X(G),$$

it suffices to show that its restriction to every finite subgroup H of G is a virtual character (cf. §13, #14). So consider

$$r_{G \to H}(i_{\Gamma \to G}\chi)$$

or still, take in the above $\Gamma_1 = H$, $\Gamma_2 = \Gamma$, $\phi = \chi$, and consider

$$\sum_{s \in S} \chi^{\#}(U_s)\Phi_s(\chi).$$

Here

$$\Phi_s(\chi) = i_{x_s \Gamma x_s^{-1} \cap H \to H}\chi_s,$$

where χ_s is the restriction of χ^s to $\Gamma(s) \equiv x_s\Gamma x_s^{-1} \cap H$, a finite group. But now

$$\chi_s \in X(\Gamma(s)) \Rightarrow i_{\Gamma(s) \to H}\chi \in X(H),$$

which finishes the proof.

<u>23:</u> <u>N.B.</u> If G is finite, then the arrow

$$i_{\Gamma \to G}:CL(\Gamma) \to CL(G)$$

sends characters of Γ to characters of G but this need not be true if $\dim G > 0$

(cf. #13) (1_Γ is a character of Γ but the induced class function

$$i_\Gamma \to G^{1_\Gamma}$$

is identically zero, a virtual character, not a character).

24: REMARK Let G be a compact connected semisimple Lie group, $T \subset G$ a maximal torus -- then

$$\hat{G} \longleftrightarrow d\hat{T} \cap \bar{C} \quad \text{(cf. §10, \#23).}$$

While the theory developed above gives rise to an arrow

$$X(T) \to X(G),$$

it does not respect the foregoing parameterization which can only be accomplished by a more sophisticated version of the preceding process.

APPENDIX

There is a different approach to induction which is suggested by A, II, §9, #1. So let G be a compact Lie group, $\Gamma \subset G$ a closed Lie subgroup.

CONSTRUCTION Let (θ,E) be a finite dimensional unitary representation of Γ and denote by $E^G_{\Gamma,\theta}$ the space of all E-valued measurable functions f on G such that $f(x\gamma) = \theta(\gamma^{-1})f(x)$ $(x \in G, \gamma \in \Gamma)$ subject to

$$\int_{G/\Gamma} ||f||^2 d_{G/\Gamma} < \infty.$$

Then the prescription

$$(\text{Ind}^G_{\Gamma,\theta}(x)f)(y) = f(x^{-1}y)$$

defines a representation $\text{Ind}^G_{\Gamma,\theta}$ of G on $E^G_{\Gamma,\theta}$, the representation of G induced by θ.

N.B. The inner product

$$\langle f,g \rangle_\theta = \int_{G/\Gamma} \langle f,g \rangle d_{G/\Gamma}$$

equips $E_{\Gamma,\theta}^G$ with the structure of a Hilbert space and $\text{Ind}_{\Gamma,\theta}^G$ is a unitary representation.

EXAMPLE Take θ to be the trivial representation of Γ on $E = \mathbb{C}$ -- then $E_{\Gamma,\theta}^G = L^2(G/\Gamma)$.

[Note: When $\Gamma = \{e\}$, $E_{\Gamma,\theta}^G = L^2(G)$ and

$$\text{Ind}_{\Gamma,\theta}^G = L,$$

the left translation representation of G (cf. §1, #5).]

§1. *ORBITAL INTEGRALS*

Let G be a compact group.

<u>1:</u> DEFINITION Given $f \in C(G)$ and $\gamma \in G$, put

$$\mathcal{O}(f,\gamma) = \int_G f(x\gamma x^{-1})d_G(x),$$

the <u>orbital integral</u> of f at γ.

<u>2:</u> LEMMA The function $\mathcal{O}(f)$ defined by the assignment

$$\gamma \to \mathcal{O}(f,\gamma)$$

is a continuous class function on G, i.e., is an element of $CL(G)$.

<u>3:</u> RAPPEL If $f \in C(G)_{fin}$, then

$$\langle f,\chi_\Pi \rangle = 0$$

for all but finitely many Π.

<u>4:</u> LEMMA Suppose that $f \in C(G)_{fin}$ -- then $\forall\ \gamma \in G$,

$$\mathcal{O}(f,\gamma) = \sum_{\Pi \in \hat{G}} tr(\Pi^*(f))\chi_\Pi(\gamma) \quad \text{(cf. A, III, §1, \#3)},$$

the sum on the right being finite.

PROOF Apply I, §2, #19 to get

$$\mathcal{O}(f) = \sum_{\Pi \in \hat{G}} \langle \mathcal{O}(f),\chi_\Pi \rangle \chi_\Pi,$$

where the series converges in $L^2(G)$. But

$$\langle \mathcal{O}(f),\chi_\Pi \rangle = \int_G \mathcal{O}(f,\gamma)\overline{\chi_\Pi(\gamma)}d_G(\gamma)$$

$$= \int_G \left(\int_G f(x\gamma x^{-1}) d_G(x)\right) \overline{\chi_\Pi(\gamma)} d_G(\gamma)$$

$$= \int_G \left(\int_G f(x\gamma x^{-1}) \overline{\chi_\Pi(\gamma)} d_G(\gamma)\right) d_G(x)$$

$$= \int_G \left(\int_G f(\gamma) \overline{\chi_\Pi(x^{-1}\gamma x)} d_G(\gamma)\right) d_G(x)$$

$$= \int_G \left(\int_G f(\gamma) \overline{\chi_\Pi(\gamma)} d_G(\gamma)\right) d_G(x)$$

$$= \int_G \langle f, \chi_\Pi \rangle d_G(x) = \langle f, \chi_\Pi \rangle .$$

Therefore $\langle \mathcal{O}(f), \chi_\Pi \rangle = 0$ for all but finitely many Π, thus the almost everywhere equality

$$\mathcal{O}(f) = \sum_{\Pi \in \hat{G}} \langle \mathcal{O}(f), \chi_\Pi \rangle \chi_\Pi$$

is that of two continuous functions, thus is valid everywhere. Finally, from the definitions,

$$\langle f, \chi_\Pi \rangle = \int_G f(x) \overline{\chi_\Pi(x)} d_G(x)$$

$$= \int_G f(x) \chi_{\Pi^*}(x) d_G(x)$$

$$= \int_G f(x) \operatorname{tr}(\Pi^*(x)) d_G(x)$$

$$= \operatorname{tr}(\Pi^*(f)) .$$

§2. *KERNELS*

Let (X,μ), (Y,ν) be σ-finite measure spaces.

1: NOTATION Given $K \in L^2(X \times Y)$, define $T_K:L^2(Y) \to L^2(X)$ by

$$(T_K\phi)(x) = \int_Y K(x,y)\phi(y)d\nu(y).$$

2: THEOREM The map $K \to T_K$ is a linear isometry of $L^2(X \times Y)$ onto $L_{HS}(L^2(Y),L^2(X))$.

3: NOTATION Given

$$\begin{bmatrix} K_1 \in L^2(X \times Y) \\ K_2 \in L^2(Y \times Z), \end{bmatrix}$$

define their convolution

$$K_1 * K_2 \in L^2(X \times Z)$$

by

$$(K_1 * K_2)(x,z) = \int_Y K_1(x,y)K_2(y,z)d\nu(y).$$

[Note: The underlying measure-theoretic assumption is again σ-finiteness (which is needed infra for Fubini).]

4: THEOREM

$$T_{K_1} \circ T_{K_2} = T_{K_1 * K_2}.$$

5: APPLICATION Take X = Y = Z -- then

$$\begin{cases} T_{K_1}:L^2(X) \to L^2(X) \\ T_{K_2}:L^2(X) \to L^2(X) \end{cases}$$

are Hilbert-Schmidt, hence

$$T_{K_1 * K_2}:L^2(X) \to L^2(X)$$

is trace class.

6: LEMMA Take X = Y = Z and put $K = K_1 * K_2$ -- then

$$\mathrm{tr}(T_K) = \int_X K(x,x)\,d\mu(x).$$

PROOF

$$\mathrm{tr}(T_K) = \mathrm{tr}(T_{K_1} \circ T_{K_2})$$

$$= \langle T_{K_2}, T_{K_1}^* \rangle_{HS}$$

$$= \langle K_2, K_1^* \rangle$$

$$= \int_X \int_X K_2(y,x)\overline{K_1^*(y,x)}\,d\mu(y)\,d\mu(x)$$

$$= \int_X \int_X K_2(y,x)\overline{\overline{K_1(x,y)}}\,d\mu(y)\,d\mu(x)$$

$$= \int_X \int_X K_1(x,y)K_2(y,x)\,d\mu(y)\,d\mu(x)$$

$$= \int_X K_1 * K_2(x,x)\,d\mu(x)$$

$= \int_X K(x,x)d\mu(x).$

7: REMARK It can happen that $K_1 = K_2$ a.e. (so $T_{K_1} = T_{K_2}$), yet

$$\int_X K_1(x,x)d\mu(x) \neq \int_X K_2(x,x)d\mu(x).$$

[E.g.: Take $X = Y = [0,1]$, $K_1 \equiv 0$, $K_2 = \chi_\Delta$ (Δ the diagonal).]

8: THEOREM Let X be a locally compact Hausdorff space, μ a σ-finite Radon measure on X. Suppose that $K \in L^2(X \times X)$ is separately continuous and T_K is trace class -- then the function

$$x \to K(x,x)$$

is integrable on X and

$$\mathrm{tr}(T_K) = \int_X K(x,x)d\mu(x).$$

APPENDIX

LEMMA Let M be a compact C^∞ manifold, μ a smooth measure on M, $T:L^2(M) \to C^{2k}(M)$ $(k > \frac{1}{4}\dim M)$ -- then T is trace class.

PROOF Let Δ be a Laplacian on M and write

$$T = (1-\Delta)^{-k}(1-\Delta)^k T.$$

Then

$$(1-\Delta)^k T(L^2(M)) \subset C(M) \subset L^\infty(M)$$

so $(1-\Delta)^k T$ is Hilbert-Schmidt. On the other hand, by Sobolev theory,

$$(1-\Delta)^{-k}L^2(M) \subset H^{2k}(M) \subset C(M),$$

thus $(1-\Delta)^{-k}$ is also Hilbert-Schmidt.

§3. *THE LOCAL TRACE FORMULA*

Let G be a compact group.

1: NOTATION Denote by $\pi_{L,R}$ the representation of $G \times G$ on $L^2(G)$ given by

$$(\pi_{L,R}(x_1,x_2)f)(x) = f(x_1^{-1}xx_2) \qquad \text{(cf. A, III, §2, \#1).}$$

2: LEMMA $\pi_{L,R}$ is unitary.

PROOF

$$||\pi_{L,R}(x_1,x_2)f||^2$$

$$= \int_G |(\pi_{L,R}(x_1,x_2)f)(x)|^2 d_G(x)$$

$$= \int_G |f(x_1^{-1}xx_2)|^2 d_G(x)$$

$$= \int_G |f(x)|^2 d_G(x) = ||f||^2.$$

Given $f_1, f_2 \in C(G)$, define $f \in C(G \times G)$ by

$$f(x_1,x_2) = f_1(x_1)f_2(x_2),$$

and let

$$\pi_{L,R}(f) = \int_G \int_G f_1(x_1)f_2(x_2)\pi_{L,R}(x_1,x_2)d_G(x_1)d_G(x_2).$$

Then $\forall\ \phi \in L^2(G)$,

$$(\pi_{L,R}(f)\phi)(x) = \int_G \int_G f_1(x_1)f_2(x_2)\phi(x_1^{-1}xx_2)d_G(x_1)d_G(x_2)$$

$$= \int_G K_f(x,y)\phi(y)d_G(y),$$

where

$$K_f(x,y) = \int_G f_1(xz)f_2(zy)d_G(z).$$

Therefore $\pi_{L,R}(f)$ is an integral operator on $L^2(G)$ with kernel $K_f(x,y)$.

3: CONSTRUCTION

- Given $f_1 \in C(G)$, put

$$K_{f_1}(x,y) = f_1(xy^{-1}) \quad (x,y \in G).$$

Then

$$K_1 \in L^2(G \times G) \quad (K_1 = K_{f_1})$$

and

$$\begin{aligned}(T_{K_1}\phi)(x) &= \int_G K_1(x,y)\phi(y)d_G(y)\\ &= \int_G f(xy^{-1})\phi(y)d_G(y)\\ &= \int_G f(y)\phi(y^{-1}x)d_G(y)\\ &= \int_G f(y)(L(y)\phi)(x)d_G(y)\\ &= (L(f)\phi)(x).\end{aligned}$$

- Given $f_2 \in C(G)$, put

$$K_{f_2}(x,y) = f_2(x^{-1}y) \quad (x,y \in G).$$

Then

$$K_2 \in L^2(G \times G) \quad (K_2 = K_{f_2})$$

and

$$(T_{K_2}\phi)(x) = \int_G K_2(x,y)\phi(y)d_G(y)$$

$$= \int_G f_2(x^{-1}y)\phi(y)d_G(y)$$

$$= \int_G f_2(y)\phi(xy)d_G(y)$$

$$= \int_G f_2(y)(R(x)\phi)(y)d_G(y)$$

$$= (R(f)\phi)(x).$$

4: LEMMA Let $f_1,f_2 \in C(G)$ and let $f = f_1f_2$ -- then

$$K_f = K_{f_1} * K_{f_2}.$$

PROOF

$$(K_1 * K_2)(x,y) = \int_G K_1(x,z)K_2(z,y)d_G(z)$$

$$= \int_G f_1(xz^{-1})f_2(z^{-1}y)d_G(z)$$

$$= \int_G f_1(xz)f_2(zy)d_G(z).$$

Since the kernels of

$$\begin{cases} T_{K_1}:L^2(G) \to L^2(G) \\ T_{K_2}:L^2(G) \to L^2(G) \end{cases}$$

are square integrable, it follows that these operators are Hilbert-Schmidt. But

$$T_{K_1} \circ T_{K_2} = T_{K_1 * K_2} \quad \text{(cf. §2, \#4).}$$

Therefore $T_{K_1 * K_2}$ is trace class, i.e., T_{K_f} is trace class, i.e., $\pi_{L,R}(f)$ is trace class.

<u>5:</u> LEMMA

$$\operatorname{tr}(\pi_{L,R}(f)) = \int_G K_f(x,x)\, d_G(x) \quad \text{(cf. §2, \#6).}$$

<u>6:</u> RAPPEL Let

$$f \in \operatorname{span}_C(L^2(G) * L^2(G)) \subset C(G).$$

Then

$$f(e) = \sum_{\Pi \in \hat{G}} d_\Pi \operatorname{tr}(\Pi(f)) \quad \text{(cf. I, §2, \#15).}$$

We have

$$K_f(x,y) = \int_G f_1(xz) f_2(zy)\, d_G(z)$$

$$= \int_G f_1(u) f_2(x^{-1}uy)\, d_G(u)$$

$$= \int_G f_1(u) f_{2,x,y}(u)\, d_G(u).$$

Put now

$$F_{x,y}(v) = \int_G f_1(u) f_{2,x,y}(v^{-1}u)\, d_G(u).$$

Then

$$F_{x,y}(v) = \int_G f_1(u) \check{f}_{2,x,y}(u^{-1}v)\, d_G(u)$$

$$= f_1 * \check{f}_{2,x,y}(v)$$

=>

$$K_f(x,y) = F_{x,y}(e)$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \mathrm{tr}(\Pi(F_{x,y}))$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \mathrm{tr}(\Pi(f_1 * \check{f}_{2,x,y}))$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \mathrm{tr}(\Pi(f_1)\Pi(\check{f}_{2,x,y}))$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \mathrm{tr}(\Pi(f_1)\Pi(\delta_x * \check{f}_2 * \delta_{y^{-1}}))$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \mathrm{tr}(\Pi(f_1)\Pi(\delta_x)\Pi(\check{f}_2)\Pi(\delta_{y^{-1}}))$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \mathrm{tr}(\Pi(\delta_{y^{-1}})\Pi(f_1)\Pi(\delta_x)\Pi(\check{f}_2))$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \mathrm{tr}(\Pi(y^{-1})\Pi(f_1)\Pi(x)\Pi(\check{f}_2))$$

=>

$$\mathrm{tr}(\pi_{L,R}(f)) = \int_G K_f(x,x)d_G(x)$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \mathrm{tr}((\int_G \Pi(x^{-1})\Pi(f_1)\Pi(x)d_G(x)) \circ \Pi(\check{f}_2))$$

$$= \sum_{\Pi \in \hat{G}} d_\Pi \mathrm{tr}((\int_G \Pi(x)\Pi(f_1)\Pi(x^{-1})d_G(x)) \circ \Pi(\check{f}_2)).$$

<u>7:</u> SUBLEMMA $\forall\ \phi \in C(G)$, the operator

$$\int_G \Pi(x)\Pi(\phi)\Pi(x^{-1})d_G(x)$$

intertwines Π, hence is a scalar multiple of the identity (cf. I, §1, #15), call it λ_ϕ.

8: N.B.

$$\lambda_\phi = \int_G \Pi(x)\Pi(\phi)\Pi(x^{-1})d_G(x)$$

=>

$$\lambda_\phi d_\Pi = tr(\int_G \Pi(x)\Pi(\phi)\Pi(x^{-1})d_G(x))$$

=>

$$\lambda_\phi = \frac{tr(\Pi(\phi))}{d_\Pi} .$$

Therefore

$$tr(\pi_{L,R}(f))$$

$$= \sum_{\Pi\in\hat{G}} d_\Pi tr((\int_G \Pi(x)\Pi(f_1)\Pi(x^{-1})d_G(x)) \circ \Pi(\check{f}_2))$$

$$= \sum_{\Pi\in\hat{G}} d_\Pi \lambda_{f_1} tr(\Pi(\check{f}_2))$$

$$= \sum_{\Pi\in\hat{G}} d_\Pi \frac{tr(\Pi(f_1))}{d_\Pi} tr(\Pi(\check{f}_2))$$

$$= \sum_{\Pi\in\hat{G}} tr(\Pi(f_1))tr(\Pi(\check{f}_2))$$

$$= \sum_{\Pi\in\hat{G}} J(\Pi,f)$$

if

$$J(\Pi,f) = tr(\Pi(f_1))tr(\Pi(\check{f}_2)).$$

9: SUBLEMMA $\forall\ \phi \in C(G)$,

$$tr(\Pi(\check{\phi})) = tr(\int_G \check{\phi}(x)\Pi(x)d_G(x))$$

$$= tr(\int_G \phi(x^{-1})\Pi(x)d_G(x))$$

$$= tr(\int_G \phi(x)\Pi(x^{-1})d_G(x))$$

$$= tr(\int_G \phi(x)\Pi^*(x)d_G(x))$$

$$= tr(\Pi^*(\phi)).$$

10: N.B. Consequently,

$$J(\Pi,f) = tr(\Pi(f_1))tr(\Pi^*(f_2)).$$

There is another way to manipulate

$$\int_G K_f(x,x)d_G(x)$$

which then leads to a second formula for

$$tr(\pi_{L,R}(f)).$$

To wit:

$$\int_G K_f(x,x)d_G(x)$$

$$= \int_G \int_G f_1(xzx^{-1})f_2(z)d_G(z)d_G(x)$$

or still, for any $y \in G$,

$$\int_G \int_G f_1(xzx^{-1}) f_2(yzy^{-1}) d_G(z) d_G(x).$$

Now multiply through by $d_G(y)$ and integrate with respect to y:

$$\mathrm{tr}(\pi_{L,R}(f)) = \int_G \mathrm{tr}(\pi_{L,R}(f)) d_G(y)$$

$$= \int_G \int_G \int_G f_1(xzx^{-1}) f_2(yzy^{-1}) d_G(z) d_G(x) d_G(y)$$

$$= \int_G \left(\int_G f_1(xzx^{-1}) d_G(x)\right) \left(\int_G f_2(yzy^{-1}) d_G(y)\right) d_G(z)$$

$$= \int_G \mathcal{O}(f_1,z) \mathcal{O}(f_2,z) d_G(z).$$

<u>11:</u> DEFINITION Given $f = f_1 f_2$, the <u>local trace formula</u> is the relation

$$\sum_{\Pi \in \hat{G}} J(\Pi,f)$$

$$= \sum_{\Pi \in \hat{G}} \mathrm{tr}(\Pi(f_1)) \mathrm{tr}(\Pi^*(f_2))$$

$$= \int_G \mathcal{O}(f_1,z) \mathcal{O}(f_2,z) d_G(z).$$

Let G be a compact connected semisimple Lie group, $T \subset G$ a maximal torus.

<u>12:</u> RAPPEL For any continuous function $f \in C(G)$,

$$\int_G f(x) d_G(x) = \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G f(xtx^{-1}) d_G(x) d_T(t) \quad \text{(cf. I, §9, \#12)}$$

or still,

$$\int_G f(x) d_G(x) = \frac{1}{|W|} \int_T |\Delta(t)|^2 \mathcal{O}(f,t) d_T(t).$$

As above, let $f = f_1 f_2$ -- then

$$\int_G K_f(x,x)d_G(x)$$

$$= \int_G \int_G f_1(xzx^{-1})f_2(z)d_G(z)d_G(x)$$

$$= \int_G \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G f_1(xyty^{-1}x^{-1})f_2(yty^{-1})d_G(y)d_T(t)d_G(x)$$

$$= \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G (\int_G f_1(xyty^{-1}x^{-1})d_G(x))f_2(yty^{-1})d_G(y)d_T(t)$$

$$= \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G (\int_G f_1(xtx^{-1})d_G(x))f_2(yty^{-1})d_G(y)d_T(t)$$

$$= \frac{1}{|W|} \int_T |\Delta(t)|^2 (\int_G f_1(xtx^{-1})d_G(x))(\int_G f_2(yty^{-1})d_G(y))d_T(t)$$

$$= \frac{1}{|W|} \int_T |\Delta(t)|^2 \mathcal{O}(f_1,t)\mathcal{O}(f_2,t)d_T(t).$$

§1. *TOPOLOGICAL TERMINOLOGY*

1: DEFINITION A topological space X is compact if every open cover of X has a finite subcover.

2: DEFINITION A topological space X is locally compact if every point in X has a neighborhood basis consisting of compact sets.

3: LEMMA A Hausdorff space X is locally compact iff every point in X has a compact neighborhood.

4: APPLICATION Every compact Hausdorff space is locally compact.

5: EXAMPLE R is a locally compact Hausdorff space.

6: EXAMPLE Q is a Hausdorff space but it is not locally compact (Q is first category while a locally compact Hausdorff space is second category).

7: LEMMA An open subset of a locally compact Hausdorff space is locally compact.

8: LEMMA A closed subset of a locally compact Hausdorff space is locally compact.

9: LEMMA In a locally compact Hausdorff space, the intersection of an open set with a closed set is locally compact.

10: EXAMPLE The semiclosed, semiopen interval [0,1[is locally compact. [In fact,

$$[0,1[\; = \;]-1,1[\; \cap \; [0,1].]$$

11: DEFINITION A topological group is a Hausdorff topological space G equipped with a group structure such that the function from $G \times G$ to G defined by $(x,y) \to xy^{-1}$ is continuous or still, as is equivalent:

- The function $G \times G \to G$ that sends (x,y) to xy is continuous.
- The function $G \to G$ that sends x to x^{-1} is continuous.

If G is a topological group and if $H \subset G$ is a subgroup, then the set G/H is to be given the quotient topology.

12: LEMMA The space G/H is Hausdorff iff H is closed.

13: DEFINITION A locally compact (compact) group is a topological group G that is both locally compact (compact) and Hausdorff.

14: LEMMA If G is a locally compact group and if H is a closed subgroup, then G/H is a locally compact Hausdorff space.

15: LEMMA If G is a locally compact group and if H is a closed normal subgroup, then G/H is a locally compact group.

16: LEMMA If G is a locally compact group and if H is a locally compact subgroup, then H is closed in G.

17: LEMMA If G is a locally compact group, then a subgroup H is open iff the quotient G/H is discrete.

18: LEMMA If G is a compact group, then a subgroup H is open iff the quotient G/H is finite.

<u>19:</u> LEMMA If G is a locally compact group, then every open subgroup of G is closed and every finite index closed subgroup of G is open.

<u>20:</u> DEFINITION A topological space X is <u>totally disconnected</u> if the connected components of X are singletons.

<u>21:</u> EXAMPLE Q is totally disconnected.

<u>22:</u> LEMMA If G is a totally disconnected locally compact group, then {e} has a neighborhood basis consisting of open-compact subgroups.

<u>23:</u> LEMMA If G is a totally disconnected compact group, then {e} has a neighborhood basis consisting of open-compact normal subgroups.

<u>24:</u> DEFINITION A topological space X is <u>0-dimensional</u> if every point of X has a neighborhood basis consisting of open-closed sets.

<u>25:</u> EXAMPLE Q is 0-dimensional.

<u>26:</u> LEMMA A locally compact Hausdorff space is 0-dimensional iff it is totally disconnected.

[Note: In such a space, every point has a neighborhood basis consisting of open-compact sets.]

<u>27:</u> REMARK It is false that the continuous image of a 0-dimensional locally compact Hausdorff space is again 0-dimensional.

[To see this, recall that every compact metric space is the continuous image of the Cantor set.]

<u>28:</u> LEMMA If G is a locally compact 0-dimensional group and if H is a closed subgroup of G, then G/H is 0-dimensional.

<u>29:</u> LEMMA A 0-dimensional T_1 space is totally disconnected.

<u>30:</u> REMARK There are totally disconnected metric spaces which are not 0-dimensional.

§2. *INTEGRATION THEORY*

Let X be a locally compact Hausdorff space.

<u>1:</u> DEFINITION A <u>Radon measure</u> is a measure μ defined on the Borel σ-algebra of X subject to the following conditions.

1. μ is finite on compacta, i.e., for every compact set $K \subset X$, $\mu(K) < \infty$.
2. μ is outer regular, i.e., for every Borel set $A \subset X$,

$$\mu(A) = \inf_{U \supset A} \mu(U),$$

where $U \subset X$ is open.

3. μ is inner regular, i.e., for every open set $A \subset X$,

$$\mu(A) = \sup_{K \subset A} \mu(K),$$

where $K \subset X$ is compact.

<u>2:</u> RAPPEL If X is a locally compact Hausdorff space and if X is second countable, then for any open subset $U \subset X$, there exist compact sets $K_1 \subset K_2 \subset \cdots$ such that $U = \bigcup_{n=1}^{\infty} K_n$.

<u>3:</u> APPLICATION If (X,μ) is a Radon measure space and if X is second countable, then X is σ-finite.

<u>4:</u> RIESZ REPRESENTATION THEOREM Let X be a locally compact Hausdorff space. Suppose that $\Lambda:C_c(X) \to \mathbf{C}$ is a positive linear functional -- then there exists a unique Radon measure μ on X such that $\forall\ f \in C_c(X)$,

$$\Lambda f = \int_X f(x)d\mu(x).$$

Let G be a locally compact group.

5: DEFINITION A left Haar measure on G is a Radon measure $\mu_G \neq 0$ which is left invariant, i.e., $\forall\ x \in G$ and $\forall$ Borel set $A \subset G$, $\mu_G(xA) = \mu_G(A)$.

[Note: Equivalently, a Radon measure $\mu \neq 0$ is a left Haar measure on G if $\forall\ f \in C_c(G)$ and $\forall\ y \in G$,

$$\int_G f(yx)d\mu(x) = \int_G f(x)d\mu(x).]$$

6: THEOREM G admits a left Haar measure and if μ_{G_1}, μ_{G_2} are two such, then $\mu_{G_1} = c\mu_{G_2}$ $(\exists\ c > 0)$.

7: LEMMA Every nonempty open subset of G has positive left Haar measure.

8: LEMMA Every compact subset of G has finite left Haar measure.

9: N.B. The definition of a right Haar measure on G is analogous.

Given $x \in G$ and a Borel set $A \subset X$, let

$$\mu_{G,x}(A) = \mu_G(Ax).$$

Then $\mu_{G,x}$ is a left Haar measure on G:

$$\mu_{G,x}(yA) = \mu_G(yAx) = \mu_G(Ax) = \mu_{G,x}(A).$$

The uniqueness of left Haar measure now implies that there is a unique positive real number $\Delta_G(x)$ such that

$$\mu_{G,x} = \Delta_G(x)\mu_G.$$

10: LEMMA $\Delta_G:G \to R_{>0}^{\times}$ is independent of the choice of μ.

11: LEMMA $\Delta_G : G \to R_{>0}^{\times}$ is a continuous homomorphism.

12: DEFINITION Δ_G is called the modular function of G.

So, $\forall\ f \in C_c(G)$ and $\forall\ y \in G$,

$$\int_G f(xy^{-1})d\mu_G(x) = \Delta_G(y) \int_G f(x)d\mu_G(x).$$

13: LEMMA $\forall\ f \in C_c(G)$,

$$\int_G \check{f}(x)d\mu_G(x) = \int_G \frac{f(x)}{\Delta_G(x)} d\mu_G(x).$$

[Note: As usual, $\check{f}(x) = f(x^{-1})$.]

14: N.B. The positive linear functional that assigns to each $f \in C_c(G)$ the common value of the two members of this equality is a right Haar integral.

15: LEMMA If $\phi : G \to G$ is a topological automorphism, then there is a unique positive real number $\delta_G(\phi)$ such that $\forall\ f \in C_c(G)$,

$$\int_G f(\phi^{-1}(x))d\mu_G(x) = \delta_G(\phi) \int_G f(x)d\mu_G(x).$$

[The positive linear functional

$$f \to \int_G f(\phi^{-1}(x))d\mu_G(x)$$

is a left Haar integral.]

[Note: The arrow $\phi \to \delta_G(\phi)$ is a homomorphism: $\delta_G(\phi_1\phi_2) = \delta_G(\phi_1)\delta_G(\phi_2)$.]

16: EXAMPLE If V is a real finite dimensional vector space and if

$T:V \to V$ is an invertible linear transformation, then per "Lebesgue measure",

$$\int_V f(T^{-1}(x))dx = |\det T| \int_V f(x)dx,$$

so here

$$\delta_V(T) = |\det T|.$$

17: EXAMPLE Define $I_y:G \to G$ by $I_y(x) = yxy^{-1}$ -- then

$$\int_G f(I_y^{-1}(x))d\mu_G(x) = \int_G f(y^{-1}xy)d\mu_G(x)$$

$$= \int_G f(xy)d\mu_G(x)$$

$$= \Delta_G(y^{-1}) \int_G f(x)d\mu_G(x),$$

which implies that

$$\delta_G(I_y) = \Delta_G(y^{-1}).$$

18: LEMMA If $\phi:G \to G$ is a topological automorphism, then $\forall\ y \in G$,

$$\Delta_G(\phi(y)) = \Delta_G(y).$$

[On the one hand,

$$\int_G f(\phi(xy^{-1}))d\mu_G(x)$$

$$= \Delta_G(y) \int_G f(\phi(x))d\mu_G(x)$$

$$= \Delta_G(y)\delta_G(\phi^{-1}) \int_G f(x)d\mu_G(x)$$

and, on the other hand,

$$\int_G f(\phi(xy^{-1}))d\mu_G(x)$$

$$= \int_G f(\phi(x)\phi(y^{-1}))d\mu_G(x)$$

$$= \delta_G(\phi^{-1}) \int_G f(x\phi(y)^{-1})d\mu_G(x)$$

$$= \delta_G(\phi^{-1})\Delta_G(\phi(y)) \int_G f(x)d\mu_G(x).$$

Therefore

$$\Delta_G(y) = \Delta_G(\phi(y)).]$$

19: LEMMA If G_1, G_2 are locally compact groups and if μ_{G_1}, μ_{G_2} are left Haar measures per G_1, G_2, then $\mu_{G_1} \times \mu_{G_2}$ is a left Haar measure per $G_1 \times G_2$ and

$$\Delta_{G_1 \times G_2}(x_1,x_2) = \Delta_{G_1}(x_1)\Delta_{G_2}(x_2).$$

Let G be a locally compact group, X and Y two closed subgroups of G.

20: DEFINITION The pair (X,Y) is admissible if the following conditions are satisfied.

- The intersection $X \cap Y$ is compact.
- The multiplication $X \times Y \to G$ is an open map.
- The set of products XY exhausts G up to a set of Haar measure 0 (left or right).

21: EXAMPLE Using the notation of #19, work with $G_1 \times G_2$ and take $X = G_1 \times \{e_2\}$, $Y = \{e_1\} \times G_2$ -- then the pair (X,Y) is admissible.

22: THEOREM Suppose that the pair (X,Y) is admissible. Fix left Haar measures μ_X, μ_Y on X,Y -- then there is a unique left Haar measure μ_G on G such that $\forall$ $f \in C_c(G)$,

$$\int_G f d\mu_G = \int_{X \times Y} f(xy) \frac{\Delta_G(y)}{\Delta_Y(y)} d\mu_X(x) d\mu_Y(y).$$

<u>23:</u> <u>N.B.</u> Specializing the setup to that of #21 leads back to #19.

[Note that

$$\Delta_{G_1 \times G_2}(e_1, x_2) = \Delta_{G_1}(e_1)\Delta_{G_2}(x_2)$$

$$= \Delta_{G_2}(x_2)$$

thereby cancelling the factor in the denominator.]

<u>24:</u> LEMMA If G is a locally compact group and if $H \subset G$ is a closed normal subgroup, then $\Delta_G | H = \Delta_H$.

<u>25:</u> APPLICATION In the setup of #22, assume in addition that Y is normal -- then $\forall$ $f \in C_c(G)$,

$$\int_G f d\mu_G = \int_{X \times Y} f(xy) d\mu_X(x) d\mu_Y(y).$$

[Note: Given $x \in X$, the restriction

$$\begin{cases} I_{x^{-1}}: Y \to Y \\ \quad y \to x^{-1} y x \end{cases}$$

is an automorphism of Y and

$$\Delta_G(xy) = \Delta_X(x)\Delta_Y(y)\delta_Y(I_{x^{-1}}).]$$

Let G be a locally compact group, X and Y two closed subgroups of G.

26: DEFINITION G is the topological semidirect product of X and Y if every element $z \in G$ can be expressed in a unique manner as a product $z = xy$ ($x \in X$, $y \in Y$) and if the multiplication $X \times Y \to G$ is a homeomorphism.

27: N.B. A priori, the multiplication $X \times Y \to G$ is a continuous bijection, thus the condition is satisfied if the multiplication $X \times Y \to G$ is an open map, this being automatic whenever G is second countable.

[Under these circumstances, G is the union of a sequence of compact sets (cf. #2), so the same is true of $X \times Y$. But G is a locally compact Hausdorff space, hence is a Baire space.]

[Note: If A is a Baire space and if $\{A_n : n \in \mathbb{N}\}$ is a closed covering of A, then at least one A_n must contain an open set.]

If G is the topological semidirect product of X and Y, then $X \cap Y = \{e\}$ and the pair (X,Y) is admissible. Therefore the theory is applicable in this situation.

28: N.B. In general, the arrow $(x,y) \to xy$ is not an isomorphism of groups but this will be the case if every element of X commutes with every element of Y or, equivalently, if X and Y are normal subgroups of G, i.e., if G is the topological direct product of X and Y.

§3. UNIMODULARITY

Let G be a locally compact group, μ_G a left Haar measure on G.

1: DEFINITION G is <u>unimodular</u> if $\Delta_G \equiv 1$.

2: N.B. G is unimodular iff μ_G is a right Haar measure on G.

3: EXAMPLE Take for G the group of all real matrices of the form

$$\begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} \quad (y \neq 0) \text{ -- then}$$

$$\Delta_G \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} = |y|,$$

thus G is not unimodular.

4: LEMMA G is unimodular iff $\forall\ f \in C_c(G)$,

$$\int_G f(x^{-1})d\mu_G(x) = \int_G f(x)d\mu_G(x) \quad (\text{cf. §2, \#13}).$$

5: LEMMA

- Every locally compact abelian group is unimodular.
- Every compact group is unimodular.
- Every discrete group is unimodular.

6: LEMMA Every locally compact group that coincides with its closed commutator subgroup is unimodular.

<u>7:</u> LEMMA Every open subgroup of a unimodular locally compact group is unimodular.

<u>8:</u> LEMMA Every closed normal subgroup of a unimodular locally compact group is unimodular.

[Note: A closed subgroup of a unimodular locally compact group is not necessarily unimodular.]

<u>9:</u> LEMMA Let G be a locally compact group, Z(G) its center -- then G is unimodular iff G/Z(G) is unimodular.

Let G be a locally compact group, $H \subset G$ a closed subgroup (H is then a locally compact subgroup) (cf. §1, #8).

<u>10:</u> DEFINITION H is a <u>cocompact</u> subgroup if the quotient G/H is compact.

<u>11:</u> LEMMA If G admits a unimodular cocompact subgroup $H \subset G$, then G is unimodular.

§4. *INTEGRATION ON HOMOGENEOUS SPACES*

Let G be a locally compact group, $H \subset G$ a closed subgroup.

<u>1:</u> <u>N.B.</u> The quotient G/H is a locally compact Hausdorff space (cf. §1, #14).

Fix left Haar measures

$$\left[\begin{array}{l} \mu_G \text{ on } G \\ \mu_H \text{ on } H. \end{array}\right.$$

<u>2:</u> NOTATION Given $f \in C_c(G)$, define $f^H \in C_c(G/H)$ by the rule

$$f^H(xH) = \int_H f(xy)\,d\mu_H(y).$$

<u>3:</u> LEMMA The arrow

$$f \to f^H$$

sends $C_c(G)$ onto $C_c(G/H)$.

<u>4:</u> DEFINITION A Radon measure $\mu \neq 0$ on the Borel σ-algebra of G/H is said to be an <u>invariant measure</u> if $\forall\, x \in G$ and $\forall$ Borel set $A \subset G/H$, $\mu(xA) = \mu(A)$.

[Note: If $H = \{e\}$, then "invariant measure" = "left Haar measure".]

<u>5:</u> THEOREM There exists an invariant measure $\mu_{G/H}$ on G/H iff $\Delta_G|H = \Delta_H$ and when this is so, $\mu_{G/H}$ is unique up to a positive scalar factor.

[Note: Matters are automatic if H is compact or if both G and H are unimodular.]

6: N.B. If H is a normal closed subgroup of G, then $\Delta_G|H = \Delta_H$.

[For a left Haar measure on G/H is an invariant measure.]

7: THEOREM There is a unique choice for $\mu_{G/H}$ such that $\forall\ f \in C_c(G)$,

$$\int_G f(x)d\mu_G(x) = \int_{G/H} f^H(\dot{x})d\mu_{G/H}(\dot{x}) \qquad (\dot{x} = xH).$$

[Note: Bear in mind that μ_G, μ_H have been fixed at the beginning.]

8: N.B. This formula is valid for all $f \in L^1(G)$.

9: LEMMA Let $H_1 \subset G$, $H_2 \subset G$ be closed subgroups of G with $H_1 \subset H_2$ -- then G/H_2 and H_2/H_1 admit finite invariant measures iff G/H_1 admits a finite invariant measure.

10: APPLICATION If G/H has a finite invariant measure and if H is unimodular, then G is unimodular.

[Let K be the kernel of Δ_G -- then $\Delta_G|H = \Delta_H \equiv 1$, thus $H \subset K$ and so G/K has a finite invariant measure (as does K/H). But G/K is a locally compact group. Therefore G/K is actually a compact group (its Haar measure being finite) and this implies that $\Delta_G(G)$ is a compact subgroup of $R_{>0}^{\times}$, hence $\Delta_G(G) = \{1\}$, i.e., G is unimodular.]

11: N.B. Suppose that $H \subset G$ is a unimodular cocompact subgroup -- then G/H admits a finite invariant measure $\mu_{G/H}$.

[In fact, G is necessarily unimodular (cf. §3, #11), from which the existence of $\mu_{G/H}$. But $\mu_{G/H}$ is Radon, hence finite on compacta, hence in particular,

$$\mu_{G/H}(G/H) < \infty.]$$

[Note: Take $G = SL(2,R)$ and let

$$H = \{X : X = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in SL(2,R)\}.$$

Then G is unimodular but H is not unimodular. Therefore G/H does not admit an invariant measure even though H is a cocompact subgroup.]

<u>12:</u> LEMMA Let $H_1 \subset G$, $H_2 \subset G$ be closed subgroups of G with H_1 normalizing H_2 and H_1H_2 closed in G -- then the following are equivalent.

- H_1H_2/H_1 admits a finite invariant measure.
- $H_2/H_1 \cap H_2$ admits a finite invariant measure.

[Note: There is a commutative diagram

$$\begin{array}{ccc} H_2 & = & H_2 \\ \downarrow & & \downarrow \\ H_2/H_1 \cap H_2 & \xrightarrow[\phi]{} & H_1H_2/H_1, \end{array}$$

where

$$\phi(x_2(H_1 \cap H_2)) = x_2H_1.$$

The vertical arrows are continuous and open. Therefore the bottom horizontal arrow is a homeomorphism.]

<u>13:</u> APPLICATION Suppose that G is the topological semidirect product

of X and Y (cf. §2, #26) and take Y normal -- then $G = XY$ and $X \cap Y = \{e\}$. Therefore G/X has a finite invariant measure iff Y has a finite invariant measure.

§5. INTEGRATION ON LIE GROUPS

Suppose that M is an orientable n-dimensional C^∞ manifold which we take to be second countable. Let ω be a positive n-form on M -- then the theory leads to a positive linear functional

$$f \to \int_M f\omega \quad (f \in C_c(M))$$

from which a Radon measure μ_ω.

Assume now that G is a Lie group with Lie algebra $\mathfrak{g}$. Let $L_x: G \to G$ be left translation $y \to xy$ by x.

<u>1:</u> DEFINITION A differential form ω on G is <u>left invariant</u> if $\forall\, x \in G$, $L_x^*\omega = \omega$.

<u>2:</u> NOTATION Given $X \in \mathfrak{g}$, let $\tilde{X}$ be the corresponding left invariant vector field on G.

Let $n = \dim G$ $(= \dim \mathfrak{g})$ and fix a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. Define 1-forms $\omega^1, \ldots, \omega^n$ on G by the condition $\omega^i(\tilde{X}_j) = \delta^i_j$.

<u>3:</u> LEMMA The ω^i are left invariant.

Put

$$\omega = \omega^1 \wedge \ldots \wedge \omega^n.$$

Then $\forall\, x \in G$,

$$L_x^*\omega = L_x^*(\omega^1 \wedge \ldots \wedge \omega^n)$$

$$= L_x^*\omega^1 \wedge \ldots \wedge L_x^*\omega^n$$

$$= \omega^1 \wedge \ldots \wedge \omega^n = \omega.$$

I.e.: ω is a left invariant n-form on G.

4: LEMMA ω is nowhere vanishing on G.

5: LEMMA G can be oriented so as to render ω positive.

[Note: The orientation of G depends on the choice of a basis for $\mathfrak{g}$. If $Y_1, \ldots, Y_n$ is another basis, then the resulting orientation of G does not change iff the linear transformation $X_i \to Y_i$ $(1 \leq i \leq n)$ has positive determinant.]

6: SCHOLIUM The assignment

$$f \to \int_G f\omega \quad (f \in C_c(G))$$

is a positive linear functional.

7: LEMMA The Radon measure μ_ω is a left Haar measure.

PROOF $\forall\, x \in G$, $L_x: G \to G$ is an orientation preserving diffeomorphism, so $\forall\, f \in C_c(G)$,

$$\int_G f d\mu_\omega = \int_G f\omega = \int_G (f \circ L_x) L_x^* \omega$$

$$= \int_G (f \circ L_x)\omega = \int_G (f \circ L_x) d\mu_\omega.$$

8: REMARK Any subset S of G which is contained in an at most countable union of smooth images of C^∞ manifolds of dimension < dim G has zero left Haar measure.

9: THEOREM $\forall\, x \in G$,

$$\Delta_G(x) = \frac{1}{|\det \mathrm{Ad}(x)|}.$$

10: EXAMPLE Every connected nilpotent Lie group N is unimodular.

[If $X \in \mathfrak{n}$ (the Lie algebra of N), then ad(X) is nilpotent, thus tr(ad(X)) = 0 and so

$$\det \operatorname{Ad}(\exp X) = \det e^{\operatorname{ad}(X)}$$

$$= e^{\operatorname{tr}(\operatorname{ad}(X))} = 1.]$$

11: LEMMA A 1-dimensional representation of a connected semisimple Lie group is trivial.

12: APPLICATION The restriction of Δ_G to any semisimple analytic subgroup of G is $\equiv 1$.

13: THEOREM Suppose that G is a reductive Lie group in the Harish-Chandra class -- then G is unimodular.

PROOF First decompose G as the product ${}^0G \times V$, where V is a central vector group (possibly trivial) and

$${}^0G = \bigcap_\chi \operatorname{Ker} \chi,$$

the χ running through the set of continuous homomorphisms $G \to \mathbf{R}_{>0}^{\times}$. This done, take for a left Haar measure on G the product of the left Haar measures on ${}^0G$ and V. Since V is unimodular, it will be enough to deal with ${}^0G$ (cf. §2, #19). Fix a maximal compact subgroup K of G -- then K is a maximal compact subgroup of ${}^0G$ and ${}^0G = KG_{ss}$, thus $\forall\ k \in K$, $\forall\ x \in G_{ss}$,

$$\Delta_{{}^0G}(kx) = \Delta_{{}^0G}(k)\Delta_{{}^0G}(x) = 1 \cdot 1 = 1.$$

[Note: G_{ss} is the analytic subgroup of G corresponding to $\mathfrak{g}_{ss}$ (the ideal in $\mathfrak{g}$ spanned by $[\mathfrak{g},\mathfrak{g}]$). It is closed and normal.]

Maintaining the supposition that G is a reductive Lie group in the Harish-Chandra class, consider an Iwasawa decomposition G = KAN.

<u>14:</u> <u>N.B.</u> N is a normal subgroup of AN and AN is the topological semi-direct product of A and N.

[Note: AN is second countable so there are no technical issues.]

<u>15:</u> LEMMA

$$\Delta_{AN}(an) = \frac{1}{|\det \mathrm{Ad}(an)|} = \frac{1}{e^{2\rho(\log a)}}.$$

[Note: Here 2ρ is the sum of the positive roots of $(\mathfrak{g},\mathfrak{a})$ counted with multiplicities.]

Since the pair (K,AN) is admissible and since $\Delta_G \equiv 1$, it follows from §2, #22 that $\forall\ f \in C_c(G)$,

$$\begin{aligned}\int_G f d\mu_G &= \int_{K \times AN} f(kan) \frac{\Delta_G(an)}{\Delta_{AN}(an)} d\mu_K(k) d\mu_{AN}(an) \\ &= \int_{K \times AN} f(kan) \frac{1}{\Delta_{AN}(an)} d\mu_K(k) d\mu_{AN}(an) \\ &= \int_{K \times AN} f(kan) e^{2\rho(\log a)} d\mu_K(k) d\mu_{AN}(an) \\ &= \int_{K \times A \times N} f(kan) e^{2\rho(\log a)} d\mu_K(k) d\mu_A(a) d\mu_N(n).\end{aligned}$$

[Note: To be completely precise, fix left Haar measures μ_K, μ_A, μ_N on K, A, N -- then there is a unique determination of the left Haar measure μ_G on G such that for any $f \in C_c(G)$, the function

$$(k,a,n) \rightarrow f(kan)$$

lies in

$$C_c(K \times A \times N)$$

and

$$\int_G f d\mu_G = \int_{K \times A \times N} f(kan) e^{2\rho(\log a)} d\mu_K(k) d\mu_A(a) d\mu_N(n).]$$

<u>16:</u> LEMMA

$$\Delta_{AN}(an) = \Delta_A(a)\Delta_N(n)\delta_N(I_{a^{-1}}) \qquad \text{(cf. §2, \#25)}$$

$$= \delta_N(I_{a^{-1}}).$$

[Note: A is abelian and N is nilpotent... .]

So, $\forall f \in C_c(G)$,

$$\int_{K \times N \times A} f(kna) d\mu_K(k) d\mu_N(n) d\mu_A(a)$$

$$= \int_{K \times N \times A} f(kaa^{-1}na) d\mu_K(k) d\mu_N(n) d\mu_A(a)$$

$$= \int_{K \times N \times A} f(kaI_{a^{-1}}(n)) d\mu_K(k) d_N(n) d\mu_A(a)$$

$$= \int_{K \times N \times A} f(kan) \delta_N(I_a) d\mu_K(k) d\mu_A(a) d\mu_N(n) \qquad \text{(cf. §2, \#15)}$$

$$= \int_{K \times A \times N} f(kan) \Delta_{AN}(a^{-1}n) d\mu_K(k) d\mu_A(a) d\mu_N(n)$$

$$= \int_{K \times A \times N} f(kan) e^{2\rho(\log a)} d\mu_K(k) d\mu_A(a) d\mu_N(n)$$

$$= \int_G f d\mu_G.$$

[Note: As a corollary,

$$\int_{A \times N \times K} f(ank) d\mu_A(a) d\mu_N(n) d\mu_K(k)$$

$$= \int_{K \times N \times A} \check{f}(k^{-1}n^{-1}a^{-1}) d\mu_K(k) d\mu_N(n) d\mu_A(a)$$

$$= \int_{K \times N \times A} \check{f}(kna) d\mu_K(k) d\mu_N(n) d\mu_A(a) \qquad \text{(cf. §3, \#4)}$$

$$= \int_G \check{f} d\mu_G = \int_G f d\mu_G,$$

G being unimodular (cf. #13).]

Let M be the centralizer of $\mathfrak{a}$ in K and put $\bar{N} = \theta N$ -- then the map

$$(\bar{n}, m, a, n) \rightarrow \bar{n}man$$

is an open bijection of $\bar{N} \times M \times A \times N$ onto an open submanifold $\bar{N}MAN \subset G$.

17: LEMMA The complement of $\bar{N}MAN$ in G is a set of Haar measure 0.

[Using the Bruhat decomposition, the said complement is seen to be a finite union of smooth images of C^∞ manifolds of dimension < dim G so one can quote #8.]

The pair ($\bar{N}$, MAN) is therefore admissible, hence $\forall f \in C_c(G)$ (cf. §2,#22),

$$\int_G f d\mu_G$$

$$= \int_{\bar{N} \times MAN} f(\bar{n}man) \frac{\Delta_G(man)}{\Delta_{MAN}(man)} d\mu_{\bar{N}}(\bar{n}) d\mu_{MAN}(man)$$

$$= \int_{\bar{N} \times MAN} f(\bar{n}man) \frac{1}{\Delta_{MAN}(man)} d\mu_{\bar{N}}(\bar{n}) d\mu_{MAN}(man)$$

$$= \int_{\bar{N} \times MAN} f(\bar{n}man) e^{2\rho(\log a)} d\mu_{\bar{N}}(\bar{n}) d\mu_{MAN}(man)$$

$$= \int_{\bar{N} \times M \times A \times N} f(\bar{n}man) e^{2\rho(\log a)} d\mu_{\bar{N}}(\bar{n}) d\mu_M(m) d\mu_A(a) d\mu_N(n).$$

<u>18:</u> RAPPEL Let V be a finite dimensional real Hilbert space -- then the canonical Haar measure dV on V is that in which the parallelepiped determined by an orthonormal basis has unit measure.

[Spelled out, if $\{X_1,\ldots,X_n\}$ is an orthonormal basis for V and if Q is the set of all points $X = \sum_{i=1}^{d} c_i X_i$ $(c_i \in R)$ with $0 \le c_i \le 1$. then

$$\int_Q dV = 1.]$$

[Note: Matters are independent of the particular choice of an orthonormal basis since the transition matrix between any two such is orthogonal, hence the absolute value of its determinant is 1.]

<u>19:</u> SUBLEMMA Let V be a finite dimensional real Hilbert space; let $V_1 \subset V$, $V_2 \subset V$ be subspaces. Suppose that $T:V_1 \to V_2$ is a bijective linear transformation -- then $\forall\ \phi \in C_c(V_2)$,

$$\int_{V_2} \phi \, dV_2 = |\det T| \int_{V_1} \phi \circ T \, dV_1,$$

where the determinant is computed relative to an orthonormal basis in V_1 and an orthonormal basis in V_2.

20: N.B. Symbolically,

$$dV_2 = |\det T|\, dV_1.$$

21: CONVENTION Extend the Killing form on $\mathfrak{g}_{ss} \times \mathfrak{g}_{ss}$ to a nondegenerate symmetric bilinear form $B:\mathfrak{g} \times \mathfrak{g} \to R$ with the following properties:

- B is Ad G invariant.
- B is θ-invariant.
- $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal under B.
- B is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$.

22: N.B. The bilinear form

$$(X,Y)_\theta = -B(X,\theta Y) \quad (X,Y \in \mathfrak{g})$$

equips $\mathfrak{g}$ with the structure of a real Hilbert space.

Relative to this data, any subspace $\mathfrak{l}$ of $\mathfrak{g}$ carries a canonical Haar measure $d\mathfrak{l}$, an instance being the Lie algebra $\mathfrak{l}$ of a closed Lie subgroup L of G.

23: EXAMPLE $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal and $d\mathfrak{g} = d\mathfrak{k}d\mathfrak{p}$.

[Note: The orthogonal projections $E_{\mathfrak{k}}$, $E_{\mathfrak{p}}$ of $\mathfrak{g}$ onto $\mathfrak{k},\mathfrak{p}$ are given by

$$\begin{cases} E_{\mathfrak{k}} = \dfrac{1+\theta}{2} \\[2ex] E_{\mathfrak{p}} = \dfrac{1-\theta}{2} \end{cases}$$

respectively.]

24: CONSTRUCTION Choose an open neighborhood N_0 of 0 in $\mathfrak{l}$ and an open neighborhood N_e of e in L such that exp is an analytic diffeomorphism of N_0 onto N_e. Normalize the left Haar measure μ_L on L in such a way that $\forall\ f \in C_c(N_e)$,

$$\int_{N_e} f d\mu_L = \int_{N_0} F dl,$$

where

$$F(X) = f(\exp X) \left| \det\left[\frac{1-e^{-ad(x)}}{ad(X)} \right] \right| .$$

This fixes μ_L uniquely, call it dL, and its definition is independent of the choice of N_0.

25: N.B. If L is compact, put

$$\mathrm{vol}(L) = \int_L dL$$

and term $\frac{1}{\mathrm{vol}(L)}\, dL$ the normalized Haar measure of L.

Now write after Iwasawa G = KAN, thus $\forall\ f \in C_c(G)$,

$$\int_G f d\mu_G = \int_{K \times A \times N} f(kan) e^{2\rho(\log a)} d\mu_K(k) d\mu_A(a) d\mu_N(n).$$

On the right hand side, take

$$d\mu_K(k) = \frac{1}{\mathrm{vol}(K)}\, dK,\ d\mu_A(a) = dA,\ d\mu_N(n) = dN.$$

Then these choices determine $d\mu_G$ uniquely, denote it by the symbol $d_{st}G$ and refer to it as the standard Haar measure of G.

26: LEMMA

$$dG = 2^{-\frac{1}{2}\dim N}\, e^{2\rho(\log a)} dK dA dN.$$

PROOF It suffices to show that

$$d\mathfrak{g} = 2^{-\frac{1}{2}\dim N} dk\,d\mathfrak{a}\,d\mathfrak{n}.$$

To establish this, write

$$\mathfrak{p} = \mathfrak{a} + E_{\mathfrak{p}}\mathfrak{n},$$

the sum being orthogonal, hence

$$\begin{aligned} d\mathfrak{g} = dk\,d\mathfrak{p} &= dk\,d\mathfrak{a}\,dE_{\mathfrak{p}}\mathfrak{n} \\ &= |\det E_{\mathfrak{p}}|\mathfrak{n}|\,dk\,d\mathfrak{a}\,d\mathfrak{n}. \end{aligned}$$

Choose an orthonormal basis Z_i for $\mathfrak{n}$ -- then

$$(E_{\mathfrak{p}}Z_i, E_{\mathfrak{p}}Z_j)_\theta = \delta_{ij}/2$$

which implies that $\sqrt{2}\,E_{\mathfrak{p}}Z_i$ is an orthonormal basis for $E_{\mathfrak{p}}\mathfrak{n}$, so

$$|\det E_{\mathfrak{p}}\mathfrak{n}| = \begin{vmatrix} \frac{1}{\sqrt{2}} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \frac{1}{\sqrt{2}} \end{vmatrix} = 2^{-\frac{1}{2}\dim N}.$$

[Note:

$$\dim N = \dim G/K - \operatorname{rank} G/K.]$$

Therefore

$$d_{st}G = e^{2\rho(\log a)}\left(\frac{dK}{\operatorname{vol}(K)}\right) dA\,dN$$

$$= \frac{1}{\mathrm{vol}(K)} \frac{2^{-\frac{1}{2}\dim N}}{2^{-\frac{1}{2}\dim N}} e^{2\rho(\log a)} dKdAdN$$

$$= \frac{1}{\mathrm{vol}(K)} 2^{\frac{1}{2}\dim N} dG.$$

§1. *TRANSVERSALS*

Let G be a locally compact group.

1: SUBLEMMA Fix $x \in G$ -- then for any open neighborhood U of e there exists an open neighborhood V of x such that $V^{-1}V \subset U$.

2: DEFINITION A subgroup $\Gamma \subset G$ is a discrete subgroup if the relative topology on Γ is the discrete topology.

3: LEMMA A subgroup $\Gamma \subset G$ is discrete iff there exists an open neighborhood U of e (in G) such that $\Gamma \cap U = \{e\}$.

4: THEOREM Suppose that $\Gamma \subset G$ is a discrete subgroup -- then Γ is closed in G, hence G/Γ is a locally compact Hausdorff space (cf. I, §1, #14).

5: EXAMPLE

- Take $G = \mathbf{R}$, $\Gamma = \mathbf{Z}$.
- Take $G = \mathbf{A}$, $\Gamma = \mathbf{Q}$.
- Take $G = \mathbf{I}$, $\Gamma = \mathbf{Q}^\times$.

6: LEMMA Let Γ be a discrete subgroup of G -- then there exists an open neighborhood U_0 of e such that $U_0\gamma \cap U_0 = \emptyset$ for all $\gamma \neq e$ in Γ.

PROOF First choose U per #3. This done, choose V per #1 (with x = e) and put $U_0 = V$. Assume now that $u_0' \in U_0\gamma \cap U_0$, thus $u_0' = u_0\gamma$ ($\exists\ u_0 \in U_0$), so

$$\gamma = u_0^{-1}u_0' \in U_0^{-1}U_0 = V^{-1}V \subset U$$

$$\Rightarrow \gamma = e.$$

7: SUBLEMMA Let H be a closed subgroup of G and give G/H the quotient topology -- then the projection $\pi:G \to G/H$ is an open map.

[Let $U \subset G$ be a nonempty open set, the claim being that $\pi(U) \subset G/H$ is a nonempty open set. But $\pi(U)$ is open iff $\pi^{-1}(\pi(U))$ is open. And

$$\pi^{-1}(\pi(U)) = UH = \bigcup_{h \in H} Uh$$

which is a union of open sets.]

8: THEOREM Suppose that $\Gamma \subset G$ is a discrete subgroup -- then the projection $\pi:G \to G/\Gamma$ is a local homeomorphism.

PROOF Fix $x \in G$ and choose U_0 per #6 to get an open neighborhood xU_0 of x with the property that $\forall\ \gamma \neq e$ in Γ,

$$xU_0\gamma \cap xU_0 = x(U_0\gamma \cap U_0) = \emptyset.$$

Therefore the arrow $xU_0 \to \pi(xU_0)$ is a continuous bijection, hence is a homeomorphism (cf. #7).

9: DEFINITION Let Γ be a discrete subgroup of G -- then a Borel subset $\mathfrak{T} \subset G$ is a transversal for G/Γ if the restriction of π to $\mathfrak{T}$ is bijective.

10: N.B. In other words, a transversal $\mathfrak{T}$ for G/Γ is a Borel subset of G which meets each coset exactly once.

11: THEOREM Suppose that $\Gamma \subset G$ is a discrete subgroup. Assume: G is second countable -- then G/Γ admits a transversal $\mathfrak{T}$.

12: REMARK A transversal $\mathfrak{T}$ for G/Γ gives rise to a unique section $\tau:G/\Gamma \to \mathfrak{T} \subset G$ $(\pi \circ \tau = \mathrm{id})$ which is Borel measurable if G is second countable.

13: N.B. Tacitly, Lie groups are assumed to be second countable (cf. I, §5), hence σ-compact (cf. I, §2, #2).

[Note: Still, in this situation it is not claimed (nor is it true in general) that smooth sections exist.]

14: EXAMPLE Take $G = R$, $\Gamma = Z$ -- then $[0,1[$ is a transversal for R/Z.

15: EXAMPLE Take $G = A$, $\Gamma = Q$ -- then $\prod_p Z_p \times [0,1[$ is a transversal for A/Q.

16: EXAMPLE Take $G = I$, $\Gamma = Q^\times$ -- then $\prod_p Z_p^\times \times R_{>0}^\times$ is a transversal for $I/Q^\times$.

17: CONVENTION The Haar measure on a discrete group Γ is the counting measure:

$$\int_\Gamma f(\gamma)d_\Gamma(\gamma) = \sum_{\gamma\in\Gamma} f(\gamma).$$

[Note: Γ is unimodular (being discrete).]

18: LEMMA If $\Gamma \subset G$ is a discrete subgroup and if G is second countable, then Γ is at most countable.

19: LEMMA If $\Gamma \subset G$ is a discrete subgroup, if G is second countable and if $\mathfrak{T}$ is a transversal for G/Γ, then

$$G = \bigcup_{\gamma\in\Gamma} \mathfrak{T}\gamma \quad \text{(disjoint union)},$$

$$\int_G f d\mu_G = \sum_{\gamma\in\Gamma} \int_{\mathfrak{T}\gamma} f d\mu_G$$

$$= \int_{\mathfrak{T}} f^\Gamma \circ \pi d\mu_G.$$

[Note: $\forall\, x \in \mathfrak{T}$,

$$(f^{\Gamma} \circ \pi)(x) = f^{\Gamma}(x\Gamma)$$

$$= \int_{\Gamma} f(x\gamma)d\mu_{\Gamma}(\gamma)$$

$$= \sum_{\gamma\in\Gamma} f(x\gamma).]$$

<u>20:</u> RAPPEL If G is unimodular and if μ_G is fixed, then G/Γ admits an invariant measure $\mu_{G/\Gamma}$ characterized by the condition that for all $f \in C_c(G)$,

$$\int_G f(x)d\mu_G(x) = \int_{G/\Gamma} f^{\Gamma}(\dot{x})d\mu_{G/\Gamma}(\dot{x}) \quad (\dot{x} = x\Gamma) \qquad \text{(cf. I, §4, #7).}$$

<u>21:</u> THEOREM If $\Gamma \subset G$ is a discrete subgroup, if G is second countable, if $\mathfrak{T}$ is a transversal for G/Γ, if G is unimodular and if μ_G is fixed, then $\forall\, f \in C_c(G)$,

$$\int_{G/\Gamma} f^{\Gamma}d\mu_{G/\Gamma} = \int_{\mathfrak{T}} f^{\Gamma} \circ \pi d\mu_G.$$

[Simply assemble the foregoing data.]

[Note: Since the f^{Γ} $(f \in C_c(G))$ exhaust $C_c(G/\Gamma)$ (cf. I, §4, #3), it follows that $\forall\, \phi \in C_c(G/\Gamma)$,

$$\int_{G/\Gamma} \phi d\mu_{G/\Gamma} = \int_{\mathfrak{T}} \phi \circ \pi d\mu_G.$$

In particular, this holds for all ϕ if G/Γ is compact.]

<u>22:</u> DEFINITION Let Γ be a discrete subgroup of G -- then a Borel subset $\mathcal{F} \subset G$ is a <u>fundamental domain</u> for G/Γ if it differs from a transversal by a set of Haar measure 0 (left or right).

<u>23:</u> EXAMPLE Take $G = R$, $\Gamma = Z$ -- then $[0,1]$ is a fundamental domain for R/Z.

<u>24:</u> <u>N.B.</u> What was said in #21 goes through verbatim if "transversal" is replaced by "fundamental domain".

§2. *LATTICES*

Let G be a second countable locally compact group, $\Gamma \subset G$ a discrete subgroup.

<u>1:</u> NOTATION Given a finite subset $\Delta \subset \Gamma$, let G_Δ denote the centralizer of Δ in G.

<u>2:</u> <u>N.B.</u> G_Δ is closed in G.

<u>3:</u> LEMMA $G_\Delta\Gamma$ is closed in G.

PROOF Let $x_n \in G_\Delta$ and $\gamma_n \in \Gamma$ be sequences such that $x_n\gamma_n$ converges to a limit x -- then the claim is that $x \in G_\Delta\Gamma$. To begin with, $\forall\ \gamma \in \Delta$,

$$x^{-1}\gamma x = \lim_{n \to \infty} (\gamma_n^{-1}x_n^{-1}\gamma x_n\gamma_n)$$

$$= \lim_{n \to \infty} \gamma_n^{-1}\gamma\gamma_n.$$

Since Γ is discrete, $\exists\ n_0(\gamma)$:

$$n \geq n_0(\gamma) \Rightarrow \gamma_n^{-1}\gamma\gamma_n = \gamma_{n+1}^{-1}\gamma\gamma_{n+1}$$

$$\Rightarrow \gamma_{n+1}\gamma_n^{-1} \in G_\Delta.$$

But Δ is finite, thus $\exists\ n_0$ independent of the choice of γ such that

$$n \geq n_0 \Rightarrow \gamma_n = y_n\gamma_{n_0} (y_n \in G_\Delta)$$

$\Rightarrow$

$$x_n\gamma_n = x_ny_n\gamma_{n_0} = z_n\gamma_{n_0} \quad (z_n \in G_\Delta)$$

=>

$$z_n = x_n \gamma_n \gamma_{n_0}^{-1} \to x\gamma_{n_0}^{-1} \quad (n \to \infty)$$

=>

$$x\gamma_{n_0}^{-1} \in G_\Delta \Rightarrow x \in G_\Delta \Gamma.$$

4: NOTATION Given $\gamma \in \Gamma$, G_γ is its centralizer in G and Γ_γ ($= G_\gamma \cap \Gamma$) is its centralizer in Γ.

5: N.B. G_γ is a closed subgroup of G, as is Γ_γ (cf. §1, #4).

6: LEMMA $G_\gamma \Gamma$ is closed in G (cf. #3 (take $\Delta = \{\gamma\}$)).

7: SUBLEMMA If H is a closed subgroup of G, if $\pi: G \to G/H$ is the projection and if F is a closed subset of G that is the union of cosets xH, then $\pi(F)$ is closed in G/H.

8: APPLICATION The image of

$$G_\gamma \Gamma = \bigcup_{x \in G_\gamma} x\Gamma$$

in G/Γ is closed, hence is a locally compact Hausdorff space.

9: REMARK The projection $\pi: G \to G/\Gamma$ is an open map (cf. §1, #7) but, in general, it is not a closed map.

[Take G = R, Γ = Z and view R/Z as [0,1[equipped with the topology in which an open basis consists of all sets]a,b[$(0 < a < b < 1)$ and of all sets

$[0,a[\ \cup\ ]b,1[$ $(0 < a < b < 1)$ -- then

$$A = \{\frac{3}{2}, \frac{9}{4}, \ldots, n+2^{-n}, \ldots\}$$

is closed in R but

$$\pi(A) = \{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\}$$

is not closed in $[0,1[$.]

Considered as families of subsets of G, $G_\gamma\Gamma/\Gamma$ and $\pi(G_\gamma)$ are identical: The elements of $G_\gamma\Gamma/\Gamma$ are the cosets $x\Gamma$ with $x \in G_\gamma\Gamma$ and the elements of $\pi(G_\gamma)$ are the cosets $x\Gamma$ with $x \in G_\gamma$.

10: LEMMA The identity map

$$\{x\Gamma : x \in G_\gamma\Gamma\} \to \{x\Gamma : x \in G_\gamma\}$$

is a homeomorphism.

[Note: That is to say, the two topologies are the same.]

11: N.B. One may then identify $\pi(G_\gamma)$ with $G_\gamma\Gamma/\Gamma$ which is therefore closed in G/Γ (cf. #8).

12: NOTATION Let

$$r : G_\gamma\Gamma/\Gamma \to G_\gamma/G_\gamma \cap \Gamma$$

be the arrow defined by

$$r(x\Gamma) = x(G_\gamma \cap \Gamma).$$

13: N.B. r is bijective.

14: THEOREM r is a homeomorphism.

This is not completely obvious and it will be best to break the proof into two parts.

15: LEMMA r carries open subsets of $G_\gamma\Gamma/\Gamma$ onto open subsets of $G_\gamma/G_\gamma \cap \Gamma$.

PROOF An open subset of $G_\gamma\Gamma/\Gamma$ is a subset $\{x\Gamma : x \in X\}$, where $X \subset G_\gamma$, such that $X\Gamma$ is open in $G_\gamma\Gamma$ viewed as a subspace of G. Since

$$X(G_\gamma \cap \Gamma) = X\Gamma \cap G_\gamma,$$

it follows that $X(G_\gamma \cap \Gamma)$ is an open subset of G_γ in its relative topology as a subspace of G, thus by the very definition of the topology on $G_\gamma/G_\gamma \cap \Gamma$,

$$r\{x\Gamma : x \in X\} = \{x(G_\gamma \cap \Gamma) : x \in X\}$$

is an open subset of $G_\gamma/G_\gamma \cap \Gamma$.

16: LEMMA r^{-1} carries open subsets of $G_\gamma/G_\gamma \cap \Gamma$ onto open subsets of $G_\gamma\Gamma/\Gamma$.

PROOF Let $\{y(G_\gamma \cap \Gamma) : y \in Y\}$ $(Y \subset G_\gamma)$ be an open subset of $G_\gamma/G_\gamma \cap \Gamma$ -- then $Y(G_\gamma \cap \Gamma)$ is an open subset of G_γ, so

$$\pi(Y(G_\gamma \cap \Gamma)) = \{y\Gamma : y \in Y\}$$

is open in $G_\gamma\Gamma/\Gamma$ (see the Appendix infra) or still,

$$\{y\Gamma : y \in Y\} = r^{-1}\{y(G_\gamma \cap \Gamma) : y \in Y\}$$

is open in $G_\gamma\Gamma/\Gamma$.

17: EXAMPLE Take $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$ and $H = \sqrt{2}\,\mathbb{Z}$ -- then the argument used in

#15 is applicable if the G_γ there is replaced by H, thus the map

$$H/H \cap \Gamma \to H + \Gamma/\Gamma$$

is continuous. Nevertheless, it is not a homeomorphism.

[$H \cap \Gamma$ is trivial so $H/H \cap \Gamma$ is isomorphic to Z and carries the discrete topology. Meanwhile, $H + \Gamma = \sqrt{2}\, Z + Z$ is dense in R, hence

$$H + \Gamma/\Gamma = \sqrt{2}\, Z + Z/Z$$

is dense in $R/Z \approx T$. It is isomorphic to Z as a group but it is not discrete since every nonempty open subset of T intersects it in an infinite set implying thereby that none of its finite subsets are open.]

[Note: The difference here is this: $G_\gamma \Gamma/\Gamma$ is locally compact but $H + \Gamma/\Gamma$ is not locally compact.]

<u>18:</u> DEFINITION Γ is said to be a <u>lattice</u> if G/Γ admits a finite invariant measure (cf. I, §4, #4), Γ being termed <u>uniform</u> or <u>nonuniform</u> according to whether G/Γ is compact or not.

<u>19:</u> <u>N.B.</u> If there is a lattice in G, then G is necessarily unimodular (cf. I, §3, #11 and I, §4, #10).

[Note: A discrete cocompact subgroup is necessarily a uniform lattice... .]

<u>20:</u> EXAMPLE Z is a uniform lattice in R.

<u>21:</u> EXAMPLE $SL(2,Z)$ is a nonuniform lattice in $SL(2,R)$.

<u>22:</u> THEOREM Suppose that $\Gamma \subset G$ is a uniform lattice -- then $\forall\, \gamma \in \Gamma$, G_γ/Γ_γ is compact.

PROOF $G_\gamma \Gamma/\Gamma$ is closed in G/Γ, hence is compact (this being the case of G/Γ). On the other hand,

$$r: G_\gamma \Gamma/\Gamma \to G_\gamma / G_\gamma \cap \Gamma \quad (= G_\gamma/\Gamma_\gamma)$$

is a homeomorphism (cf. #14).

[Note: Consequently, $\Gamma_\gamma \subset G_\gamma$ is a uniform lattice and G_γ is unimodular.]

23: NOTATION $[\Gamma]$ is a set of representatives for the Γ-conjugacy classes in Γ.

Put

$$\mathcal{S} = \coprod_{\gamma \in [\Gamma]} G/\Gamma_\gamma \times \{\gamma\}$$

and define $\psi: \mathcal{S} \to G$ by the rule

$$\psi(x\Gamma_\gamma, \gamma) = x\gamma x^{-1}.$$

24: N.B. Γ_γ is a discrete subgroup of G, thus Γ_γ is closed in G (cf. §1, #4) and therefore the quotient G/Γ_γ is a locally compact Hausdorff space from which it follows that $\mathcal{S}$ is a locally compact Hausdorff space.

25: DEFINITION Let X and Y be locally compact Hausdorff spaces, $f: X \to Y$ a continuous function -- then f is proper if for every compact subset K of Y, the inverse image $f^{-1}(K)$ is a compact subset of X.

26: THEOREM Suppose that $\Gamma \subset G$ is a uniform lattice -- then ψ is a proper map.

27: NOTATION Given $\gamma \in \Gamma$, let

$$[\gamma]_G = \{x\gamma x^{-1} : x \in G\}.$$

28: APPLICATION In the uniform situation, for any compact subset $K \subset G$,

$$\{\gamma \in [\Gamma] : [\gamma]_G \cap K \neq \emptyset\}$$

is finite.

29: LEMMA A proper map $f:X \to Y$ is closed:

$$S \subset X \text{ closed} \Rightarrow f(S) \subset Y \text{ closed}.$$

30: APPLICATION In the uniform situation, $\forall\ \gamma \in \Gamma$, $[\gamma]_G$ is closed.

[In fact,

$$[\gamma]_G = \psi\Big(\bigcup_{\gamma_0 \in [\gamma]_G \cap [\Gamma]} G/\gamma_0 \times \{\gamma_0\}\Big).]$$

31: N.B. Accordingly, $[\gamma]_G$ is a locally compact Hausdorff space and the canonical arrow

$$G/G_\gamma \to [\gamma]_G$$

is a homeomorphism.

APPENDIX

Denote by $\pi|G_\gamma$ the restriction of $\pi:G \to G/\Gamma$ to G_γ.

CRITERION Suppose that there exist nonempty open sets

$$U \subset G_\gamma,\ V \subset G_\gamma\Gamma/\Gamma$$

such that the restriction of $\pi|G_\gamma$ to U is an open continuous map of U onto V -- then $\pi|G_\gamma$ is open.

PROOF Given $x \in G_\gamma$ and an open neighborhood W of x in G_γ, it suffices to show that $(\pi|G_\gamma)(W)$ contains an open neighborhood N_x of $(\pi|G_\gamma)(x)$. So fix a point $y \in U$ and put

$$\tilde{U} = U \cap yx^{-1}W,$$

an open neighborhood of y in U, thus the image $(\pi|G_\gamma)(\tilde{U})$ is an open subset of $G_\gamma\Gamma/\Gamma$ or still, $\tilde{U}\Gamma$ is an open subset of $G_\gamma\Gamma$, hence

$$xy^{-1}\tilde{U}\Gamma = (xy^{-1}U \cap W)\Gamma$$

is an open subset of $G_\gamma\Gamma$ and

$$x \in xy^{-1}U \cap W.$$

Put now

$$N_x = \{z\Gamma : z \in xy^{-1}U \cap W\}.$$

Then N_x is an open subset of $G_\gamma\Gamma/\Gamma$ contained in

$$(\pi|G_\gamma)(W) = \{w\Gamma : w \in W\}$$

to which $(\pi|G_\gamma)(x)$ belongs.

There is a commutative diagram

$$\begin{array}{ccc} G_\gamma & \xrightarrow{\ \pi|G_\gamma\ } & G_\gamma\Gamma/\Gamma \\ \Vert & & \uparrow{\scriptstyle r^{-1}} \\ G_\gamma & \xrightarrow[\ \pi_\gamma\ ]{} & G_\gamma/G_\gamma \cap \Gamma \end{array}$$

and $G_\gamma\Gamma/\Gamma$ is a locally compact Hausdorff space (cf. #8), thus is a Baire space.

LEMMA $\pi|G_\gamma$ is an open map.

PROOF The quotient $G_\gamma/G_\gamma \cap \Gamma$ is second countable, hence σ-compact, hence

$$G_\gamma/G_\gamma \cap \Gamma = \bigcup_{n=1}^{\infty} K_n,$$

where $K_1, K_2, \ldots$ are compact. In view of #15,

$$r^{-1}:G_\gamma/G_\gamma \cap \Gamma \rightarrow G_\gamma\Gamma/\Gamma$$

is continuous and one-to-one, so $\forall$ n the restriction of r^{-1} to K_n is a homeomorphism of K_n onto $L_n \equiv r^{-1}(K_n)$:

$$G_\gamma\Gamma/\Gamma = \bigcup_{n=1}^{\infty} L_n,$$

a countable union of compacta. Being Baire, it therefore follows that $\exists\ n \in N$ and a nonempty open subset V of $G_\gamma\Gamma/\Gamma$ such that $V \subset r^{-1}(L_n)$. Put

$$U = (\pi|G_\gamma)^{-1}(V).$$

Then $U \subset G_\gamma$ is nonempty and open and the restriction of π_γ to U is an open continuous map of U onto r(V) or still, the restriction of $\pi|G_\gamma$ to U is an open continuous map of U onto V.

§3. *UNIFORMLY INTEGRABLE FUNCTIONS*

Let G be a unimodular locally compact group and, generically, let $\mathfrak{U}$ be a compact symmetric neighborhood of the identity in G.

1: NOTATION Given a continuous function f on G, put

$$f_{\mathfrak{U}}(y) = \sup_{x,z\in\mathfrak{U}} |f(xyz)| \qquad (y \in G).$$

2: LEMMA $f_{\mathfrak{U}} \in C(G)$, i.e., is a continuous function on G.

3: DEFINITION A continuous function f on G is said to be uniformly integrable if there exists a $\mathfrak{U}$ such that $f_{\mathfrak{U}} \in L^1(G)$.

4: N.B. Since $|f| \leq f_{\mathfrak{U}}$, it is clear that if f is uniformly integrable, then f is integrable: $f \in L^1(G)$.

5: NOTATION Write $C_{UN}(G)$ for the set of continuous functions on G that are uniformly integrable.

6: LEMMA

$$C_c(G) \subset C_{UN}(G) \subset C_0(G).$$

[Note: As usual, $C_c(G)$ is the set of continuous functions on G that are compactly supported and $C_0(G)$ is the set of continuous functions on G that vanish at infinity.]

7: LEMMA

$$C_{UN}(G) \subset L^2(G).$$

[Integrable functions in $C_0(G)$ are square integrable.

8: EXAMPLE Take $G = R$ -- then $f(x) = e^{-x^2}$ is uniformly integrable.

9: LEMMA If $f,g \in C_{UN}(G)$, then $f*g \in C_{UN}(G)$.

[Working with a common $\mathcal{U}$,

$$(f*g)_{\mathcal{U}}(y) = \sup_{x,z\in\mathcal{U}} \left|\int_G f(u)g(u^{-1}xyz)d\mu_G(u)\right|$$

$$= \sup_{x,z\in\mathcal{U}} \left|\int_G f(xu)g(u^{-1}yz)d\mu_G(u)\right|$$

$$\leq \sup_{x,z\in\mathcal{U}} \int_G |f(xu)g(u^{-1}yz)|d\mu_G(u)$$

$$\leq \int_G f_{\mathcal{U}}(u)g_{\mathcal{U}}(u^{-1}y)d\mu_G(u)$$

$$= (f_{\mathcal{U}}*g_{\mathcal{U}})(y),$$

which suffices.

[Note: The convolution $f*g$ is continuous.]

Let $H \subset G$ be a closed subgroup and assume that H is unimodular and cocompact.

10: NOTATION $L^2(G/H)$ is the Hilbert space associated with $\mu_{G/H}$ (the invariant measure on G/H per I, §4, #5).

11: NOTATION $L_{G/H}$ is the left translation representation of G on $L^2(G/H)$.

12: THEOREM Let $f \in C_{UN}(G)$ -- then

$$L_{G/H}(f) = \int_G f(x)L_{G/H}(x)d\mu_G(x)$$

is an integral operator on $L^2(G/H)$ with continuous kernel

$$K_f(x,y) = \int_H f(xhy^{-1})d\mu_H(h).$$

Since

$$C(G/H \times G/H) \subset L^2(G/H \times G/H),$$

it follows that $\forall f \in C_{UN}(G)$, $L_{G/H}(f)$ is Hilbert-Schmidt, hence is compact.

<u>13:</u> SUBLEMMA Let U be a unitary representation of G on a Hilbert space $\mathcal{H}$ with the property that $\forall f \in C_c(G)$, the operator

$$U(f) = \int_G f(x)U(x)d\mu_G(x)$$

is compact -- then U is discretely decomposable, a given irreducible unitary representation of G occurring at most a finite number of times in the orthogonal decomposition of U.

[Note: If G is a Lie group, then one can replace $C_c(G)$ by $C_c^\infty(G)$.]

<u>14:</u> <u>N.B.</u> If G is second countable, then $\mathcal{H}$ is separable.

<u>15:</u> APPLICATION Take $\mathcal{H} = L^2(G/H)$, $U = L_{G/H}$ -- then there exist non-negative integers $m(\Pi, L_{G/H})$ $(\Pi \in \hat{G})$ such that

$$L_{G/H} = \hat{\bigoplus_{\Pi \in \hat{G}}} m(\Pi, L_{G/H})\Pi.$$

§4. *THE SELBERG TRACE FORMULA*

Let G be a second countable locally compact group, $\Gamma \subset G$ a discrete subgroup. Assume: Γ is a uniform lattice -- then G/Γ is cocompact and G is necessarily unimodular (cf. §2, #19).

Working with $L^2(G/\Gamma)$, there is an orthogonal decomposition

$$L_{G/\Gamma} = \widehat{\bigoplus_{\Pi \in \hat{G}}} m(\Pi, L_{G/\Gamma})\Pi \quad \text{(cf. §3, #15)},$$

the multiplicities $m(\Pi, L_{G/\Gamma})$ being certain nonnegative integers.

1: RAPPEL $\forall f \in C_{UN}(G)$, $L_{G/\Gamma}(f)$ is an integral operator on $L^2(G/\Gamma)$ with continuous kernel

$$K_f(x,y) = \sum_{\gamma \in \Gamma} f(x\gamma y^{-1}) \quad \text{(cf. §3, #12)}.$$

[Note: This implies that $L_{G/\Gamma}(f)$ is Hilbert-Schmidt.]

2: CONVENTION Fix a Haar measure μ_G on G, take the counting measure on Γ, and normalize the invariant measure $\mu_{G/\Gamma}$ on G/Γ by the stipulation

$$\int_G = \int_{G/\Gamma} \int_\Gamma \; (= \int_{G/\Gamma} \sum_\Gamma).$$

If $f = g * g^*$ ($g \in C_{UN}(G)$), then $f \in C_{UN}(G)$ (cf. §3, #9),

$$L_{G/\Gamma}(f) = L_{G/\Gamma}(g) L_{G/\Gamma}(g)^*$$

is trace class and (cf. B, II, §2, #8)

$$\text{tr}(L_{G/\Gamma}(f)) = \int_{G/\Gamma} K_f(\dot{x},\dot{x})\, d\mu_{G/\Gamma}(\dot{x}) \quad (\dot{x} = x\Gamma).$$

3: REMARK The assumption that $f = g * g^*$ ($g \in C_{UN}(G)$) is not restrictive. For if $f = g * h^*$ ($g,h \in C_{UN}(G)$), put

$$T(g,h) = g * h^*$$

and using the same letter for the diagonal, note that

$$T(g,h) = \frac{1}{4}(T(g+h) - T(g-h)$$

$$- \sqrt{-1}\, T(g - \sqrt{-1}\, h) + \sqrt{-1}\, T(g + \sqrt{-1}\, h)).$$

Let $\chi_{G/\Gamma}$ be the characteristic function of G/Γ, i.e., the function $\equiv 1$. Choose $\alpha \in C_c(G):\alpha^\Gamma = \chi_{G/\Gamma}$ (cf, I, §4, #3), thus $\forall\, x \in G$,

$$\alpha^\Gamma(x\Gamma) = \sum_{\gamma\in\Gamma} \alpha(x\gamma) = 1.$$

One can then write

$$\sum_{\Pi\in\hat{G}} m(\Pi, L_{G/\Gamma})\operatorname{tr}(\Pi(f))$$

$$= \operatorname{tr}(L_{G/\Gamma}(f))$$

$$= \int_{G/\Gamma} K_f(\dot{x},\dot{x})\, d\mu_{G/\Gamma}(\dot{x})$$

$$= \int_{\mathfrak{T}} K_f(x\Gamma,x\Gamma)\, d\mu_G(x) \qquad \text{(cf. §1, \#21)}$$

$$= \int_{\mathfrak{T}} \Big(\sum_{\gamma\in\Gamma} \alpha(x\gamma)\Big) K_f(x\Gamma,x\Gamma)\, d\mu_G(x)$$

$$= \int_{\mathfrak{T}} \sum_{\gamma\in\Gamma} \alpha(x\gamma) K_f(x\gamma,x\gamma)\, d\mu_G(x)$$

$$= \int_{\mathfrak{T}} (\alpha K_f)^\Gamma \circ \pi(x)\, d\mu_G(x)$$

$$= \int_G \alpha(x) K_f(x,x)\, d\mu_G(x)$$

$$= \int_G \alpha(x) \sum_{\gamma\in\Gamma} f(x\gamma x^{-1})\, d\mu_G(x)$$

$$= \sum_{\gamma\in\Gamma} \int_G \alpha(x) f(x\gamma x^{-1})\, d\mu_G(x).$$

4: NOTATION For any $\gamma \in \Gamma$,

$$\begin{cases} G_\gamma = \text{centralizer of } \gamma \text{ in } G \\ \Gamma_\gamma = \text{centralizer of } \gamma \text{ in } \Gamma. \end{cases}$$

5: RAPPEL Γ_γ is a uniform lattice in G_γ (cf. §2, #22).

[Note: Consequently, G_γ is unimodular.]

6: NOTATION For any $\gamma \in \Gamma$,

$$\begin{cases} [\gamma]_\Gamma = \text{conjugacy class of } \gamma \text{ in } \Gamma \\ [\gamma]_G = \text{conjugacy class of } \gamma \text{ in } G. \end{cases}$$

7: RAPPEL There are canonical bijections

$$\begin{cases} \Gamma/\Gamma_\gamma \to [\gamma]_\Gamma \\ G/G_\gamma \to [\gamma]_G. \end{cases}$$

Returning to the computation, break the sum over Γ into conjugacy classes in Γ, the contribution from

$$[\gamma]_\Gamma = \{\delta\gamma\delta^{-1} : \delta \in \Gamma/\Gamma_\gamma\}$$

being

$$\sum_{\delta\in\Gamma/\Gamma_\gamma} \int_G \alpha(x) f(x\delta\gamma\delta^{-1}x^{-1})\, d\mu_G(x)$$

$$= \sum_{\delta\in\Gamma/\Gamma_\gamma} \int_G \alpha(x\delta^{-1}) f(x\delta x^{-1})\, d\mu_G(x)$$

$$= \int_G \Big(\sum_{\delta\in\Gamma/\Gamma_\gamma} \alpha(x\delta^{-1})\Big) f(x\gamma x^{-1})\, d\mu_G(x).$$

8: CONVENTION Supplementing the agreements in #2, fix a Haar measure μ_{G_γ} on G_γ, take the counting measure on Γ_γ, and normalize the invariant measure $\mu_{G_\gamma/\Gamma_\gamma}$ on G_γ/Γ_γ by the stipulation

$$\int_{G_\gamma} = \int_{G_\gamma/\Gamma_\gamma} \int_{\Gamma_\gamma} \Big(= \int_{G_\gamma/\Gamma_\gamma} \sum_{\Gamma_\gamma}\Big).$$

Next, fix μ_{G/G_γ} via

$$\int_G = \int_{G/G_\gamma} \int_{G_\gamma}.$$

Finally, make the identification

$$G/\Gamma_\gamma \approx (G/G_\gamma) \times (G_\gamma/\Gamma_\gamma)$$

and put

$$\int_{G/\Gamma_\gamma} = \int_{G/G_\gamma} \int_{G_\gamma/\Gamma_\gamma}.$$

Moving on,

$$\int_G \Big(\sum_{\delta\in\Gamma/\Gamma_\gamma} \alpha(x\delta^{-1})\Big) f(x\gamma x^{-1})\, d\mu_G(x)$$

$$= \int_{G/G_\gamma} \int_{G_\gamma} \int_{\Gamma/\Gamma_\gamma} \cdots$$

$$= \int_{G/G_\gamma} \int_{G_\gamma/\Gamma_\gamma} \int_{\Gamma_\gamma} \int_{\Gamma/\Gamma_\gamma} \cdots .$$

But

$$\int_{\Gamma_\gamma} \int_{\Gamma/\Gamma_\gamma} \alpha(x\eta\delta^{-1})$$

is $\equiv 1$, leaving

$$\int_{G/G_\gamma} \int_{G_\gamma/\Gamma_\gamma} \cdots = \int_{G/\Gamma_\gamma} \cdots .$$

Summary:

$$\operatorname{tr}(L_{G/\Gamma}(f)) = \sum_{\gamma \in [\Gamma]} \int_{G/\Gamma_\gamma} f(x\gamma x^{-1})\, d\mu_{G/\Gamma_\gamma}(\dot{x}),$$

the sum being taken over a set of representatives for the Γ-conjugacy classes in Γ (cf. §2, #23).

<u>9:</u> <u>N.B.</u>

$$\int_{G/\Gamma_\gamma} f(x\gamma x^{-1})\, d\mu_{G/\Gamma_\gamma}(\dot{x})$$

$$= \int_{G/G_\gamma} \left(\int_{G_\gamma/\Gamma_\gamma} f(x\eta\gamma\eta^{-1}x^{-1})\, d\mu_{G_\gamma/\Gamma_\gamma}(\dot{\eta})\right) d\mu_{G/G_\gamma}(\dot{x})$$

$$= \int_{G/G_\gamma} \left(\int_{G_\gamma/\Gamma_\gamma} f(x\gamma x^{-1})\, d\mu_{G_\gamma/\Gamma_\gamma}(\dot{\eta})\right) d\mu_{G/G_\gamma}(\dot{x})$$

$$= \int_{G/G_\gamma} f(x\gamma x^{-1}) \left(\int_{G_\gamma/\Gamma_\gamma} d\mu_{G_\gamma/\Gamma_\gamma}\right) d\mu_{G/G_\gamma}(\dot{x})$$

$$= \operatorname{vol}(G_\gamma/\Gamma_\gamma) \int_{G/G_\gamma} f(x\gamma x^{-1})\, d\mu_{G/G_\gamma}(\dot{x}).$$

<u>10:</u> DEFINITION Given $f \in C_{UN}(G) * C_{UN}(G)$, the <u>Selberg trace formula</u> is

the relation

$$\sum_{\Pi \in \hat{G}} m(\Pi, L_{G/\Gamma}) \mathrm{tr}(\Pi(f))$$

$$= \sum_{\gamma \in [\Gamma]} \mathrm{vol}(G_\gamma/\Gamma_\gamma) \int_{G/G_\gamma} f(x\gamma x^{-1}) d\mu_{G/G_\gamma}(\dot{x}),$$

their common value being

$$\mathrm{tr}(L_{G/\Gamma}(f)).$$

<u>11:</u> REMARK Suppose that G is a Lie group -- then

$$C_c^\infty(G) * C_c^\infty(G) = C_c^\infty(G) \quad \text{(Dixmier-Malliavin).}$$

Since

$$C_c^\infty(G) \subset C_{UN}(G),$$

it follows that the Selberg trace formula is valid for all $f \in C_c^\infty(G)$.

Let G be a second countable locally compact group, $\Gamma \subset G$ a uniform lattice.

<u>12:</u> LEMMA Let $\chi: G \to \mathbb{T}$ be a unitary character -- then the multiplicity of χ in $L^2(G/\Gamma)$ is 1 if $\chi(\Gamma) = \{1\}$ and is 0 otherwise.

Now take G abelian and identify $\hat{G}$ with the unitary character group of $G: \Pi \longleftrightarrow \chi$, the Fourier transform being defined by

$$\mathrm{tr}(\Pi(f)) = \hat{f}(\chi) = \int_G f(x)\chi(x) d\mu_G(x).$$

<u>13:</u> NOTATION Let

$$\Gamma^\perp = \{\chi \in \hat{G} : \chi(\gamma) = 1 \ \forall \ \gamma \in \Gamma\}.$$

14: N.B. Therefore

$$\begin{cases} \chi \in \Gamma^{\perp} \Rightarrow m(\chi, L_{G/\Gamma}) = 1 \\ \chi \in \Gamma^{\perp} \Rightarrow m(\chi, L_{G/\Gamma}) = 0. \end{cases}$$

The Selberg trace formula thus simplifies:

- Matters on the "spectral side" reduce to

$$\sum_{\gamma \in \Gamma^{\perp}} \hat{f}(\chi).$$

- Matters on the "geometric side" reduce to

$$\mathrm{vol}(G/\Gamma) \sum_{\gamma \in \Gamma} f(\gamma).$$

15: DEFINITION The relation

$$\sum_{\chi \in \Gamma^{\perp}} \hat{f}(\chi) = \mathrm{vol}(G/\Gamma) \sum_{\gamma \in \Gamma} f(\gamma)$$

is the Poisson summation formula (cf. A, III, §4, #7) (in that situation

$$\mathrm{vol}(G/\Gamma) = \frac{|G|}{|\Gamma|}).$$

§5. *FUNCTIONS OF REGULAR GROWTH*

Let G be a second countable locally compact group, $\Gamma \subset G$ a uniform lattice. While $C_{UN}(G)$ is theoretically convenient, there is a larger class of functions that can be fed into the Selberg trace formula.

1: DEFINITION Let $\phi \in C(G) \cap L^1(G)$ be nonnegative -- then ϕ is said to be of regular growth if there is a compact symmetric neighborhood $\mathfrak{U}$ of the identity in G and a positive constant C (depending on ϕ and $\mathfrak{U}$) such that $\forall\, y \in G$,

$$\phi(y) \leq C \int_U \phi(xy)\, d\mu_G(x).$$

2: N.B. In terms of the characteristic function $\chi_{\mathfrak{U}}$ of $\mathfrak{U}$, $\forall\, y \in G$,

$$\begin{aligned}(\chi_U * \phi)(y) &= \int_G \chi_{\mathfrak{U}}(x)\phi(x^{-1}y)\, d\mu_G(x) \\ &= \int_{\mathfrak{U}} \phi(x^{-1}y)\, d\mu_G(x) \\ &= \int_{\mathfrak{U}^{-1}} \phi(xy)\, d\mu_G(x) \\ &= \int_{\mathfrak{U}} \phi(xy)\, d\mu_G(x).\end{aligned}$$

3: EXAMPLE Take $G = R^n$ and fix a real number $r > 0$ such that

$$\int_{R^n} \frac{1}{(1 + ||Y||)^r}\, dY < \infty.$$

Given $\mathfrak{U}$, fix a real number $N > 0$ such that $\forall\, X \in \mathfrak{U}$,

$$(1 + ||Y||)^{-r} \leq N(1 + ||X + Y||)^{-r}.$$

Then

$$\frac{1}{\mathrm{vol}(\mathfrak{U})} \int_{\mathfrak{U}} \frac{dX}{(1 + ||X + Y||)^r}$$

$$\geq \frac{1}{\mathrm{vol}(\mathfrak{U})} \int_{\mathfrak{U}} \frac{dX}{(1 + ||Y||)^r}$$

$$= (1 + ||Y||)^{-r}.$$

Therefore

$$\phi(Y) = (1 + ||Y||)^{-r}$$

is of regular growth

<u>4:</u> EXAMPLE Let G be a connected semisimple Lie group with finite center and fix a real number $r > 0$ such that

$$\int_G \Xi^2(y) \ (1 + \sigma(y))^{-r} d_G(y) < \infty.$$

Given $\mathfrak{U}$, fix a real number $M > 0$ such that $\forall\ x \in \mathfrak{U}$,

$$\Xi^2(y) \leq M \ \Xi^2(xy)$$

and fix a real number $N > 0$ such that $\forall\ x \in \mathfrak{U}$,

$$(1 + \sigma(y))^{-r} \leq N(1 + \sigma(xy))^{-r}.$$

Then

$$\frac{MN}{\mathrm{vol}(\mathfrak{U})} \int_{\mathfrak{U}} \Xi^2(xy)\,(1 + \sigma(xy))^{-r} d\mu_G(x)$$

$$\geq \frac{1}{\mathrm{vol}(\mathfrak{U})} \int_{\mathfrak{U}} \Xi^2(y)\,(1 + \sigma(y))^{-r} d\mu_G(x)$$

$$= \Xi^2(y)\,(1 + \sigma(y))^{-r}.$$

Therefore

$$\phi(y) = |\text{---}\circ\text{---}|^2(y)(1 + \sigma(y))^{-r}$$

is of regular growth.

5: DEFINITION Let f be a continuous function on G -- then f is admissible if there exists a function ϕ of regular growth such that $\forall\, y \in G$,

$$|f(y)| \leq \phi(y) \; (\leq C(\chi_{\mathcal{U}} * \phi)(y)).$$

[Note: Admissible functions are integrable.]

6: EXAMPLE The rapidly decreasing functions on R^n are admissible (cf. #3).

7: LEMMA If $f \in C_{\text{UN}}(G)$, then f is admissible.

PROOF $\forall\, y \in G$, $|f(y)| \leq f_{\mathcal{U}}(y)$. And

$$f_{\mathcal{U}}(y) = \sup_{u,z\in\mathcal{U}} |f(uyz)|$$

$$\leq \sup_{u,z\in\mathcal{U}} |f(uxyz)| \quad (x \in \mathcal{U})$$

=>

$$f_{\mathcal{U}}(y) = \frac{\text{vol}(\mathcal{U})}{\text{vol}(\mathcal{U})} f_{\mathcal{U}}(y)$$

$$\leq \frac{1}{\text{vol}(\mathcal{U})} \int_{\mathcal{U}} f_{\mathcal{U}}(y)\, d\mu_G(x)$$

$$\leq \frac{1}{\text{vol}(\mathcal{U})} \int_{\mathcal{U}} \sup_{x,z\in\mathcal{U}} |f(uxyz)|\, d\mu_G(x)$$

$$= \frac{1}{\text{vol}(\mathcal{U})} \int_{\mathcal{U}} f_{\mathcal{U}}(xy)\, d\mu_G(x).$$

Therefore f_{11} is of regular growth, hence f is admissible.

8: LEMMA Suppose that $|f| \leq |g|$, where g is admissible, say $|g| \leq \psi$ -- then f is admissible (clear) as is $f * f$.

[For

$$|f * f| \leq |f| * |f|$$
$$\leq |g| * |g|$$
$$\leq \psi * |g|.$$

And $\psi * |g|$ is of regular growth:

$$\psi * |g| \leq (C\chi_{11} * \psi) * |g|$$
$$= C(\chi_{11} * (\psi * |g|)).$$

The condition of admissible is then met by

$$\phi = \psi * |g|.]$$

[Note: If $f_1, f_2 \in C(G) \cap L^1(G)$ and if f_1 is admissible, then $f_1 * f_2$ is admissible. Proof:

$$|f_1 * f_2| \leq |f_1| * |f_2|$$
$$\leq \phi_1 * |f_2|$$
$$\leq C(\chi_{11} * (\phi_1 * |f_2|)).]$$

9: DEFINITION A series of functions $f_1, f_2, \ldots$ on a locally compact Hausdorff space X is locally dominantly absolutely convergent (ldac) if for every compact set $K \subset X$ there exists a positive constant M_K such that $\forall\ k \in K$,

$$\sum_n |f_n(k)| < M_K.$$

10: CRITERION Let $f \in C(G) \cap L^1(G)$. Assume: The operator $L_{G/\Gamma}(f)$ is trace class and the series

$$\sum_{\gamma\in\Gamma} f(x\gamma y^{-1})$$

is ldac on $G \times G$ to a separately continuous function -- then the Selberg trace formula obtains:

$$\mathrm{tr}(L_{G/\Gamma}(f)) = \sum_{\gamma\in[\Gamma]} \mathrm{vol}(G_\gamma/\Gamma_\gamma) \int_{G/G_\gamma} f(x\gamma x^{-1})d\mu_{G/G_\gamma}(\dot{x}),$$

the sum on the right hand side being absolutely convergent.

[First of all,

$$\mathrm{tr}(L_{G/\Gamma}(f)) = \int_{G/\Gamma} \sum_{\gamma\in\Gamma} f(x\gamma x^{-1})d\mu_{G/\Gamma}(\dot{x}) \quad \text{(cf. B, II, §2, \#8)}.$$

Proceeding, fix a compact set $K \subset G$: $K\Gamma = G$ (cf. #11 infra) and choose $M_K > 0$:

$$k,\ell \in K \Rightarrow \sum_{\gamma\in\Gamma} |f(k\gamma\ell^{-1})| < M_K.$$

Here, of course, the ldac condition is per $K \times K^{-1} \subset G \times G$. Given $x,y \in G$, $\exists\ \gamma_x,\gamma_y \in \Gamma: x\gamma_x, y\gamma_y \in K$, so

$$\sum_{\gamma\in\Gamma} |f(x\gamma y^{-1})| = \sum_{\gamma\in\Gamma} |f(x\gamma_x\gamma\gamma_y^{-1}y^{-1})| < M_K,$$

from which

$$M_K\,\mathrm{vol}(G/\Gamma) \geq \int_{G/\Gamma} \sum_{\gamma\in\Gamma} |f(x\gamma y^{-1})|d\mu_{G/\Gamma}(\dot{x}).$$

Now interchange sum and integral, the ensuing formal manipulation being justified by Fubini.]

11: SUBLEMMA There exists a compact set $K \subset G$ such that $K\Gamma = G$.

[Let U be an open neighborhood of e such that $\bar{U}$ is compact -- then the collection $\{\pi(xU):x \in G\}$ is an open covering of G/Γ, thus there is a finite subcollection

$$\pi(x_1U),\ \pi(x_2U),\ldots,\pi(x_nU)$$

that covers G/Γ and one may take

$$K = x_1\bar{U} \cup x_2\bar{U} \cup \cdots \cup x_n\bar{U}.$$

Indeed,

$$G/\Gamma = \{k\Gamma:k \in K\},$$

so given $x \in G$,

$$x\Gamma = k\Gamma\ (\exists\ k) \Rightarrow x = k\gamma\ (\exists\ \gamma) \Rightarrow x \in K\Gamma.]$$

[Note: It can be shown that K contains a transversal $\mathfrak{T}$ which is therefore relatively compact.]

Suppose that f is admissible -- then $\forall\ x,y \in G$,

$$|f(x\gamma y^{-1})| \leq \phi(x\gamma y^{-1})$$

$$\leq C\int_{\mathfrak{U}} \phi(ux\gamma y^{-1})d\mu_G(u).$$

<u>12:</u> LEMMA Fix $x,y \in G$ -- then $\forall\ \gamma_1,\gamma_2 \in \Gamma$,

$$\mathfrak{U}x\gamma_1y^{-1} \cap \mathfrak{U}x\gamma_2y^{-1} \neq \emptyset$$

iff

$$\gamma_2\gamma_1^{-1} \in x^{-1}\mathfrak{U}^{-1}\mathfrak{U}x.$$

[In one direction,

$$u_1 x\gamma_1 y^{-1} = u_2 x\gamma_2 y^{-1}$$

=>

$$u_1 x\gamma_1 = u_2 x\gamma_2$$

=>

$$u_1 x = u_2 x\gamma_2\gamma_1^{-1}$$

=>

$$u_2^{-1} u_1 x = x\gamma_2\gamma_1^{-1}$$

=>

$$x^{-1} u_2^{-1} u_1 x = \gamma_2\gamma_1^{-1}.]$$

Since Γ is discrete, the compact set $x^{-1}\mathfrak{U}^{-1}\mathfrak{U}x$ contains a finite number N of elements of Γ. So, for fixed x,y, not more than N of the $\mathfrak{U}x\gamma_2 y^{-1}$ can intersect $\mathfrak{U}x\gamma_1 y^{-1}$.

<u>13:</u> <u>N.B.</u> Consider the case when N = 1. Since it is always true that $e \in x^{-1}\mathfrak{U}^{-1}\mathfrak{U}x$, in this situation the $\mathfrak{U}x\gamma y^{-1}$ are disjoint, hence

$$\sum_{\gamma\in\Gamma} \int_{\mathfrak{U}} \phi(ux\gamma y^{-1})d\mu_G(u)$$

$$\le \int_G \phi d\mu_G < \infty.$$

<u>14:</u> RAPPEL If μ is a measure, then

$$\sum_{i=1}^{n} \mu(X_i) = \mu(\bigcup_{i=1}^{n} X_i) + \mu(\bigcup_{\substack{i=1 \\ i<j}}^{n} \bigcup_{j=1}^{n} X_i \cap X_j)$$

$$+ \mu\left(\bigcup_{i=1}^{n} \bigcup_{j=1}^{n} \bigcup_{\substack{k=1 \\ i<j<k}}^{n} X_i \cap X_j \cap X_k\right) + \cdots + \mu\left(\bigcap_{i=1}^{n} X_i\right).$$

<u>15:</u> LEMMA Fix $x,y \in G$ -- then

$$\sum_{\gamma \in \Gamma} |f(x\gamma y^{-1})| \leq NC \int_G \phi d\mu_G < \infty.$$

<u>16:</u> <u>N.B.</u> More is true: The series

$$\sum_{\gamma \in \Gamma} f(x\gamma y^{-1})$$

is ldac on $G \times G$ to a continuous function.

[The point is that the preceding estimate is uniform in x and y if these variables are confined to compacta K_x and K_y.]

[Note: Consequently,

$$f \text{ admissible} \Rightarrow L_{G/\Gamma}(f) \text{ Hilbert-Schmidt.}]$$

<u>17:</u> THEOREM If f is admissible and if $L_{G/\Gamma}(f)$ is trace class, then the Selberg trace formula obtains (cf. #10).

<u>18:</u> <u>N.B.</u>

$$f \text{ admissible} \Rightarrow f * f \text{ admissible (cf. \#8).}$$

Therefore

$$L_{G/\Gamma}(f * f) = L_{G/\Gamma}(f)L_{G/\Gamma}(f)$$

is trace class and the foregoing is applicable.

Specialize now to the case when G is a connected semisimple Lie group with finite center.

<u>19:</u> RAPPEL $\mathcal{C}^1(G)$ is the L^1-Schwartz space of G. It is closed under convolution and contains $C_c^\infty(G)$ as a dense subspace.

Let $f \in \mathcal{C}^1(G)$ and take $r > 0$ per #4 -- then there exists a constant $C > 0$ such that

$$|f(y)| \leq C \; |{-\!\!\circ\!\!-}|^2(y)(1 + \sigma(y))^{-r} \qquad (y \in G).$$

Therefore f is admissible.

<u>20:</u> LEMMA $L_{G/\Gamma}(f)$ is trace class.

[Using the theory of the parametrix, write

$$f = g * \mu + f * \nu,$$

where $g \in \mathcal{C}^1(G)$ (a certain derivative of f), $\mu \in C_c^p(G)$, $\nu \in C_c^\infty(G)$, so

$$L_{G/\Gamma}(f) = L_{G/\Gamma}(g)L_{G/\Gamma}(\mu) + L_{G/\Gamma}(f)L_{G/\Gamma}(\nu).$$

The functions

$$f, g, \mu, \nu$$

are admissible, hence the operators

$$L_{G/\Gamma}(f),\ L_{G/\Gamma}(g),\ L_{G/\Gamma}(\mu),\ L_{G/\Gamma}(\nu)$$

are Hilbert-Schmidt.]

<u>21:</u> SCHOLIUM $\forall\ f \in \mathcal{C}^1(G)$, the Selberg trace formula obtains.

22: N.B. The assignment

$$f \to \mathrm{tr}(L_{G/\Gamma}(f))$$

is continuous in the topology of $C^1(G)$.

[Note: Analogously, the assignment

$$f \to \mathrm{tr}(L_{G/\Gamma}(f))$$

is continuous in the topology of $C_c^\infty(G)$, i.e., is a distribution on G.]

APPENDIX

By way of reconciliation, consider the case when G is finite and use the notation of A, III, §3 and §4 -- then given $f \in C(G)$, $\phi \in C(G/\Gamma)$, we have

$$(L_{G/\Gamma}(f)\phi)(x) = \sum_{y \in G} K_f(x,y)\phi(y),$$

where in this context

$$K_f(x,y) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma y^{-1}).$$

Here

$$\begin{cases} \mu_G = \text{counting measure on } G \\ \mu_\Gamma = \text{counting measure on } \Gamma. \end{cases}$$

Write

$$G = \coprod_{k=1}^{n} x_k\Gamma.$$

Then for any $f \in C(G)$,

$$\int_G f d\mu_G = \sum_{x \in G} f(x)$$

$$= \int_{G/\Gamma} f^{\Gamma} d\mu_{G/\Gamma}$$

$$= \sum_{k=1}^{n} \sum_{\gamma\in\Gamma} f(x_k\gamma),$$

so $\mu_{G/\Gamma}$ is counting measure on G/Γ.

Now explicate matters:

$$\begin{aligned}(L_{G/\Gamma}(f)\phi)(x) &= \sum_{y\in G} K_f(x,y)\phi(y) \\ &= \sum_{k=1}^{n} \sum_{\gamma\in\Gamma} K_f(x,x_k\gamma)\phi(x_k\gamma) \\ &= \sum_{k=1}^{n} \sum_{\gamma\in\Gamma} K_f(x,x_k)\phi(x_k) \\ &= \sum_{k=1}^{n} |\Gamma| \cdot K_f(x,x_k)\phi(x_k) \\ &= \sum_{k=1}^{n} |\Gamma| \cdot \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} f(x\gamma x_k^{-1})\phi(x_k) \\ &= \sum_{k=1}^{n} \sum_{\gamma\in\Gamma} f(x\gamma x_k^{-1})\phi(x_k)\end{aligned}$$

which establishes that $L_{G/\Gamma}(f)$ is an integral operator on $C(G/\Gamma)$ with kernel

$$\sum_{\gamma\in\Gamma} f(x\gamma y^{-1}),$$

this being the "K_f" of §4, #1.

There is more to be said. Thus given $f \in C(G)$, we have

$$\mathrm{tr}(L_{G/\Gamma}(f)) = \sum_{x\in G} \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} f(x\gamma x^{-1}) \qquad \text{(cf. A, III, §3, \#8)}$$

$$= \sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} \mathcal{O}(f,\gamma_i) \quad \text{(cf. A, III, §4, #2).}$$

Here

$$[\Gamma] = \{\gamma_1,\ldots,\gamma_n\}$$

while

$$\mathcal{O}(f,\gamma_i) = \sum_{x\in G} f(x\gamma_i x^{-1})$$

$$= |G_{\gamma_i}| \sum_{x\in G/G_{\gamma_i}} f(x\gamma_i x^{-1}).$$

Therefore

$$\mathrm{vol}(G_{\gamma_i}/\Gamma_{\gamma_i}) = \frac{|G_{\gamma_i}|}{|\Gamma_{\gamma_i}|}$$

$$= [G_{\gamma_i}:\Gamma_{\gamma_i}].$$

N.B. The Haar measures μ_G (or μ_{G_γ}) and μ_Γ (or μ_{Γ_γ}) are counting measures, hence the invariant measure $\mu_{G/\Gamma}$ (or $\mu_{G_\gamma/\Gamma_\gamma}$) is counting measure, hence the invariant measure μ_{G/G_γ} per

$$\int_G = \int_{G/G_\gamma} \int_{G_\gamma}$$

is counting measure, its total volume being

$$[G:G_\gamma] = \frac{|G|}{|G_\gamma|}.$$

Finally, the invariant measure μ_{G/Γ_γ} per

$$\int_{G/\Gamma_\gamma} = \int_{G/G_\gamma} \int_{G_\gamma/\Gamma_\gamma}$$

is counting measure and

$$\mathrm{vol}(G/\Gamma_\gamma) = \mathrm{vol}(G/G_\gamma)\mathrm{vol}(G_\gamma/\Gamma_\gamma),$$

i.e.,

$$[G:\Gamma_\gamma] = [G:G_\gamma][G_\gamma:\Gamma_\gamma],$$

i.e.,

$$\frac{|G|}{|\Gamma_\gamma|} = \frac{|G|}{|G_\gamma|}[G_\gamma:\Gamma_\gamma]$$

=>

$$\frac{|G_\gamma|}{|\Gamma_\gamma|} = [G_\gamma:\Gamma_\gamma].$$

Matters are thus consistent, so the bottom line is that the global trace formula of A, III, §4, #6 is in this context the Selberg trace formula.

§6. *DISCRETE SERIES*

Let G be a unimodular locally compact group.

1: DEFINITION Let Π be an irreducible unitary representation of G on a Hilbert space $V(\Pi)$ -- then Π is *square integrable* if $\exists\ v \neq 0$ in $V(\Pi)$ such that the coefficient

$$x \to \langle \Pi(x)v,v\rangle$$

is square integrable on G.

2: THEOREM If Π is square integrable, then for all $v_1,v_2 \in V(\Pi)$, the coefficient

$$x \to \langle \Pi(x)v_1,v_2\rangle$$

lies in $L^2(G)$ and there exists a unique positive real number d_Π (depending on the normalization of the Haar measure on G but independent of v_1,v_2) such that

$$\int_G |\langle \Pi(x)v_1,v_2\rangle|^2 \, d\mu_G(x) = \frac{1}{d_\Pi} ||v_1||^2 ||v_2||^2.$$

3: DEFINITION d_Π is called the *formal dimension* of Π.

[Note: If G is compact, then every irreducible unitary representation of G is square integrable and d_Π is the dimension of Π in the usual sense provided $\int_G d\mu_G = 1$.

4: NOTATION $\hat{G}_d$ is the subset of $\hat{G}$ comprised of the square integrable representations and is called the *discrete series* for G.

[Note: $\hat{G}_d$ may very well be empty (e.g., take $G = R$).]

5: REMARK If $\hat{G}_d$ is not empty, then the center of G is compact (the converse being false).

6: N.B. The elements of $\hat{G}_d$ are precisely those irreducible unitary representations of G which occur as irreducible subrepresentations of the left translation representation of G on $L^2(G)$.

7: NOTATION Given a $\Pi \in \hat{G}_d$, let

$$\phi_{\cdot,\cdot}(x) = \langle \Pi(x)\cdot,\cdot \rangle \quad (x \in G)$$

stand for a generic coefficient.

8: THEOREM Suppose that Π is square integrable -- then $\forall\ v_1, v_2$, $\forall\ w_1, w_2$ in $V(\Pi)$,

$$\int_G \phi_{v_1,v_2}(x)\ \overline{\phi_{w_1,w_2}(x)}\,d\mu_G(x)$$

$$= \frac{1}{d_\Pi} \langle v_1, w_1 \rangle \overline{\langle v_2, w_2 \rangle}.$$

9: APPLICATION

$$\phi_{v_1,v_2} * \phi_{w_1,w_2} = \frac{1}{d_\Pi} \langle v_1, w_2 \rangle \phi_{w_1,v_2}.$$

[Note: If $v_1 = v_2 = w_1 = w_2$ is a unit vector, call it v and abbreviate $\phi_{v,v}$ to ϕ, then

$$||\phi||_2^2 = \frac{1}{d_\Pi} \text{ and } \phi * \phi = \frac{1}{d_\Pi}\, \phi.]$$

10: DEFINITION Let Π be an irreducible unitary representation of G on a Hilbert space $V(\Pi)$ -- then Π is integrable if $\exists$ $v \neq 0$ in $V(\Pi)$ such that the coefficient

$$x \to \langle \Pi(x)v,v \rangle$$

is integrable on G.

11: N.B. The coefficient

$$x \to \langle \Pi(x)v,v \rangle$$

is bounded and L^1, hence is L^2. Therefore

$$\text{"}\Pi \text{ integrable"} \Rightarrow \text{"}\Pi \text{ square integrable"}$$

but the converse is false.

12: THEOREM If Π is integrable, then there exists a dense subspace $V(\Pi)^\sim$ of $V(\Pi)$ such that for all v_1,v_2 in $V(\Pi)^\sim$ the coefficient

$$x \to \langle \Pi(x)v_1,v_2 \rangle$$

lies in $L^1(G)$.

[Note: If $\phi_{v,v} \in L^1(G)$, then one can take

$$V(\Pi)^\sim = \Pi(C_c(g))v.]$$

Take G second countable and assume that $\Pi \in \hat{G}$ is integrable, say $\phi_{v,v} \in L^1(G)$ -- then $\forall$ $f \in C_c(G)$,

$$\phi_{\Pi(f)v,\Pi(f)v} \in L^1(G) \quad \text{(cf. \#12)}.$$

Put $v_0 = \Pi(f)v$, normalized by $||v_0|| = 1$, and let

$$\phi_0 = d_\Pi \phi_{v_0,v_0}.$$

13: N.B.

$$\phi_0 * \phi_0 = d_\Pi \phi_{v_0,v_0} * d_\Pi \phi_{v_0,v_0}$$

$$= d_\Pi^2 \phi_{v_0,v_0} * \phi_{v_0,v_0}$$

$$= d_\Pi^2 \frac{1}{d_\Pi} \phi_{v_0,v_0} \qquad \text{(cf. \#9)}$$

$$= d_\Pi \phi_{v_0,v_0} = \phi_0.$$

It is also clear that $\phi_0^* = \phi_0$ and $\Pi(\phi_0)v_0 = v_0$.

14: NOTATION If Π is an irreducible unitary representation of G and if π is a unitary representation of G, then

$$I_G(\Pi,\pi)$$

is the set of intertwining operators between Π and π.

15: LEMMA For any unitary representation π of G, $\pi(\bar{\phi}_0)$ is the orthogonal projection onto

$$\{Tv_0 : T \in I_G(\Pi,\pi)\}.$$

[Note: It's $\pi(\bar{\phi}_0)$, not $\pi(\phi_0)$... .]

Suppose that $\Gamma \subset G$ is a uniform lattice and take $\pi = L_{G/\Gamma}$.

16: APPLICATION

$$L_{G/\Gamma}(\bar{\phi}_0)$$

is trace class and

$$\text{tr}(L_{G/\Gamma}(\bar{\phi}_0)) = \dim I_G(\Pi, L_{G/\Gamma}) = m(\Pi, L_{G/\Gamma}).$$

17: THEOREM The series

$$\sum_{\gamma\in\Gamma} \phi_0(x\gamma y^{-1})$$

is ldac on $G \times G$ to a separately continuous function.

PROOF Let $K \subset G$ be compact and let

$$n(K) = |\Gamma \cap K^{-1}\mathrm{spt}(f)\,\mathrm{spt}(f)^{-1}K|.$$

Then the arrow

$$\mathrm{spt}(f)^{-1}K \to G/\Gamma$$

is at most $n(K)$-to-1 and $\forall\ x \in K$,

$$\sum_{\gamma\in\Gamma} |\phi_0(x\gamma y^{-1})| = d_\Pi \sum_{\gamma\in\Gamma} |\langle\Pi(x\gamma y^{-1})v_0,v_0\rangle|$$

$$= d_\Pi \sum_{\gamma\in\Gamma} |\langle\Pi(x\gamma y^{-1})v_0,\Pi(f)v\rangle|$$

$$= d_\Pi \sum_{\gamma\in\Gamma} |\langle\Pi(f)v,\Pi(x\gamma y^{-1})v_0\rangle|$$

$$= d_\Pi \sum_{\gamma\in\Gamma} |\int_G f(z)\langle\Pi(z)v,\Pi(x\gamma y^{-1})v_0\rangle d\mu_G(z)|$$

$$= d_\Pi \sum_{\gamma\in\Gamma} |\int_G f(z)\langle v,\Pi(z^{-1}x\gamma y^{-1})v_0\rangle d\mu_G(z)|$$

$$= d_\Pi \sum_{\gamma\in\Gamma} |\int_G f(z^{-1})\langle v,\Pi(zx\gamma y^{-1})v_0\rangle d\mu_G(z)|$$

$$= d_\Pi \sum_{\gamma\in\Gamma} |\int_G f(z^{-1})\overline{\langle\Pi(zx\gamma y^{-1})v_0,v\rangle} d\mu_G(z)|$$

$$\leq d_\Pi \sum_{\gamma\in\Gamma} \int_G |f(z^{-1})|\,|\overline{\langle\Pi(zx\gamma y^{-1})v_0,v\rangle}|d\mu_G(z)$$

$$= d_\Pi \sum_{\gamma\in\Gamma} \int_G |f(z^{-1})|\,|<\Pi(zx\gamma y^{-1})v_0,v>|\,d\mu_G(z)$$

$$\leq d_\Pi\ ||f||_\infty \sum_{\gamma\in\Gamma} \int_{spt(f)^{-1}K} |<\Pi(z\gamma y^{-1})v_0,v>|\,d\mu_G(z)$$

$$\lesssim d_\Pi\ ||f||_\infty\ n(K) \int_{G/\Gamma} \sum_{\gamma\in\Gamma} |<\Pi(\dot{z}\gamma y^{-1})v_0,v>|\,d\mu_{G/\Gamma}(\dot{z})$$

$$= d_\Pi\ ||f||_\infty\ n(K) \int_G |<\Pi(zy^{-1})v_0,v>|\,d\mu_G(z)$$

$$= d_\Pi\ ||f||_\infty\ n(K) \int_G |<\Pi(z)v_0,v>|\,d\mu_G(z)$$

$$= d_\Pi\ ||f||_\infty\ n(K)\ ||\phi_{v_0,v}||_1.$$

And

$$||\phi_{v_0,v}||_1 \leq \int_G \int_G |f(y)<\Pi(xy)v,v>|\,d\mu_G(x)\,d\mu_G(y)$$

$$\leq ||f||_1\ ||\phi_{v,v}||_1 < \infty,$$

thereby settling the ldac condition (and then some (no restriction on "y")), leaving the claim of separate continuity which can be left to the reader.

The operator $L_{G/\Gamma}(\bar{\phi}_0)$ is trace class (cf. #16). So, in view of what has been said above, the criterion of §5, #10 is applicable.

<u>18:</u> SCHOLIUM

$$m(\Pi,L_{G/\Gamma}) = \sum_{\gamma\in[\Gamma]} \mathrm{vol}(G_\gamma/\Gamma_\gamma) \int_{G/G_\gamma} \overline{\phi_0(x\gamma x^{-1})}\,d\mu_{G/G_\gamma}(\dot{x}),$$

the sum on the right hand side being absolutely convergent.

19: REMARK There are circumstances in which the integral

$$\int_{G/G_\gamma} \overline{\phi_0(x\gamma x^{-1})}\, d\mu_{G/G_\gamma}(\dot{x})$$

vanishes for all γ except $\gamma = e$, hence then

$$\begin{aligned} m(\Pi, L_{G/\Gamma}) &= \mathrm{vol}(G/\Gamma)\overline{\phi_0(e)} \\ &= \mathrm{vol}(G/\Gamma) d_\Pi \langle \Pi(e) v_0, v_0\rangle \\ &= \mathrm{vol}(G/\Gamma) d_\Pi \langle v_0, v_0\rangle \\ &= \mathrm{vol}(G/\Gamma) d_\Pi. \end{aligned}$$

Therefore $m(\Pi, L_{G/\Gamma})$ is positive, so Π definitely occurs in $L_{G/\Gamma}$.

[Note: To run a reality check, take G finite, $\Gamma = \{e\}$ -- then $\mathrm{vol}(G/\Gamma) = \mathrm{vol}(G) = 1$ and $\forall\ \Pi \in \hat{G}$,

$$m(\Pi, L_{G/\Gamma}) = d_\Pi \text{ (cf. A, II, §5, #8 and A, III, §3, #15).]}$$

20: N.B. The situation envisioned in #19 is realized if G is a connected semisimple Lie group with finite center and if Γ has no elements of finite order other than the identity.

21: LEMMA If G is a Lie group and if $f \in C_c^\infty(G)$, then the series

$$\sum_{\gamma\in\Gamma} \phi_0(x\gamma y^{-1})$$

is a C^∞ function of x,y.